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THE MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE AND COLLAPSING VOLUME

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Abstract. Let M be a complete noncompact n-manifold with collapsing volume and $Ric \geq 0$. The paper proves that M is of finite topological type under some restrictions on volume growth.

1. Introduction and main result

By the well-known Cheeger-Gromoll soul theorem [1], a complete noncompact Riemannian manifold with nonnegative sectional curvature is of finite topological type. However, complete noncompact Riemannian manifolds with nonnegative Ricci curvature may have infinite topological type.

For an n-manifold with nonnegative Ricci curvature, it is known that

$$c(n)\mathrm{vol}[B(p,1)]r \leq \mathrm{vol}[B(p,r)] \leq \omega_n r^n, \tag{1}$$

where $c(n)$ is a constant and ω_n is the volume of the unit ball in R^n.

Definition 1. Let M be a complete noncompact manifold with $Ric \geq 0$. By M is of small volume growth we mean that

$$\lim_{r\to\infty} \frac{1}{r^2}\mathrm{vol}[B(p,r)] = 0 \tag{2}$$

is true.

On the one hand, Sha and Yang [2] and Menguy [3] have constructed specific manifolds with nonnegative Ricci curvature which are of infinite topological type. Menguy's examples are with large volume growth, while Sha-Yang's examples have neither small volume growth nor large volume growth. On the other hand, Abresch-Gromoll [4], Shen-Wei [5], Shen [6], Ordway-Stephens-Yang [7], Xia [8], [9], and Zhan [10], [11] etc. have showed that under certain other geometric conditions, the manifold is of finite topological type. So it is very interesting to probe whether a manifold with nonnegative Ricci curvature is of finite topological type or not.

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However, many of the above results are under the assumption that the manifolds do not have collapsing volume, i.e.

$$v_M = \inf_{x\in M} \mathrm{vol}[B(x,1)] > 0. \tag{3}$$

If the manifold has collapsing volume, i.e., there is a sequence of points $\{m_i\} \subset M$ such that

$$\lim_{i\to\infty} \mathrm{vol}[B(m_i,1)] = 0, \tag{4}$$

the corresponding problem becomes more difficult. There are few results in studying the finite topology type in this case that we are able to find. Fortunately, by generalizing an important lemma of Shen-Wei [5], we succeed in making some progress in this problem. First, we quote the following definition.

Definition 2. Let M be a complete noncompact manifold and let $p \in M$ be a point such that

$$v_p(r) = \inf_{x\in S(p,r)} \mathrm{vol}[B(x,1)] = O(\frac{1}{r^\alpha}). \tag{5}$$

Then we say that M has α-order collapsing volume.

The main result in this paper is the following.

Theorem 3. *Let M be a complete noncompact n-manifold with $Ric \geq 0$, $K_M \geq -K$ and α-order collapsing volume. There is a positive constant c and a point $p \in M$ such that if*

$$\lim_{r\to\infty} \frac{\mathrm{vol}[B(p,r)]}{r^{1+\frac{1}{n}-\alpha}} < c, \tag{6}$$

then M is of finite topological type. Here $0 \leq \alpha \leq \frac{1}{n}$ and K is a positive constant.

When $\alpha = 0$, Theorem 3 has been proved in Shen-Wei [5].

2. Finite topological type

A manifold M is said to have finite topological type if there is a compact domain Ω whose boundary $\partial\Omega$ is a topological manifold such that $M\backslash\Omega$ is homeomorphic to $\partial\Omega \times [0,\infty)$. The fundamental notion involved in such a finite topological type result is that of the critical point of a distance function introduced by Grove and Shiohama. For this and the following fundamental lemma, the reader can refer to [14].

Lemma 4 (Isotopy Lemma)**.** *If $r_1 \leq r_2 \leq \infty$ and if a connected component C of $\overline{B(p,r_2)}\backslash B(p,r_1)$ is free of critical points of p, then C is homeomorphic to $C_1 \times [r_1,r_2]$, where C_1 is a topological submanifold without boundary.*

Thus, in order to prove that a complete Riemannian manifold M has a finite topological type, one only needs to show that there are no critical points outside a compact subset with respect to a fixed point $p_0 \in M$. For $r > 0$ and a point p on a complete manifold M, let

$$R(p,r) = \{\gamma(r) : \gamma \text{ is a ray from } p\}.$$

$R(p,r)$ consists of the points of intersections of the geodesic sphere of radius r with all the rays emanating from p. Let

$$R_p(x) = d(x, R(p,r)), \text{ where } r = d(p,x).$$

By the estimate on the excess functions, Shen-Wei [5] obtained the following important proposition.

Proposition 5. *Let M be a complete noncompact n-manifold with $Ric \geq 0$ and its sectional curvature bounded below, i.e. $K_M \geq -K$ $(K > 0)$. If $d(p,x) \geq \frac{10}{\sqrt{K}}$ and*

$$R_p(x) \leq \frac{1}{16} K^{-\frac{n-1}{2n}} d(p,x)^{\frac{1}{n}}, \tag{7}$$

then x is not a critical point of p.

Now, according to Shen-Wei [5], we introduce a notion of essential diameter of ends (compare [15]) . Let M be a complete manifold and $p \in M$. For any $r > 0$, the essential diameter of ends at distance r from p is defined by

$$D(p,r) = \sup_{\sum} \operatorname{diam}(\sum), \tag{8}$$

where the supremum is taken over all boundary components $\sum$ of $M \backslash \overline{B(p,r)}$, with $\sum \cap R(p,r) \neq \emptyset$. Let $\sum_r$ be a boundary component of $M \backslash \overline{B(p,r)}$ with $\sum_r \cap R(p,r) \neq \emptyset$. By the definition of $R_p(x)$, one has that for any $x \in \sum_r$, $R_p(x) \leq D(p,r)$.

Let M be a complete open n-manifold. For a given point $p \in M$, set

$$v_p(A,r) = \inf_{x \in S(p,r)} \operatorname{vol}[B(x, \frac{A}{2})], \quad 0 < A < \frac{r}{2}. \tag{9}$$

Lemma 6. *Let M be a complete noncompact n-manifold. Then*

$$D(p,r) \leq \frac{8}{v_p(A,r)} \operatorname{vol}[B(p,r+A) \backslash B(p,r-A)] \cdot A. \tag{10}$$

Proof. Let $\sum_r$ be a boundary component of $M \backslash \overline{B(p,r)}$ with $\sum_r \cap R(p,r) \neq \emptyset$. Then there is a ray γ_p such that $\gamma_p(r) \in \sum_r$. Let $\{B(p_j, \frac{A}{2})\}$ be a maximal set of disjoint balls with radius $\frac{A}{2}$ and center $p_j \in \sum_r$. Then

$$\bigcup_{j=1}^{N} B(p_j, A) \supset \sum_r$$

and

$$N \leq \frac{1}{v_p(A,r)} \operatorname{vol}[B(p,r+A) \backslash B(p,r-A)].$$

By the connectedness of $\sum_r$, one can show that for any point $x \in \sum_r$, there is a subset of $\{p_i\}$, say, $q_1, q_2, \cdots, q_k$, $k \leq N$, such that $x \in B(q_1, A)$, $\gamma_p(r) \in B(q_k, A)$, and

$$B(q_t, A) \cap B(q_{t+1}, A) \neq \emptyset, \ 1 \leq t \leq k-1.$$

Now one can easily construct a piecewise smooth geodesic c joining x and $\gamma_p(r)$ through $q_i^{\cdot}s$. Thus

$$d(x, \gamma_p(r)) \leq L(c) \leq 4N \cdot A \leq \frac{4}{v_p(A,r)} \operatorname{vol}[B(p,r+A) \backslash B(p,r-A)] \cdot A,$$

where $L(c)$ is the length of c. Thus

$$D(p,r) \leq \frac{8}{v_p(A,r)} \operatorname{vol}[B(p,r+A) \backslash B(p,r-A)] \cdot A. \qquad \square$$

Lemma 7 ([5])**.** *Let M be a complete noncompact n-manifold with $Ric \geq 0$. Then there is a constant c_1 such that for $\forall R \geq r$,*

$$\mathrm{vol}[B(p,R)\backslash(p,r)] \leq c_1 \int_r^R \frac{1}{s}\mathrm{vol}[B(p,s)]ds. \tag{11}$$

By Lemma 6 and Lemma 7, clearly we have

$$D(p,r) \leq c\frac{\mathrm{vol}[B(p,r+A)]}{v_p(A,r)(r-A)}. \tag{12}$$

We shall prove a more general theorem than Theorem 3.

Theorem 8. *Let M be a complete noncompact n-manifold with $Ric \geq 0$, $K_M \geq -K$. If there is a positive constant c and a given point $p \in M$ such that*

$$\lim_{r\to\infty} \frac{\mathrm{vol}[B(p,r)]}{v_p(r)r^{1+\frac{1}{n}}} \leq c, \tag{13}$$

then M is of finite topological type, where $v_p(r) = v_p(2,r)$.

Proof. (i) If there is a line in M, then by the Cheeger-Gromoll splitting theorem [17], $M = N \times R$, where N is an $(n-1)$-dimensional manifold with $Ric \geq 0$. If there is a line in N, then $N = N_1 \times R$, i.e. $M = N_1 \times R^2$. So one can assume that $M = N \times R^k$, $1 \leq k \leq n-1$, and there is no line in N. Just as in the following discussion (ii), one knows that N is of finite topological type under the conditions of the theorem. Thus M is of finite topological type.

(ii) If there is no line in M, it is easy to see that M has only one end, which means that $D(p,r) = \mathrm{diam}\ (S(p,r))$. By (12), there is a positive constant c_1 such that

$$\begin{aligned} D(p,r)) &\leq c_1\frac{\mathrm{vol}[B(p,r+2)]}{v_p(r)(r-2)} = c_1\frac{\mathrm{vol}[B(p,r+2)]}{\omega_n(r+2)^n}\cdot\frac{\omega_n(r+2)^n}{v_p(r)(r-2)} \\ &\leq c_1\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n}\cdot\frac{\omega_n(r+2)^n}{v_p(r)(r-2)}. \end{aligned}$$

Now, it is easy to see that

$$\lim_{r\to\infty}\frac{D(p,r)}{\sqrt[n]{r}} \leq c_1 \lim_{r\to\infty}\frac{\mathrm{vol}[B(p,r)]}{v_p(r)r^{1+\frac{1}{n}}}.$$

Thus there is a constant $c > 0$, such that as long as (13) is true and

$$cc_1 < \frac{1}{16}K^{-\frac{n-1}{2n}},$$

then

$$\lim_{r\to\infty}\frac{D(p,r)}{\sqrt[n]{r}} < \frac{1}{16}K^{-\frac{n-1}{2n}}.$$

By Proposition 5, one gets the result. □

Corollary 9. *Theorem 3 is true.*

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