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DYNAMIC APPROACH TO A STOCHASTIC DOMINATION: THE FKG AND BRASCAMP-LIEB INEQUALITIES

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Abstract. A coupling based on a pair of stochastic differential equations is introduced to show a stochastic domination for a system with continuous spins, from which the FKG and Brascamp-Lieb like inequalities follow.

1. Introduction

The FKG inequality and Brascamp-Lieb momentum inequality have their origins in statistical mechanics or quantum field theory, and play an important role as basic tools. The former implies positive correlations under the Gibbs measures with attractive interaction potentials; see [2], [4], [6], [9], [12], [13]. The latter shows that the centered moments of a distribution having a log-concave density with respect to a Gaussian measure μ are not larger than those of μ; see [3], [4], [7]. The Gaussian measure represents a free field or a harmonic crystal in physics.

The purpose of this paper is to give a new proof for these inequalities or their variants, especially in the case of continuous spins, based on a stochastic domination which is shown via a coupling of stochastic differential equations with the help of an ergodic theorem. A similar idea was employed by Holley [9] to prove the FKG inequality in the case of discrete spins due to a coupling of Markov chains. Bakry and Michel [2] gave another dynamic proof of the FKG inequality for a system with continuous spins. On the other hand, Brascamp-Lieb inequality was recently extended by Caffarelli [4] with a deep insight in optimal mass transportation and using the Monge-Ampére equation (cf. [1]). Our approach is different, much simpler (at least for probabilists) and applicable to the derivation of certain related inequalities.

We shall work on the d-dimensional Euclidean space $\mathbb{R}^d$, which represents the configuration space of continuous spins on d sites. The class of all Borel probability measures on $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$. The space $\mathbb{R}^d$ is equipped with a natural partial order $x \le y$ for $x = (x_i)_{i=1}^d, y = (y_i)_{i=1}^d \in \mathbb{R}^d$ defined by $x_i \le y_i$ for every $1 \le i \le d$. For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, we say ν stochastically dominates μ and write $\mu \le \nu$ if $E^\mu[F] \le E^\nu[F]$ holds for all bounded non-decreasing (in the above partial order) functions F on $\mathbb{R}^d$. Note that the stochastic domination $\mu \le \nu$ is equivalent to

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the existence of two $\mathbb{R}^d$-valued random variables X and Y, which are realized on a common probability space, being distributed under μ and ν (we denote $X \sim \mu$ and $Y \sim \nu$), respectively, and satisfying $X \leq Y$ a.s.; see Theorem 2.4 on p. 72 of [12] (at least for compact state spaces; an extension to $\mathbb{R}^d$ is easy).

Section 2 shows Holley's stochastic domination and the FKG inequality as its corollary under a continuous setting. Section 3 discusses the domination to the origin and Brascamp-Lieb type inequalities. The Gaussian property of μ is not assumed in general. The convexity of potentials is also relaxed. Section 4 proposes an extension of the Brascamp-Lieb inequality for a genuinely non-convex potential obtained as a small perturbation of a convex one, which includes a potential of double-well type when $d = 1$.

2. Holley's stochastic domination and FKG inequality

Holley [9] gave a sufficient condition for the stochastic domination and applied it to show the FKG inequality under a discrete setting; see also Chapter II-2 of [12]. It was then generalized by Preston [13] including the case of continuous spins. We use a dynamic approach based on stochastic differential equations to establish Holley's stochastic domination under a continuous setting.

Let μ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ be probability measures of the following forms:

$$\begin{aligned} \mu(dx) &= \frac{1}{Z_\mu} e^{-V(x)} dx, \\ \nu(dx) &= \frac{1}{Z_\nu} e^{-V(x)-U(x)} dx, \end{aligned} \tag{2.1}$$

where Z_μ and Z_ν are finite normalizing constants, and the measurable functions V and U on $\mathbb{R}^d$ satisfy the following condition:

$$V(x) + V(y) - V(x \vee y) - V(x \wedge y) \geq U(x \vee y) - U(x), \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$, where $x \vee y = (x_i \vee y_i)_{i=1}^d$ and $x \wedge y = (x_i \wedge y_i)_{i=1}^d$. In general, for μ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ being absolutely continuous with respect to dx, the condition (2.2) is equivalent to Holley's condition (cf. [13]):

$$g(x \vee y)\, f(x \wedge y) \geq g(x)\, f(y), \tag{2.3}$$

where f and g are the probability density functions of μ and ν, respectively.

Proposition 2.1. *Assume that $V, U \in C^1(\mathbb{R}^d)$ and $\nabla V, \nabla U$ are both Lipschitz continuous. Consider the pair of stochastic differential equations on $\mathbb{R}^d$:*

$$\begin{aligned} dX_t &= -\frac{1}{2} \nabla V(X_t) dt + dw_t, \\ dY_t &= -\frac{1}{2} \{\nabla V(Y_t) + \nabla U(Y_t)\} dt + dw_t, \end{aligned}$$

driven by a common d-dimensional Brownian motion $w = \{w_t; 0 \leq t < \infty\}$. Then, $X_0 \leq Y_0$ implies $X_t \leq Y_t$ for all $t \geq 0$, a.s.

Proof. Since $X_0 \le Y_0$, we have by Itô's formula

$$\begin{aligned}\phi_t &\equiv \sum_{i=1}^d (X_t^i - Y_t^i)^2 1_{\{X_t^i > Y_t^i\}} \\ &= \sum_{i=1}^d \int_0^t (X_s^i - Y_s^i) 1_{\{X_s^i > Y_s^i\}} \big(\nabla_i V(Y_s) + \nabla_i U(Y_s) - \nabla_i V(X_s)\big) ds,\end{aligned} \tag{2.4}$$

where $X_t = (X_t^i)_{i=1}^d$ and $\nabla_i V = \partial V / \partial x_i$, etc. However, on $\{x_i > y_i\}$, we have

$$\begin{aligned}&\nabla_i V(y) + \nabla_i U(y) - \nabla_i V(x) \\ &\quad \le \nabla_i V(y) + \nabla_i U(y) - \nabla_i V(x \vee y) - \nabla_i U(x \vee y) \\ &\quad = \sum_{k=1}^d \Big\{ \nabla_i V((x \vee y)^{(k+1)}) - \nabla_i V((x \vee y)^{(k)}) \\ &\qquad\qquad + \nabla_i U((x \vee y)^{(k+1)}) - \nabla_i U((x \vee y)^{(k)}) \Big\} 1_{\{x_k > y_k\}} \\ &\quad \le C \sum_{k=1}^d (x_k - y_k) 1_{\{x_k > y_k\}},\end{aligned} \tag{2.5}$$

where $(x \vee y)^{(k)}, 1 \le k \le d+1$, are defined by

$$((x \vee y)^{(k)})_j = x_j \vee y_j \text{ (for } j \ge k), \;\; y_j \text{ (for } j \le k-1).$$

Indeed, for the second line, we have used the condition (2.2) which implies

$$U((x + \epsilon e^i) \vee y) - U(x \vee y) \le \big(V(x + \epsilon e^i) - V(x)\big) - \big(V((x + \epsilon e^i) \vee y) - V(x \vee y)\big)$$

for $\epsilon > 0$ on $\{x_i > y_i\}$; take $x + \epsilon e^i$ and $x \vee y$ for x and y, respectively, in (2.2), where $e^i \in \mathbb{R}^d$ stands for the ith unit vector. For the third line, note that $(x \vee y)^{(1)} = x \vee y$, $(x \vee y)^{(d+1)} = y$ and $x_k \le y_k$ implies $(x \vee y)^{(k+1)} = (x \vee y)^{(k)}$. The last line follows from the Lipschitz continuity of ∇V and ∇U. Substituting (2.5) for (2.4), we have

$$\begin{aligned}\phi_t &\le C \sum_{i=1}^d \int_0^t (X_s^i - Y_s^i) 1_{\{X_s^i > Y_s^i\}} \sum_{k=1}^d (X_s^k - Y_s^k) 1_{\{X_s^k > Y_s^k\}} \, ds \\ &\le Cd \int_0^t \sum_{i=1}^d (X_s^i - Y_s^i)^2 1_{\{X_s^i > Y_s^i\}} \, ds \\ &= Cd \int_0^t \phi_s \, ds.\end{aligned}$$

This implies $\phi_t = 0$ with the help of Gronwall's lemma, and the proof is complete. □

The coupling introduced in Proposition 2.1 deduces Holley's stochastic domination for continuous spins:

Theorem 2.2. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be the probability measures satisfying the condition* (2.2) *or equivalently* (2.3)*. Then ν stochastically dominates μ.*

Proof. The probability measures μ and ν can be approximated weakly by sequences $\{\mu_n \in \mathcal{P}(\mathbb{R}^d)\}_{n=1,2,\dots}$ and $\{\nu_n \in \mathcal{P}(\mathbb{R}^d)\}_{n=1,2,\dots}$ having the forms:

$$\mu_n(dx) = \frac{1}{Z_{\mu_n}} e^{-V_n(x)} dx,$$
$$\nu_n(dx) = \frac{1}{Z_{\nu_n}} e^{-V_n(x)-U_n(x)} dx,$$

respectively, where V_n and U_n satisfy (2.2) together with the conditions for V and U stated in Proposition 2.1. We can find such V_n and U_n by smearing V and U, respectively; note that condition (2.2) is not harmed under the convolutions. Once Theorem 2.2 is shown for μ_n and ν_n instead of μ and ν, one can prove it for general μ and ν by taking the limit.

Let X_t and Y_t be as in Proposition 2.1 satisfying $X_0 = Y_0$. Then $X_t \le Y_t$ holds for all $t \ge 0$, a.s. But μ and ν are invariant (in fact, reversible) measures for X_t and Y_t, respectively (see [5], [10], [14] for example), and the ergodicity shows that the distributions on $\mathbb{R}^d$ of X_t and Y_t weakly converge to μ and ν, respectively, as $t \to \infty$. The proof is concluded by letting $t \to \infty$ in the inequalities $E[F(X_t)] \le E[F(Y_t)]$, which hold for all bounded continuous non-decreasing functions F on $\mathbb{R}^d$. □

The FKG inequality follows immediately from Theorem 2.2:

Corollary 2.3. *Let f be a probability density function on $\mathbb{R}^d$ satisfying*

$$f(x \vee y)\, f(x \wedge y) \ge f(x)\, f(y), \tag{2.6}$$

for all $x, y \in \mathbb{R}^d$, and let $\mu(dx) = f(x)dx \in \mathcal{P}(\mathbb{R}^d)$. Then, for all non-decreasing functions F and G on $\mathbb{R}^d$, we have

$$E^\mu[F; G] = E^\mu[FG] - E^\mu[F]E^\mu[G] \ge 0.$$

Proof. The proof is standard. Indeed, without loss of generality, we may assume $G > 0$ (by approximating it by functions bounded below and adding a constant) and apply Theorem 2.2 with the choice of $f(x) = e^{-V(x)}$ and $G(x) = e^{-U(x)}$. Note that (2.6) implies $V(x)+V(y)-V(x\vee y)-V(x\wedge y) \ge 0$ and therefore the condition (2.2). □

Remark 2.4. Corollary 2.3 can be directly shown via the coupling introduced in the proof of Proposition 2.1. In fact, noting that $\nabla_i U \le 0$, the estimate (2.5) on $\{x_i > y_i\}$ may be replaced by a simpler one:

$$\begin{aligned}
&\nabla_i V(y) + \nabla_i U(y) - \nabla_i V(x) \\
&\quad \le \sum_{k=1}^{d} \Big\{ \nabla_i V((x \vee y)^{(k+1)}) - \nabla_i V((x \vee y)^{(k)}) \Big\} 1_{\{x_k > y_k\}} \\
&\quad \le C \sum_{k=1}^{d} (x_k - y_k) 1_{\{x_k > y_k\}}.
\end{aligned}$$

3. Stochastic domination and Brascamp-Lieb type inequalities

Brascamp and Lieb [3] proved an inequality for a concentration of random variables around their means. The following theorem is in the form extended by Caffarelli [4]:

Theorem 3.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be a centered Gaussian measure and let*

$$\nu(dx) = \frac{1}{Z_\nu} e^{-U(x)} \mu(dx) \in \mathcal{P}(\mathbb{R}^d), \tag{3.1}$$

be normalizable, i.e., $Z_\nu = \int_{\mathbb{R}^d} e^{-U(x)} \mu(dx) < \infty$. Then, if $U : \mathbb{R}^d \to \mathbb{R}$ is convex, for every $v \in \mathbb{R}^d$ and $\psi : \mathbb{R} \to \mathbb{R}$, convex and bounded below, we have

$$E^\nu\big[\psi\big((v, Y) - E^\nu[(v, Y)]\big)\big] \le E^\mu\big[\psi\big((v, X)\big)\big],$$

where $X \sim \mu$, $Y \sim \nu$ and (v, X) denotes the inner product of v and X in $\mathbb{R}^d$.

This theorem was first proved by Brascamp and Lieb [3] for $\psi(a) = |a|^p, p \ge 1$. Giacomin [7] showed a related stochastic domination:

Theorem 3.2. *Let μ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ be as in Theorem 3.1. Then one can construct two $\mathbb{R}^d$-valued random variables X and Y such that $X \sim \mu$, $Y \sim \nu$ and a stochastic domination*

$$[(v, Y) - M_v]_+ \le [(v, X)]_+, \quad a.s.$$

holds for every $v \in \mathbb{R}^d$, where $[a]_+ = \max\{a, 0\}$ and M_v denotes the median of (v, Y).

Our goal is to extend the Brascamp-Lieb inequality in a spirit of stochastic domination. Our result does not completely cover Theorem 3.1, but, in a sense, it is more general. For instance, we only require the radial symmetry of μ, and its Gaussian property is unnecessary in Theorem 3.3. We also notice that conditions (3.2) or (3.4) required below for the potential U is slightly different from conventional one; in our setting, it need not be convex. We can apply the following theorems, e.g., for a square-well potential, which is non-convex.

In the case that μ is Gaussian, the argument by Prékopa and Leindler (for marginal distributions) actually reduces the proof of Theorem 3.1 to the one-dimensional case; see [3], [4], [7]. From this viewpoint, the following theorem, stated generally on d dimensional spaces, might have its own interest already when $d = 1$.

Theorem 3.3. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be radially symmetric and let $\nu \in \mathcal{P}(\mathbb{R}^d)$ be given as in (3.1). In addition, we assume that $U \in C^1(\mathbb{R}^d)$ and it satisfies*

$$(x, \nabla U(x)) \ge 0, \tag{3.2}$$

for all $x \in \mathbb{R}^d$. Then, one can construct two $\mathbb{R}^d$-valued random variables X and Y such that $X \sim \mu, Y \sim \nu$ and a stochastic domination

$$|Y| \le |X| \quad a.s.$$

holds. In particular, for every $\psi : \mathbb{R}^d \to \mathbb{R}$, radially symmetric and non-decreasing in $|x|$, we have

$$E^\nu[\psi(Y)] \le E^\mu[\psi(X)]. \tag{3.3}$$

Proof. It suffices to prove the theorem under the additional assumption that μ and ν have the forms in (2.1), respectively, with $U \in C_b^\infty(\mathbb{R}^d)$ and radially symmetric $V \in C_b^\infty(\mathbb{R}^d)$. Define $\tilde{V} : [0, \infty) \to \mathbb{R}$ such that $\tilde{V}(|x|) = V(x)$. By approximation we can further assume that $|\tilde{V}'(r)/r|$ is bounded on $(0, \infty)$.

Choosing an $O(d)$-valued function $(p_{ij}(x))_{1\le i,j\le d}$ such that $p_{i1}(x) = x^i/|x|$ $(x \neq 0)$ and δ_{i1} $(x = 0)$, we consider a pair of stochastic differential equations:

$$dX_t = -\frac{1}{2}\nabla V(X_t)dt + p(X_t)dw_t,$$
$$dY_t = -\frac{1}{2}\{\nabla V(Y_t) + \nabla U(Y_t)\}dt + p(Y_t)dw_t,$$

with a common d-dimensional Brownian motion $w_t = (w_t^i)_{i=1}^d$. Then both processes X_t and Y_t are ergodic and their unique invariant measures are μ and ν, respectively. By Itô's formula applied for $x_t = |X_t|^2$ and $y_t = |Y_t|^2$, we have

$$dx_t = \{-\tilde{V}'(\sqrt{x_t})\sqrt{x_t} + d\}dt + 2\sqrt{x_t}dw_t^1,$$
$$dy_t = \{-\tilde{V}'(\sqrt{y_t})\sqrt{y_t} - (Y_t, \nabla U(Y_t)) + d\}dt + 2\sqrt{y_t}dw_t^1.$$

Now, condition (3.2) ensures, with the help of the comparison theorem for one dimensional stochastic differential equations (see Chapter VI of [11]) that, if $y_0 \le x_0$, then $y_t \le x_t$ and therefore $|Y_t| \le |X_t|$ for all $t \ge 0$ a.s.; note that the pathwise uniqueness holds for the stochastic differential equation for x_t by Yamada-Watanabe's theorem. The ergodicity completes the proof. □

Remark 3.4. (1) If the symmetrization $\tilde{U}$ of U defined by

$$e^{-\tilde{U}(x)} = \int_{S^{d-1}} e^{-U(\sigma x)}\, d\sigma, \quad x \in \mathbb{R}^d,$$

satisfies condition (3.2), then (3.3) holds, where $d\sigma$ stands for the uniform measure on $S^{d-1} = \{\sigma; |\sigma| = 1\}$. Indeed, we have $E^\nu[\psi(Y)] = E^{\tilde{\nu}}[\psi(Y)]$ for $\tilde{\nu}$ determined from $\tilde{U}$.

(2) If $\psi : \mathbb{R}^d \to \mathbb{R}$ is radially symmetric and convex, it is non-decreasing in $|x|$.

(3) At least when $d = 1$, (3.3) can be seen directly from the FKG inequality, [8]. Note that, when $d = 1$, condition (3.2) is equivalent to the fact that U is non-decreasing for $x \ge 0$ and non-increasing for $x \le 0$.

Corollary 3.5. *Let $U \in C^2(\mathbb{R}^d)$ be convex and attaining its minimal value at 0, and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be radially symmetric and non-decreasing in $|x|$. Then, we have*

$$E^\nu[\psi(Y)] \le E^\mu[\psi(X)].$$

Proof. Since $\nabla U(0) = 0$, we have $(x, \nabla U(x)) = (x, \operatorname{Hess} U(\hat{x})x) \ge 0$ for some $\hat{x}$ by the mean value theorem. Hence, condition (3.2) holds. □

If μ is centered Gaussian and $v \in \mathbb{R}^d$ is an eigenvector of its covariance matrix, we have the following theorem:

Theorem 3.6. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be a centered Gaussian measure with covariance matrix A^{-1} and let $\nu \in \mathcal{P}(\mathbb{R}^d)$ be determined as in* (3.1) *from U and μ. Let $e \in \mathbb{R}^d$ be an eigenvector of A. Then, if $U \in C^1(\mathbb{R}^d)$ satisfies*

$$(e, \nabla U(x))(e, x) \ge 0, \tag{3.4}$$

for all $x \in \mathbb{R}^d$, one can construct two $\mathbb{R}^d$-valued random variables X and Y such that $X \sim \mu$, $Y \sim \nu$ and a stochastic domination

$$|(e, Y)| \le |(e, X)| \quad a.s.$$

holds. In particular, for every $\psi : [0, \infty) \to \mathbb{R}$, non-decreasing, we have

$$E^\nu[\psi(|(e, Y)|)] \le E^\mu[\psi(|(e, X)|)].$$

Proof. We can assume that $U \in C_b^\infty(\mathbb{R}^d)$ as in the proof of Theorem 3.3. For the eigenvector $e = (e_i)_{i=1}^d \in \mathbb{R}^d$ of A such that $|e| = 1$, choose $p = (p_{ij})_{1\le i,j\le d} \in O(d)$ in such a way that $p_{i1} = e_i,\ 1 \le i \le d$, and consider a pair of stochastic differential equations on $\mathbb{R}^d$:

$$dX_t = -\frac{1}{2}AX_t dt + p dw_t,$$
$$dY_t = -\frac{1}{2}\{AY_t + \nabla U(Y_t)\}dt + p dw_t,$$

with a common d-dimensional Brownian motion $w_t = (w_t^i)_{i=1}^d$. Then both processes X_t and Y_t are ergodic, and their unique invariant measures are μ and ν, respectively. Noting that $(e, pw_t) = w_t^1$, by Itô's formula applied for $x_t = (e, X_t)^2$ and $y_t = (e, Y_t)^2$, we have

$$dx_t = \{-\lambda x_t + d\}\, dt + 2\sqrt{x_t} dw_t^1,$$
$$dy_t = \{-\lambda y_t - (e, \nabla U(Y_t))(e, Y_t) + d\}\, dt + 2\sqrt{y_t} dw_t^1,$$

where $\lambda > 0$ is the eigenvalue of A corresponding to e: $Ae = \lambda e$. Now, condition (3.4) ensures, with the help of the comparison theorem for one dimensional stochastic differential equations that, if $y_0 \le x_0$, then $y_t \le x_t$ and therefore $|(e, Y_t)| \le |(e, X_t)|$ for all $t \ge 0$ a.s. The ergodicity completes the proof. □

Remark 3.7. (1) Condition (3.4) holds for any $e \in \mathbb{R}^d$, if U is radially symmetric and non-decreasing in $|x|$ or if ∇U has a form $\nabla U(x) = f(x)\, x/|x|$ with $f : \mathbb{R}^d \to [0, \infty)$.

(2) If U is radially symmetric and non-decreasing in $|x|$, then one can realize a stochastic domination: $[(e, Y)]_+ \le [(e, X)]_+$ a.s. with $X \sim \mu, Y \sim \nu$. Compare this with Theorem 3.2.

4. An extension to double-well potentials

We propose some extension of Brascamp-Lieb like inequalities to a genuinely non-convex potential, including a potential of double-well type when $d = 1$.

Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be radially symmetric satisfying $0 < \mu(D) < 1$, where $D = \{x \in \mathbb{R}^d; |x| < 1\}$. We consider a potential $U_\epsilon, \epsilon \in \mathbb{R}$, of the form $U_\epsilon(x) = U_0(x) + \epsilon W(x)$, where U_0 satisfies the condition (3.2): $(x, \nabla U_0(x)) \ge 0$ together with $U_0 \equiv 0$ on $\bar{D}$, $U_0 \ge 0$ on $\mathbb{R}^d$, $\mu(\{x; U_0(x) > 0\}) > 0$, and W is a bounded and measurable function on $\mathbb{R}^d$ such that $W \equiv 0$ on $D^c \equiv \mathbb{R}^d \setminus D$. Define $\nu_\epsilon \in \mathcal{P}(\mathbb{R}^d)$ by

$$\nu_\epsilon(dx) = \frac{1}{Z_\epsilon} e^{-U_\epsilon(x)} \mu(dx).$$

Corollary 4.1. *For $|\epsilon|$ sufficiently small, there exists $c = c_\epsilon < 1$ such that*

$$E^{\nu_\epsilon}[\psi(|Y|)] \le c E^\mu[\psi(|X|)] \tag{4.1}$$

for every $\psi : [0, \infty) \to \mathbb{R}$, which is 0 on $[0, 1]$ and non-decreasing for $x \ge 1$.

Proof. Let $\tilde{\mu} = \mu(\cdot | D^c)$ and

$$\tilde{\nu}(dx) = \frac{1}{Z_{\tilde{\nu}}} e^{-U_0(x)} \tilde{\mu}(dx) \ \big(= \nu_\epsilon(\cdot | D^c)\big).$$

Then, for $\psi \equiv 0$ on $[0, 1]$, we have

$$E^\mu[\psi(|X|)] = \mu(D^c) E^{\tilde{\mu}}[\psi(|\tilde{X}|)],$$
$$E^{\nu_\epsilon}[\psi(|Y|)] = \nu_\epsilon(D^c) E^{\tilde{\nu}}[\psi(|\tilde{Y}|)],$$

where $X \sim \mu, Y \sim \nu_\epsilon, \tilde{X} \sim \tilde{\mu}$ and $\tilde{Y} \sim \tilde{\nu}$. However, since $E^{\tilde{\nu}}[\psi(|\tilde{Y}|)] \leq E^{\tilde{\mu}}[\psi(|\tilde{X}|)]$ by (3.3), (4.1) holds with

$$c_\epsilon = \frac{\nu_\epsilon(D^c)}{\mu(D^c)} = \frac{M}{\mu(D^c)\{M + \int_D e^{-\epsilon W(x)}\mu(dx)\}},$$

where $M = \int_{D^c} e^{-U_0(x)}\mu(dx)$. The conditions on μ and U_0 imply $M < \mu(D^c)$ so that we have $c_0 < 1$. Moreover the dominated convergence theorem shows that c_ϵ is continuous in $\epsilon \in \mathbb{R}$. This proves the conclusion. □

Remark 4.2. The conclusion of Corollary 4.1 does not imply the stochastic domination, which seems invalid for a potential of double-well type in general.

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