PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 6, June 2007, Pages 1881–1887
S 0002-9939(07)08698-4
Article electronically published on February 2, 2007

LACK OF UNIFORMLY EXPONENTIAL STABILIZATION FOR ISOMETRIC C_0-SEMIGROUPS UNDER COMPACT PERTURBATION OF THE GENERATORS IN BANACH SPACES

FAMING GUO, KE GUO, AND CHAOLUN ZHANG

(Communicated by Joseph A. Ball)

Abstract. This paper is concerned with non-uniformly exponential stabilization for infinite-dimensional linear systems under compact feedback in Banach spaces. We prove that a compact perturbation of the generator of an isometric C_0-semigroup cannot generate a uniformly exponentially stable C_0-semigroup in a Banach space. Finally, examples are provided to illustrate our result.

1. Introduction

The controllability and stabilization problems of conservative systems by means of feedback control is very important and has been investigated extensively by many authors. See, for example, [3], [6], [7], [8], [10], [11], [13], [14] and the references cited therein. Let us consider the infinite-dimensional linear control system

$$x'(t) = Ax(t) + Bu(t), \tag{1}$$

where the state space X and the control space U are Banach spaces, $(A, D(A))$ is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X, and B is a bounded linear operator from U into X. The purpose of this paper is to study the non-uniformly exponential stabilization for an isometric C_0-semigroup under compact perturbation of its generator in the Banach space setting.

Definition 1. $\{S(t)\}_{t\geq 0}$ is said to be an isometric C_0-semigroup if it is a C_0-semigroup that satisfies $\|S(t)x\| = \|x\|$ for all $t \geq 0$ and $x \in X$.

Definition 2. A C_0-semigroup $\{S(t)\}_{t\geq 0}$ is said to be strongly stable if $S(t)x \to 0$ as $t \to \infty$ for any $x \in X$.

Definition 3. A C_0-semigroup $\{S(t)\}_{t\geq 0}$ is said to be uniformly exponentially stable if there exist constants $M \geq 1$ and $w > 0$ such that

$$\|S(t)\| \leq Me^{-wt}, \quad t \geq 0.$$

For the characterizations of the isometric C_0-semigroup and some criteria for stability of C_0-semigroup, see, for example, [1], [2], [4] and [5].

Received by the editors January 14, 2006 and, in revised form, February 23, 2006.
2000 *Mathematics Subject Classification.* Primary 47A50, 47A55.
Key words and phrases. Compact perturbation, isometric C_0-semigroup, exponential stability.
This research was supported by the Postdoctoral Science Foundation of China.

Feedback control problems of system (1) by the state feedback $u(t) = Cx(t)$ are of interest and arise in the context of control theoretic studies for infinite-dimensional linear systems, where an aim is to select the feedback operator C so as to force the corresponding feedback system

$$x'(t) = Ax(t) + Kx(t), \quad K := BC, \tag{2}$$

to possess stability properties not enjoyed by the original free system $x'(t) = Ax(t)$. Furthermore, an important feedback operator in physical applications is the compact feedback operator. However, some important systems cannot be uniformly exponentially stabilized by compact operators; see, for example, Russell [13], Gibson [3], Triggiani [15].

In Section 2, we prove our main result on the non-uniformly exponential stabilization for an isometric C_0-semigroup by quite different methods from [3], [9], [13] and [15]. More precisely, an isometric C_0-semigroup cannot be uniformly exponentially stabilized by a compact operator in a Banach space. In Section 3, we apply our result to the stabilization problem of the hybrid system where a clamped elastic beam is linked to a rigid antenna.

2. Non-uniformly exponential stabilization by compact perturbation

Russell [13], Gibson [3], and Triggiani [15] have investigated a sort of C_0-semigroup, which is strongly stable and is not uniformly exponentially stable, and cannot be uniformly exponentially stabilized by a compact operator. These results are well known and extensively invoked in the literature of feedback control systems.

Theorem 1 (Gibson [3]). *Let H be a Hilbert space, let $\{S(t)\}_{t\geq 0}$ be a strongly stable contractive C_0-semigroup with generator $(A, D(A))$ on H, and let K be a compact operator on H. Then, if the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$, generated by $(A+K,\ D(A))$, is uniformly exponentially stable, so is $\{S(t)\}_{t\geq 0}$.*

It complements a result of Russell [13] where $\{S(t)\}_{t\in R}$ was assumed to be a group, contractive for negative times. The following theorem in Triggiani [15] improves the result of Gibson by removing the assumption that $\{S(t)\}_{t\geq 0}$ be a contractive C_0-semigroup. For its generalization in the Banach space setting, see Luo, Weng and Feng [9].

Theorem 2 (Triggiani [15]). *Let H be a Hilbert space, and let $\{S(t)\}_{t\geq 0}$ be a strongly stable, not uniformly exponentially stable C_0-semigroup with generator $(A, D(A))$ on H. Then, for any compact operator K on H, the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$, generated by $(A+K,\ D(A))$, cannot be uniformly exponentially stable.*

An equivalent statement of Gibson's result is that the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$ cannot be uniformly exponentially stable if $\{S(t)\}_{t\geq 0}$ is a strongly stable, not uniformly exponentially stable contraction C_0-semigroup and K is a compact operator on H. However, $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if $\|S(t_0)\| < 1$ for some $t_0 > 0$ (Proposition 5.1.7 in [1]). Therefore, the following natural question arises: can an isometric C_0-semigroup be uniformly exponentially stabilized by a compact operator?

In the following theorem, we will answer the question, and our method presented here is quite different from those in the literature.

Theorem 3. *Let X be an infinite-dimensional Banach space, let $\{S(t)\}_{t\geq 0}$ be an isometric C_0-semigroup with the generator $(A, D(A))$ on X, and let K be a compact operator on X. Then, the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$, generated by $(A+K,\ D(A))$, cannot be uniformly exponentially stable.*

Proof. Let $U^* := \{x^* \in X^* :\ \|x^*\| \leq 1\}$ be the unit ball in X^*. Since X is an infinite-dimensional Banach space, X^* is also infinite dimensional, which implies that U^* is not a compact set. Thus, there exist $\epsilon_0 > 0$ and $y_j^* \in U^*, j = 1, 2, \dots,$ such that

$$\|y_i^* - y_j^*\| \geq \epsilon_0 > 0$$

for all $i \neq j$.

For the compactness of operator K, we conclude that $R(K)$, which is the range of operator K, is separable, and consequently, $\{S(\tau)Kx :\ x \in X\}$ is separable for each $\tau \geq 0$. Hence,

$$\bigcup_{\tau\in Q^+} \{S(\tau)Kx :\ x \in X\}$$

is also separable, where Q^+ is the set of non-negative rational numbers which is denumerable. Therefore, $V := span\{S(\tau)Kx :\ x \in X,\ \tau \geq 0\}$ is separable, or equivalently, there exist a countable dense subset $\{v_1, v_2, \dots, v_m, \dots\}$ in V. From the boundedness of $\{y_k^*\}_{k=1}^\infty$, we have that $\{y_1^*(v_1), y_2^*(v_1), \dots, y_k^*(v_1), \dots\}$ is bounded, and consequently, there is a subsequence $\{y_{k_1}^*(v_1), y_{k_2}^*(v_1), \dots, y_{k_n}^*(v_1), \dots\}$ such that $\{y_{k_n}^*(v_1)\}_{n=1}^\infty$ is convergent as $n \to \infty$. Similarly, $\{y_{k_1}^*(v_2), y_{k_2}^*(v_2), \dots,$ $y_{k_n}^*(v_2), \dots\}$ is bounded and there is a subsequence $\{y_{k_{i_1}}^*(v_2), y_{k_{i_2}}^*(v_2), \dots,$ $y_{k_{i_n}}^*(v_2), \dots\}$ such that $\{y_{k_{i_n}}^*(v_2)\}_{n=1}^\infty$ is convergent. Therefore, by the diagonal method, there is a subsequence $\{y_{j_n}^*\}_{n=1}^\infty$ of $\{y_j^*\}_{j=1}^\infty$ such that $\{y_{j_n}^*(v_m)\}_{n=1}^\infty$ is convergent for any v_m. Hence, from $\|y_{j_n}^*\| = 1$ and the denseness of $\{v_m\}_{m=1}^\infty$ in V, it follows that $\{y_{j_n}^*(x)\}_{n=1}^\infty$ is convergent for any $x \in \overline{V}$. Now, for $x \in \overline{V}$, we can define a linear functional $f_0(x) := \lim_{n\to\infty} y_{j_n}^*(x)$, which is continuous in $\overline{V}$. From the Hahn-Banach Theorem, there exists a bounded linear functional $f \in X^*$ such that

$$f|_{\overline{V}} = f_0 \ \text{ and } \ \|f\| = \|f_0\|.$$

Therefore, we have

$$K^*S^*(\tau)y_{j_n}^* \longrightarrow K^*S^*(\tau)f \ \text{ as } n \to \infty \tag{3}$$

in X^* for each $\tau \geq 0$. In fact, assume that there exist $\eta > 0,\ \tau_0 > 0$ and a subsequence $\{y_{j_{n_k}}^*\}_{k=1}^\infty$ of $\{y_{j_n}^*\}_{n=1}^\infty$ such that

$$\|K^*S^*(\tau_0)y_{j_{n_k}}^* - K^*S^*(\tau_0)f\| > \eta, \ \ k = 1, 2, \dots.$$

Hence, there exist $x_k \in X$ with $\|x_k\| = 1, k = 1, 2, \dots,$ such that

$$|(K^*S^*(\tau_0)y_{j_{n_k}}^*)(x_k) - (K^*S^*(\tau_0)f)(x_k)| > \frac{\eta}{2}, \ \ k = 1, 2, \dots. \tag{4}$$

Since K is compact, without loss of generality, there exists $y_0 \in X$ such that $Kx_k \longrightarrow y_0$ as $k \to \infty$, and then $S(\tau_0)y_0 \in \overline{V}$. Therefore, we deduce that

$$\begin{aligned}
& |(K^*S^*(\tau_0)y^*_{j_{n_k}})(x_k) - (K^*S^*(\tau_0)f)(x_k)| \\
= \quad & |y^*_{j_{n_k}}(S(\tau_0)Kx_k) - f(S(\tau_0)Kx_k)| \\
\leq \quad & |y^*_{j_{n_k}}(S(\tau_0)Kx_k) - y^*_{j_{n_k}}(S(\tau_0)y_0)| + |y^*_{j_{n_k}}(S(\tau_0)y_0) - f(S(\tau_0)y_0)| \\
& +|f(S(\tau_0)y_0) - f(S(\tau_0)Kx_k)| \\
\leq \quad & (1+\|f\|)\|Kx_k - y_0\| + |y^*_{j_{n_k}}(S(\tau_0)y_0) - f_0(S(\tau_0)y_0)| \\
\longrightarrow \quad & 0,
\end{aligned}$$

as $k \to \infty$, and there is a contradiction with (4) which shows the validity of (3).

Without loss of generality, assume $\|y^*_{j_n} - f\| > \frac{\epsilon_0}{2}$. Setting $x^*_n = \frac{y^*_{j_n} - f}{\|y^*_{j_n} - f\|}$, we have

$$\|x^*_n\| = 1 \ \text{ and } \ \|K^*S^*(\tau)x^*_n\| \to 0 \ (n \to \infty) \quad \text{for each } \tau \geq 0. \tag{5}$$

Let $f_{nm}(\tau) : \ [0, \ \infty) \mapsto X^*$ be defined by

$$f_{nm}(\tau) = \begin{cases} S^*_K(\tau)K^*S^*(m-\tau)x^*_n, & \tau \in [0, \ m], \\ 0, & \tau \in (m, \ \infty), \end{cases}$$

thus, for $m = 1, 2, \ldots$ and $\tau \in [0, \infty)$, we have

$$\|f_{nm}(\tau)\| \to 0 \ \ (n \to \infty).$$

Therefore, by the diagonal method, there exists a subsequence $\{x^*_{n_m}\}_{m=1}^{\infty}$ of $\{x^*_n\}_{n=1}^{\infty}$ such that

$$g_m(\tau) := f_{n_m m}(\tau) \to 0 \ \text{ as } m \to \infty \tag{6}$$

for $\tau \in [0, \infty)$.

Suppose on the contrary that $\{S_K(t)\}_{t \geq 0}$ is uniformly exponentially stable. Then there exist constants $M_K \geq 1$ and $w_K > 0$ such that

$$\|S_K(t)\| \leq M_K e^{-w_K t}, \quad t \geq 0.$$

Thus,

$$\begin{aligned}
\|g_m(\tau)\| \quad &= \quad \begin{cases} \|S^*_K(\tau)K^*S^*(m-\tau)x^*_{n_m}\|, & \tau \in [0, \ m], \\ 0, & \tau \in (m, \ \infty) \end{cases} \\
&\leq \quad \begin{cases} M_K\|K\|e^{-w_K\tau}, & \tau \in [0, \ m], \\ 0, & \tau \in (m, \ \infty). \end{cases} \qquad (7)
\end{aligned}$$

Hence, we conclude that

$$\lim_{m\to\infty} \int_0^{\infty} \|g_m(\tau)\| d\tau = \lim_{m\to\infty} \int_0^{m} \|S^*_K(\tau)K^*S^*(m-\tau)x^*_{n_m}\| d\tau = 0 \tag{8}$$

by the Lebesgue dominated convergence theorem.

From the bounded perturbation theorem of C_0-semigroups ([1] and [4]), it follows that

$$S_K^*(t)x^* = S^*(t)x^* + \int_0^t S_K^*(\tau)K^*S^*(t-\tau)x^*d\tau,\ x^* \in X^*,\ t \geq 0,$$

which yields that

$$\begin{aligned} & \lim_{m\to\infty} \|S^*(m)x_{n_m}^*\| \\ \leq\ & \lim_{m\to\infty} \left(\|S_K^*(m)x_{n_m}^*\| + \| \int_0^m S_K^*(\tau)K^*S^*(m-\tau)x_{n_m}^* d\tau \| \right) \\ \leq\ & \lim_{m\to\infty} \left(M_K e^{-w_K m} + \int_0^m \|S_K^*(\tau)K^*S^*(m-\tau)x_{n_m}^*\| d\tau \right) \\ =\ & 0. \end{aligned} \tag{9}$$

But $\{S(t)\}_{t\geq 0}$ is an isometric C_0-semigroup; we have $\|S^*(m)x_{n_m}^*\| = \|x_{n_m}^*\| = 1$ for all m. There is a contradiction with (9) which ends the proof of the theorem. □

3. Application

In this section, we present examples of isometric C_0-semigroups where our result above is applicable. We first study the boundary feedback stabilization of the hybrid system of elasticity in [6], which consists of a clamped elastic beam linked to a rigid antenna.

Example 1. The hybrid system can be described by the Euler-Bernoulli equation for the vibrations of the elastic beam and the Newton-Euler rigid-body equations for the oscillations of the antenna,

$$\begin{cases} y_{tt} + y_{xxxx} = 0, & t > 0,\ \ 0 < x < 1, \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ \mu_1 y_{tt}(1,t) - y_{xxx}(1,t) = L_1(y, y_t), & t > 0, \\ \mu_2 y_{xtt} + y_{xx}(1,t) = L_2(y, y_t), & t > 0, \end{cases} \tag{10}$$

where μ_1, μ_2 are positive constants, the beam is clamped at $x = 0$ and the boundary feedback operators L_1, L_2 are the following:

$$\begin{cases} L_1(y, y_t) = -a_{11}y(1,t) - a_{12}y_x(1,t) - c_{11}y_t(1,t) - c_{12}y_{xt}(1,t), \\ L_2(y, y_t) = -a_{21}y(1,t) - a_{22}y_x(1,t) - c_{21}y_t(1,t) - c_{22}y_{xt}(1,t), \end{cases} \tag{11}$$

where the coefficients $a_{ij}, c_{ij}, i, j = 1, 2$, are real numbers. For further descriptions concerning the physical structure of the hybrid system, we refer to [6].

We choose the energy space

$$\mathcal{H} := \{(y, z, \xi, \eta) \in H^2(0,1) \times L^2(0,1) \times R \times R :\ y(0) = y_x(0) = 0\}$$

with the inner product

$$\langle (y_1, z_1, \xi_1, \eta_1),\ \ (y_2, z_2, \xi_2, \eta_2) \rangle = \int_0^1 (y_{1_{xx}} y_{2_{xx}} + z_1 z_2) dx + \mu_1 \xi_1 \xi_2 + \mu_2 \eta_1 \eta_2,$$

and define the operator $(A,\ D(A))$ and B as follows:

$$D(A) = \left\{ \begin{array}{l} v = (y, z, \xi, \eta) \in H^4(0,1) \times H^2(0,1) \times R \times R : \\ \qquad y(0) = y_x(0) = z(0) = z_x(0) = 0, \ \ \xi = z(1), \ \ \eta = z_x(1) \end{array} \right\}$$

and

$$Av = (-z, y_{xxxx}, -\frac{1}{\mu_1} y_{xxx}(1), -\frac{1}{\mu_2} y_{xx}(1)), \quad v = (y, z, \xi, \eta) \in D(A),$$
$$Bv = (0, 0, b_{11} + b_{12}, b_{21} + b_{22}), \quad v = (y, z, \xi, \eta) \in \mathcal{H},$$

where

$$b_{11} = \frac{1}{\mu_1}(a_{11} y(1) + a_{12} y_x(1)), \ \ b_{12} = \frac{1}{\mu_1}(c_{11}\xi + c_{12}\eta),$$
$$b_{21} = \frac{1}{\mu_2}(a_{21} y(1) + a_{22} y_x(1)), \ \ b_{22} = \frac{1}{\mu_2}(c_{21}\xi + c_{22}\eta).$$

Now we can formulate the system (10) into the following abstract Cauchy problem:

$$\left\{ \begin{array}{l} \dfrac{d}{dt} v(t) + (A + B) v(t) = 0, \quad t > 0, \\ v(0) = v_0, \end{array} \right. \tag{12}$$

Littman and Markus [6] showed that $(A, D(A))$ generates an isometric C_0-semigroup on the energy space $\mathcal{H}$. From Sobolev's embedding theorem, the linear operator B is of finite rank, hence, compact. Therefore the system (10) or the equivalent abstract Cauchy problem (12) is not uniformly exponentially stable for any real constants $a_{ij}, c_{ij}, i, j = 1, 2$.

Example 2. Let $X := C_0(R^+)$ of all continuous functions on R^+ vanishing at infinity which is a Banach space under the supremum norm. Define the left translation operator $S_l(t)$ for $t \geq 0$ by

$$(S_l(t) f)(s) = f(t + s), \ \text{ for } \ s \in R^+ \text{ and } \ f \in X.$$

Then $\{S_l(t)\}_{t \geq 0}$ is an isometric C_0-semigroup only, not a group, called the left translation semigroup on X. The generator of $\{S_l(t)\}_{t \geq 0}$ is

$$A = \frac{d}{ds} \text{ with domain } D(A) = \{f \in X : \ f(s) \text{ is differential on } R^+ \text{ and } f' \in X\} \tag{13}$$

(for details, we refer to [1] and [4]).

Consider the infinite-dimensional linear control system on Banach space $X = C_0(R^+)$,

$$x'(t) = Ax(t) + Bu(t), \tag{14}$$

where the control space U is a Banach space, $(A, D(A))$ is as in (13), and B is a bounded linear operator from U into X. Thus, our Theorem 3 is applicable to the system (14), and we deduce that for any compact operator C on X, the system (14) cannot be uniformly exponentially stabilized by state feedback $u(t) = Cx(t)$, while Russell's, Gibson's and Triggiani's results would not be applicable in general.

Acknowledgments

The authors express their sincere thanks to the anonymous referee for his valuable suggestions.

References

[1] K.J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, 1999. MR1721989 (2000i:47075)

[2] H.O. Fattorini, *The Cauchy Problem*, Encyclopedia of Mathematics and its Applications, vol. 18, Addison-Wesley, 1983. MR0692768 (84g:34003)

[3] J.S. Gibson, A note on stabilization of infinite dimensional linear oscillators by compact linear feedback, *SIAM J. Control Optim.*, 18 (1980), 311-316. MR0569020 (81f:47040)

[4] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, 1985. MR0790497 (87c:47056)

[5] F.L. Huang, Strong asymptotic stability of linear dynamical systems in Banach spaces, *J. Differential Equations*, 104 (1993), 307-324. MR1231471 (94f:47048)

[6] W. Littman and L. Markus, Stabilization of a hybrid system of elasticity by feedback boundary damping, *Ann. Mat. Pura. Appl.,* 152 (1988), 281-330. MR0980985 (90c:93082)

[7] K. Liu, Locally distributed control and damping for the conversative system, *SIAM J. Control Optim.,* 35 (1997), 1574-1590. MR1466917 (98g:93005)

[8] K. Liu, Z. Liu, and B. Rao, Exponential stability of an abstract nondissipative linear system, *SIAM J. Control Optim.,* 40 (2001), 149-165. MR1855310 (2002g:93081)

[9] Y.F. Luo, S.P. Weng, and D.X. Feng, Stability of linear systems in Banach spaces, *Kexue Tongbao*, 43 (1998), 1676-1677 (in Chinese). MR1671371

[10] L. Miller, Controllability cost of conservative systems: resolvent conditions and tranmutation. *J. Funct. Anal.,* 218 (2005), 425-444. MR2108119 (2005i:93015)

[11] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak, A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. *J. Funct. Anal.,* 226 (2005), 193-229. MR2158180

[12] B.P. Rao, Uniform stabilization of a hybrid system of elasticity, *SIAM J. Control Optim.,* 33 (1995), 440-454. MR1318659 (95k:93089)

[13] D.L. Russell, Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods, *J. Differential Equations*, 19 (1975), 344-370. MR0425291 (54:13248)

[14] D. Tataru, Boundary controllability for conservative PDEs, *Appl. Math. Optim.,* 31 (1995), 257-295. MR1316260 (95k:93011)

[15] R. Triggiani, Lack of uniform stabilization for noncontractive semigroups under compact perturbation, *Proc. Amer. Math. Soc.*, 105 (1989), 375-383. MR0953013 (89g:47055)

[16] T.Y. Zhu and F.L. Huang, Non-exponential stability of infinite-dimensional linear systems under compact perturbation, *J. Sichuan University*, 40 (2003), 217-220. MR1976102 (2004b:47074)

Department of Applied Mathematics, College of Information Management, Chengdu University of Technology, Chengdu, 610059, People's Republic of China

E-mail address: guofm@uestc.edu.cn

Department of Applied Mathematics, College of Information Management, Chengdu University of Technology, Chengdu, 610059, People's Republic of China

Institute of Applied Mathematics, Xihua University, Chengdu, 610039, People's Republic of China