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ARTINIAN HOPF ALGEBRAS ARE FINITE DIMENSIONAL

CHIA-HSIN LIU AND JAMES J. ZHANG

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Abstract. We prove that an artinian Hopf algebra over a field is finite dimensional. This answers a question of Bergen.

A classical theorem of Connell [Co] states that if the group algebra kG of a group G over a base field k is left artinian, then G is finite. For a general Hopf algebra, Bergen asked the following question [JR, p. 373]:

If a Hopf algebra H is artinian as an algebra, must H be finite dimensional?

Based on an exercise of Sweedler, we settle the question with an easy proof. Note that the group algebra proof in [Co] is more complicated, because it relies on the characterization of prime group algebras.

Theorem. *Let H be a left (or right) noetherian Hopf algebra over a field k. If H has a minimal prime of finite codimension, then H is finite dimensional over k.*

We need the following lemma which does not use the coalgebra structure.

Lemma. *Let R be a left noetherian k-algebra containing a minimal prime ideal of finite codimension. Then R has a nonzero finite-dimensional right ideal.*

Proof. Let J be a minimal prime ideal of R of finite codimension. By [MR, Lemma 6.2.3], the left noetherian ring R has left Krull dimension, which means that the hypothesis of [GK, Lemma 2(i)] holds. By [GK, Lemma 2(i)], J (and any minimal prime ideal) is a *middle annihilator*, namely, there are ideals X and Y of R such that $XY \neq 0$ but $XJY = 0$. Pick any $x \in X$ such that the right ideal xY is nonzero. We claim that xY is finite dimensional. Since R/J is finite dimensional, the finitely generated R/J-module Y/JY is finite dimensional. Hence there is a finite-dimensional vector space $V \subset Y$ such that $Y = V + JY$. Using $xJY = 0$ we have $xY = xV + xJY = xV$. Therefore xY is finite dimensional, as required. □

Proof of the Theorem. An exercise of Sweedler [Sw, p. 108] says that if H contains a nonzero finite-dimensional right (or left) ideal, then H is finite dimensional (see [DNR, Lemma 5.3.1(i), p. 189] for a proof). The assertion follows from the above Lemma and Sweedler's exercise. □

Here is what we aim to show.

Corollary. *A left (or right) artinian Hopf algebra over k is finite dimensional.*

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Proof. Let J be the augmentation ideal $\ker \epsilon$ of a left artinian Hopf algebra H. Then J is a maximal (and hence prime) ideal of codimension 1. Since H is left artinian, the prime ideal J is a minimal prime. By Hopkins' theorem [MR, Corollary 0.1.13], H is left noetherian. The assertion follows from the Theorem. □

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We remark that in 2003, T.-K. Lee showed us some of his results [Le] using some arguments in [LZ]. Indeed, he proved that: "Let R be a left artinian algebra with 1 over a field. Then R has a proper subalgebra of finite codimension if and only if R has a non-zero finite dimensional right ideal." We would like to thank T.-K. Lee for showing us his results.

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Department of Mathematics, National Taiwan Normal University, 88 Sec. 4, Ting Chou Road, Taipei, Taiwan, ROC

E-mail address: chliu@math.ntnu.edu.tw

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195

E-mail address: zhang@math.washington.edu