

## NON-REFLEXIVITY OF THE DERIVATION SPACE FROM BANACH ALGEBRAS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let  $\Omega$  be an open connected subset of the plane, and let  $A$  be a Banach algebra of analytic functions on  $\Omega$ . We show that the space of bounded derivations from  $A$  into  $A^*$  is not reflexive. We also obtain similar results when  $A = C^{(n)}[0, 1]$  for  $n \geq 2$ .

### 1. INTRODUCTION

Let  $A$  be a Banach algebra, and let  $X$  be a Banach  $A$ -bimodule. An operator  $D$  from  $A$  into  $X$  is a local derivation if for each  $a \in A$ , there is a derivation  $D_a$  from  $A$  into  $X$  such that  $D(a) = D_a(a)$ . This concept was introduced independently by R. V. Kadison [8] and D. R. Larson [9], and it has been the topic of studies since then. The main question that one seeks is for which algebras every local derivation is a derivation, or equivalently, which algebras have an algebraically reflexive derivation space. One can also ask the topological version of this question, i.e. when the linear space of bounded derivations is reflexive [9]. These questions have been investigated for various classes of Banach algebras such as operator algebras, Banach operator algebras, group algebras, and Fourier algebras, and there have been some affirmative answers to them (e.g. [2], [4], [6]-[8], [9]-[13]).

There are also Banach algebras that have local derivations which are not derivations (e.g. [7], [12]). Johnson showed in [7] that if  $A$  is the Banach algebra of continuously differentiable functions on  $[0, 1]$ , i.e.  $A = C^1[0, 1]$ , then there is a unital Banach  $C^1[0, 1]$ -bimodule  $X$  and a bounded local derivation from  $C^1[0, 1]$  into  $X$  which is not a derivation. He, however, showed that if one just considers symmetric  $C^1[0, 1]$ -modules and assumes that the derivations  $D_a$  considered in the definition of a local derivation are bounded, then bounded local derivations are derivations [7, Theorem 6.1].

Our purpose here is to show that the preceding result will not hold if we consider certain Banach algebras of functions where the higher derivatives exist. We show that if we let  $A$  be the Banach algebra of  $n$ -times continuously differentiable functions where  $n \geq 2$ , or a Banach algebra of analytic functions, then one can construct bounded local derivations from  $A$  into  $A^*$  with the additional property that they can be determined locally by bounded derivations but they are not derivations.

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This will provide us with examples of classes of commutative semisimple Banach algebras for which the derivation space fails to be reflexive, even in the most natural case.

## 2. PRELIMINARIES

Let  $X$  and  $Y$  be Banach spaces, and let  $L(X, Y)$  and  $B(X, Y)$  be the spaces of (linear) operators and bounded operators from  $X$  into  $Y$ , respectively. Let  $\mathcal{S}$  be a linear subspace of  $L(X, Y)$ . For each  $x \in X$ , let  $\mathcal{S}x = \{S(x) \mid S \in \mathcal{S}\}$ , and let  $[\mathcal{S}x]$  be the norm-closure of  $\mathcal{S}x$ . Put

$$\text{ref}_a(\mathcal{S}) = \{T \in L(X, Y) \mid T(x) \in \mathcal{S}x, x \in X\}$$

and, if  $\mathcal{S} \subseteq B(X, Y)$ , put

$$\text{ref}(\mathcal{S}) = \{T \in B(X, Y) \mid T(x) \in [\mathcal{S}x], x \in X\}.$$

Suppose that  $\mathcal{S} \subseteq L(X, Y)$ . Then  $\mathcal{S}$  is algebraically reflexive if  $\mathcal{S} = \text{ref}_a(\mathcal{S})$ , and when  $\mathcal{S} \subseteq B(X, Y)$ , it is reflexive if  $\mathcal{S} = \text{ref}(\mathcal{S})$ .

Let  $A$  be a Banach algebra, and let  $X$  be a Banach  $A$ -bimodule. An operator  $D \in L(A, X)$  is a derivation if for all  $a, b \in A$ ,  $D(ab) = aD(b) + D(a)b$ . Let  $\mathcal{Z}^1(A, X)$  and  $\mathcal{Z}^b(A, X)$  be the linear spaces of derivations and bounded derivations from  $A$  into  $X$ , respectively.

## 3. BANACH ALGEBRAS OF ANALYTIC FUNCTIONS

Let  $\Omega$  be an open connected subset of the plane, and let  $H(\Omega)$  be the algebra of analytic functions on  $\Omega$ . A *Banach algebra of analytic functions* on  $\Omega$  is a subalgebra  $A$  of  $H(\Omega)$  such that it is a Banach algebra with respect to some norm and it contains a non-constant element. For each  $t \in \Omega$ , let  $\varphi_t$  be the character on  $A$  specified by

$$\varphi_t(u) = u(t) \quad (u \in A).$$

By [5, Theorem 2.1.29(ii)],  $\varphi_t$  is bounded and  $\|\varphi_t\| \leq 1$  ( $t \in \Omega$ ). Therefore  $\|u\|_\infty \leq \|u\|_A$  ( $u \in A$ ), where  $\|\cdot\|_\infty$  and  $\|\cdot\|_A$  are the supremum norm and the  $A$ -norm on  $A$ , respectively. This, in particular, implies that  $A$  is a commutative semisimple Banach algebra, and so, by a result of Johnson [1, Theorem 18.21], 0 is the only derivation on  $A$ . Therefore (bounded) local derivations on  $A$  are derivations. However, as we see in the following theorem, this is not always the case, even for the bounded local derivations from  $A$  into  $A^*$ .

**Theorem 3.1.** *Let  $\Omega$  be an open connected subset of the plane, and let  $A$  be a Banach algebra of analytic functions on  $\Omega$ . Then there is a bounded local derivation from  $A$  into  $A^*$  which is not a derivation. Moreover,  $\mathcal{Z}^1(A, A^*)$  is not reflexive.*

*Proof.* Let  $K$  be a closed disk in  $\Omega$ , and, for  $i = 0, 1, 2$ , let

$$K_i = \{t \in K \mid a^{(i)}(t) = 0 \text{ for all } a \in A\}.$$

We claim that for each  $i$ ,  $K_i$  is finite. Otherwise,  $K_i$  has a limit point in  $K$ , and so, in  $\Omega$ . Thus, by [3, Theorem 3.7],  $a^{(i)} = 0$  for all  $a \in A$ . Therefore the degree of each element in  $A$  is at most 1. However, if  $a$  is a non-constant element in  $A$ , then  $a^2$  has the degree of at least 2. This contradiction shows that  $K_i$  is finite. Hence there is  $t \in \Omega \setminus (K_0 \cup K_1 \cup K_2)$  and  $a, b, c \in A$  such that

$$(1) \quad a(t) \neq 0, \quad b'(t) \neq 0, \quad c''(t) \neq 0.$$

Now consider the operator  $D: A \rightarrow A^*$  defined by

$$D(u) = u''(t)\varphi_t \quad (u \in A).$$

From (1),  $\langle D(c), a \rangle = c''(t)a(t) \neq 0$ . Thus  $D$  is non-zero. We claim that  $D$  is a bounded local derivation, but it is not a derivation. We first show that  $D$  is bounded. Take  $r > 0$  such that  $\overline{B_r(t)}$ , the closed disk with the center  $t$  and the radius  $r$ , is a subset of  $\Omega$ . Let  $u \in A$ . Since  $u$  is analytic and bounded on  $\Omega$ , by [3, Cauchy's Estimate 2.14], we have

$$|u''(t)| \leq \frac{2\|u\|_\infty}{r^2} \leq \frac{2\|u\|_A}{r^2}.$$

Thus  $\|D(u)\| \leq \frac{2\|u\|_A}{r^2}$ , and so  $D$  is bounded.

We now show that  $D$  is a local derivation. Define the operators  $D_i: A \rightarrow A^*$  ( $i = 1, 2$ ) by

$$\begin{aligned} \langle D_1(v), w \rangle &= v'(t)w(t), \\ \langle D_2(v), w \rangle &= v''(t)w(t) + v'(t)w'(t). \end{aligned}$$

It is straightforward to check that  $D_1$  and  $D_2$  are derivations. Now let  $D_u = \frac{u''(t)}{u'(t)}D_1$  whenever  $u'(t) \neq 0$ , and  $D_u = D_2$  whenever  $u'(t) = 0$ . Then, for each  $u \in A$ ,  $D(u) = D_u(u)$ . Hence  $D$  is a local derivation. Moreover, by applying an argument similar to the one made to prove that  $D$  is bounded, we can show that  $D_1$  and  $D_2$  are bounded. Therefore  $D \in \text{ref}[\mathcal{Z}^1(A, A^*)]$ . Finally a simple calculation shows that

$$D(b^2) - 2bD(b) = 2[b'(t)]^2\varphi_t.$$

However, by (1),  $b'(t) \neq 0$ . Thus  $D$  is not a derivation. □

**Example 3.2.** (i) Let

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\},$$

where  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  is the open unit disc. Then  $(A(\overline{\mathbb{D}}), \|\cdot\|_\infty)$  is a Banach algebra of analytic functions on  $\mathbb{D}$ ; it is called the *disc algebra*. Also, if we let  $A^+(\overline{\mathbb{D}})$  be the set of functions  $f = \sum_{n=0}^\infty \alpha_n z^n$  in  $A(\overline{\mathbb{D}})$  which have an absolutely convergent Taylor expansion on  $\overline{\mathbb{D}}$ , then, by [5, p. 158],  $A^+(\overline{\mathbb{D}})$  with the norm

$$\|f = \sum_{n=0}^\infty \alpha_n z^n\|_1 = \sum_{n=0}^\infty |\alpha_n|$$

is a Banach algebra of analytic functions on  $\mathbb{D}$ .

(ii) Some other classes of Banach algebras of analytic functions include certain convolution algebras on the real line. Let  $\omega$  be a continuous weight function on  $\mathbb{R}^+$  [5, Definition 4.7.3], and let  $\rho_\omega = \inf\{\omega(t)^{\frac{1}{t}} : t \geq 1\}$ . Suppose that  $\rho_\omega > 0$ , and let  $\sigma_\omega = -\log \rho_\omega$ . Then, by [5, Theorem 4.7.27],  $L^1(\mathbb{R}^+, \omega)$  is a Banach algebra of analytic functions on  $\{z \in \mathbb{C} : \text{Re } z > \sigma_\omega\}$  (see [5, Section 4.7] for the details and more examples of these types).

We finish this section with the following remark which recasts Theorem 3.1 in the more general setting. This generalization was pointed out to the author by D. Hadwin.

*Remark 3.3.* Let  $\mathfrak{B}$  be an algebra over a field  $F$ , let  $\mathfrak{A}$  and  $\mathfrak{A}_1$  be subalgebras of  $\mathfrak{B}$  such that  $\mathfrak{A} \subseteq \mathfrak{A}_1$ , and let  $\varphi: \mathfrak{B} \rightarrow F$  be an algebraic homomorphism. It is easy to verify that  $L(\mathfrak{A}, F)$  turns into an  $\mathfrak{A}$ -bimodule along with the actions defined by

$$(u \cdot f)(v) = f(vu), \quad (f \cdot u)(v) = f(uv) \quad (u, v \in \mathfrak{A}, f \in L(\mathfrak{A}, F)).$$

Now suppose that there is a derivation  $\delta: \mathfrak{A}_1 \rightarrow \mathfrak{B}$  and  $a, b, c \in \mathfrak{A}$  such that  $\delta(\mathfrak{A}) \subseteq \mathfrak{A}_1$  and  $\varphi(a), \varphi(\delta(b)), \varphi(\delta^2(c))$  are all non-zero. Then the argument in the proof of Theorem 3.1 can be modified to show that the operator  $D: \mathfrak{A} \rightarrow L(\mathfrak{A}, F)$ , defined by

$$\langle D(u), v \rangle = \varphi(\delta^2(u))\varphi(v) \quad (u, v \in \mathfrak{A}),$$

is a local derivation which is not a derivation.

#### 4. BANACH ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

Let  $I$  be a compact interval of  $\mathbb{R}$ . For  $n \in \mathbb{N}$ , let  $C^{(n)}(I)$  be the algebra of  $n$ -times continuously differentiable functions on  $I$ . From [5, Theorem 4.4.1],  $C^{(n)}(I)$  is a Banach algebra with respect to the norm  $\|\cdot\|_n$ , where

$$\|f\|_n = \sum_{k=0}^n \frac{1}{k!} |f^{(k)}|_I \quad (f \in C^{(n)}(I)).$$

B. E. Johnson showed in [7, Theorem 6.1] that if  $D: C^1(I) \rightarrow C^1(I)^*$  is a bounded local derivation where, for each  $a \in A$ , there is a bounded derivation  $D_a: C^1(I) \rightarrow C^1(I)^*$  such that  $D(a) = D_a(a)$ , then  $D$  is a derivation. However, the argument in the proof of Theorem 3.1 can be employed to show that, for  $n \geq 2$  and  $t \in I$ , the operator  $D: C^{(n)}(I) \rightarrow C^{(n)}(I)^*$ , defined by

$$D(u) = u''(t)\varphi_t \quad (u \in C^{(n)}(I)),$$

is a bounded local derivation with the above property, but it is not a derivation. So, in particular,  $\mathcal{Z}^1(C^{(n)}(I), C^{(n)}(I)^*)$  is not reflexive.

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