ON THE HELTON CLASS OF \( p \)-HYPONORMAL OPERATORS

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Abstract. In this paper we show that the Helton class of \( p \)-hyponormal operators has scalar extensions. As a corollary we get that each operator in the Helton class of \( p \)-hyponormal operators has a nontrivial invariant subspace if its spectrum has its interior in the plane.

1. Introduction

Let \( H \) be a complex (separable) Hilbert space and let \( \mathcal{L}(H) \) denote the algebra of all bounded linear operators on \( H \). If \( T \in \mathcal{L}(H) \), we write \( \sigma(T) \) for the spectrum of \( T \).

An operator \( T \in \mathcal{L}(H) \) is said to be \( p \)-hyponormal, \( 0 < p \leq 1 \), if \( (T^*T)^p \geq (TT^*)^p \), where \( T^* \) is the adjoint of \( T \). If \( p = 1 \), \( T \) is called hyponormal and if \( p = \frac{1}{2} \), \( T \) is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [15]), and \( p \)-hyponormal operators for a general \( p \), \( 0 < p < 1 \), have been studied by Aluthge. Any \( p \)-hyponormal operators are \( q \)-hyponormal if \( q \leq p \) by Löwner’s theorem (see [13]). But there are examples to show that the converse of the above statement is not true (see [2]).

In [8] J. W. Helton initiated the study of operators \( T \) which satisfy an identity of the form

\[
T^m - \binom{m}{1} T^{m-1} + \cdots + (-1)^{m-1} T + 1 = 0.
\]

We need further study for this class of operators based on (1). Let \( R \) and \( S \) be in \( \mathcal{L}(H) \) and let \( C(R, S) : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \) be defined by \( C(R, S)(A) = RA - AS \). Then

\[
C(R, S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^k - j.
\]

Definition 1.1. Let \( R \in \mathcal{L}(H) \). If there is an integer \( k \geq 1 \) such that an operator \( S \) satisfies \( C(R, S)^k(I) = 0 \), we say that \( S \) belongs to the Helton class of \( R \) with order \( k \). We denote this by \( S \in \text{Helton}_k(R) \).

We remark that \( C(R, S)^k(I) = 0 \) does not imply \( C(S, R)^k(I) = 0 \) in general.
Example 1.2. Let $S$ and $R$ be operators in $\mathcal{L}(H \oplus H \oplus H)$ defined by the following:

\[ S = \begin{pmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}, \]

where $A$, $B$, $C$, and $D$ are bounded linear operators defined on $H$. Then it is easy to calculate that $C(R, S)^2(I) = 0$, but $C(S, R)^2(I) \neq 0$.

A bounded linear operator $S$ on $H$ is called scalar of order $m$ if it possesses a spectral distribution of order $m$, i.e., if there is a continuous unital morphism of topological algebras

\[ \Phi : C^m_0(C) \rightarrow \mathcal{L}(H) \]

such that $\Phi(z) = S$, where $z$ stands for the identity function on $C$, and $C^m_0(C)$ stands for the space of compactly supported functions on $C$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

In this paper we show that the Helton class of $p$-hyponormal operators has scalar extensions. As a corollary we get that the Helton class of $p$-hyponormal operators has a nontrivial invariant subspace if its spectrum has its interior in the plane.

2. Preliminaries

Let $z$ be the coordinate in the complex plane $C$ and let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space $H$ and a bounded (connected) open subset $U$ of $C$. We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \rightarrow H$, such that

\[ \|f\|_{2, U} = \left( \int_U \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty. \]

The space of functions $f \in L^2(U, H)$ which are analytic functions in $U$ (i.e., $\bar{\partial}f = 0$) is defined by

\[ A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H), \]

where $\mathcal{O}(U, H)$ denotes the Fréchet space of $H$-valued analytic functions on $U$ with respect to uniform topology. $A^2(U, H)$ is called the Bergman space for $U$. Note that $A^2(U, H)$ is a Hilbert space. We denote by $P$ the orthogonal projection of $L^2(U, H)$ onto $A^2(U, H)$.

Let $T \in \mathcal{L}(H)$. The study of the operator $T - z$ on the space $\mathcal{O}(U, H)$ led E. Bishop to fundamental results in spectral theory. Among other things he isolated in [4] the single-valued extension property, which means by definition that the operator $T - z$ acts one-to-one on $\mathcal{O}(U, H)$ for an arbitrary open subset $U$ of $C$, and the property (β), which requires that $T - z$ should be one-to-one and with closed range on $\mathcal{O}(U, H)$ for every open set $U$.

Let us define now a special Sobolev type space. Let $U$ be again a bounded open subset of $C$ and $m$ be a fixed nonnegative integer. The vector-valued Sobolev space $W^m(U, H)$ with respect to $\bar{\partial}$ and of order $m$ will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\bar{\partial}f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

\[ \|f\|_{W^m}^2 = \sum_{i=0}^m \|\bar{\partial}^i f\|_{2, U}^2, \]
\( W^m(U, H) \) becomes a Hilbert space contained continuously in \( L^2(U, H) \).

Let \( U \) be a (connected) bounded open subset of \( C \) and let \( m \) be a nonnegative integer. The linear operator \( M \) of multiplication by \( z \) on \( W^m(U, H) \) is continuous and it has a spectral distribution of order \( m \), defined by the functional calculus

\[
\Phi_M : C_0^\infty(C) \longrightarrow \mathcal{L}(W^m(U, H)), \Phi_M(f) = M_f.
\]

Therefore, \( M \) is a scalar operator of order \( m \).

3. Some properties

Let \( H \) be a complex (separable) Hilbert space with an orthonormal basis \( \{e_0, e_1, \ldots \} \) and let \( S \) be the unilateral shift defined by \( Se_n = e_{n+1} \) for all \( n \geq 0 \). If \( \{\alpha_n\}_{n=0}^\infty \) is any bounded sequence of nonnegative numbers, a unilateral weighted shift \( W \) with a weight sequence \( \{\alpha_n\} \) is defined by \( We_n = \alpha_ne_{n+1} \) for \( n \geq 0 \).

**Proposition 3.1.** Let \( S \) be the unilateral shift and let \( W \) be a unilateral weighted shift with a weight sequence \( \{\alpha_n\}_{n=0}^\infty \). Then \( W \in \text{Helton}_k(S) \) if and only if

\[
\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \alpha_n \cdots \alpha_{n+k-j-1} + 1 = 0
\]

for \( n \geq 0 \).

**Proof.** From the definition, we know that \( W \in \text{Helton}_k(S) \) if and only if

\[
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} S^j W^{k-j} = 0.
\]

Since \( S^j W^{k-j} e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+k-j-1} e_{n+k} \) and \( S^j e_n = e_{n+k} \) for \( n \geq 0 \), we get that \( W \in \text{Helton}_k(S) \) if and only if

\[
\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \alpha_n \cdots \alpha_{n+k-j-1} + 1 = 0
\]

for \( n \geq 0 \). \( \square \)

The next corollary shows that \( \text{Helton}_2(S) \) contains non-\( p \)-hyponormal operators.

**Corollary 3.2.** Let \( S \) be the unilateral shift and let \( W \) be a unilateral weighted shift with a weight sequence \( \{\alpha_n\}_{n=0}^\infty = \{2, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+2}{n+1}, \ldots \} \) for \( n \geq 1 \). Then \( W \in \text{Helton}_2(S) \), but is not a \( p \)-hyponormal operator.

**Proof.** Since a weight sequence is decreasing, \( W \) is not a \( p \)-hyponormal operator. Since \( \alpha_n \alpha_{n+1} - 2\alpha_n + 1 = 0 \) for \( n \geq 0 \), from Proposition 3.1, \( W \in \text{Helton}_2(S) \). \( \square \)

**Remark.** \( C(W, S)^k(I) \neq 0 \) for all positive integers \( k \), where \( S \) is the unilateral shift and \( W \) is the unilateral weighted shift in Corollary 3.2. Indeed, since

\[
W^j S^{k-j} e_n = \alpha_{n+k-j} \alpha_{n+k-j+1} \cdots \alpha_{n+k-1} e_{n+k}
\]
and \( \alpha_n = \frac{n+2}{n+1} \) for all \( n \geq 0 \), we obtain

\[
1 + \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \alpha_{n+k-j} \cdots \alpha_{n+k-1} = 1 + \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{n+k+1}{n+k-j+1} \neq 0.
\]

Hence, \( C(W, S)^k(I) \neq 0 \) for all positive integers \( k \).

In general, when \( R \) is a \( p \)-hyponormal operator, there exists a non-\( p \)-hyponormal operator \( T \), which satisfies \( C(R, T)^k(I) = 0 \), but \( C(T, R)^k(I) \neq 0 \) for some \( k \).

**Example 3.3.** Let \( S \) be a unilateral weighted shift with a weight sequence \( \{\alpha_n\}_{n=0}^\infty = \{\frac{1}{2}, \frac{1}{2}, \cdots\} \) and let \( W \) be a unilateral weighted shift with a weight sequence \( \{\beta_n\}_{n=0}^\infty = \{2, \frac{7}{4}, \frac{29}{16}, \frac{11}{8}, \cdots\} \) which satisfies \( \beta_{n+1} \beta_n - \frac{2(n+2)}{n+1} \beta_n + \frac{n+1}{n+1} = 0 \). Then it is easy to show that \( C(S, W)^2(I) = 0 \), but \( C(W, S)^2(I) \neq 0 \).

### 4. Subscalarity

In this section we consider the scalar extensions of the Helton class of \( p \)-hyponormal operators. We start this section with the definition of the Aluthge transform of an operator. An arbitrary operator \( T \in \mathcal{L}(H) \) has a unique polar decomposition \( T = U|T| \), where \( |T| = (T^*T)^{1/2} \) and \( U \) is the appropriate partial isometry satisfying \( \ker U = \ker |T| \) and \( \ker U^* = \ker T^* \). Associated with \( T \) is a related operator \( |T|^2U|T|^{*2} \), called the Aluthge transform of \( T \), and denoted throughout this paper by \( \tilde{T} \). Lemma 4.1 is essential for the proof of our main theorem.

**Lemma 4.1.** Let \( T \in \mathcal{L}(H) \) be a \( p \)-hyponormal operator where \( 0 < p \leq 1 \). If \( \{f_n\} \) is a sequence in \( W^{4k}(D, H) \) such that \( \lim_{n \to \infty} \|(T-z)^k \tilde{\partial}^if_n\|_{2,D} = 0 \) for \( i = 1, \cdots, 4k \), then \( \lim_{n \to \infty} \|(T-z) \tilde{\partial}^if_n\|_{2,D_0} = 0 \) for \( i = 1, \cdots, 4 \) and \( \lim_{n \to \infty} \|(I-P)f_n\|_{2,D_0} = 0 \) for every disk \( D_0 \) strictly contained in \( D \).

**Proof.** Assume that \( \{f_n\} \) is a sequence in \( W^{4k}(D, H) \) such that

\[
\lim_{n \to \infty} \|(T-z)^k \tilde{\partial}^if_n\|_{2,D} = 0 \quad \text{for } i = 1, \cdots, 4k.
\]

Using the induction, we will show that

\[
\lim_{n \to \infty} \|(T-z)^{k-1} \tilde{\partial}^if_n\|_{2,D_0} = 0
\]

for \( i = 1, \cdots, 4(k-1) \), where a disk \( D_0 \) is strictly contained in \( D \). Since \( |T|^{1/2}T = \tilde{T}|T|^{1/2} \), we have

\[
\lim_{n \to \infty} \|(\tilde{T} - z)^k |T|^{1/2} \tilde{\partial}^if_n\|_{2,D} = 0
\]

for \( i = 1, \cdots, 4k \). Since \( \tilde{T} \) is semi-hyponormal, by [10],

\[
\lim_{n \to \infty} \|(\tilde{T} - z)^{k-1} |T|^{1/2} \tilde{\partial}^if_n\|_{2,D} = 0
\]

for \( i = 1, \cdots, 4k \). By an application of [13, Proposition 2.1], we get

\[
\lim_{n \to \infty} \|(I-P)(\tilde{T} - z)^{k-1} |T|^{1/2} \tilde{\partial}^if_n\|_{2,D} = 0
\]
for \( i = 1, \ldots, 4k - 2 \). From (4) and (5), we have
\[
\lim_{n \to \infty} \| (\tilde{T} - z)P(\tilde{T} - z)^{k-1}|T|\frac{1}{z} \partial^i f_n \|_{2,D} = 0
\]
for \( i = 1, \ldots, 4k - 2 \), where \( P \) denotes the orthogonal projection of \( L^2(D,H) \) onto the Bergman space \( A^2(D,H) \). Since \( \tilde{T} \) is semihyponormal, it satisfies the property (\( \beta \)) from [11]. Hence
\[
\lim_{i \to \infty} \| P(\tilde{T} - z)^{k-1}|T|\frac{1}{z} \partial^i f_n \|_{2,D_o} = 0
\]
for \( i = 1, \ldots, 4k - 2 \), where a disk \( D_0 \) is strictly contained in \( D \). Since \( |T|^{1/2}T = \tilde{T}|T|^{1/2} \), (6) and (7) imply that \( \lim_{n \to \infty} \| (T - z)^{k-1} \partial^i f_n \|_{2,D} = 0 \) for \( i = 1, \ldots, 4k - 2 \). Since \( T = U|T| \), from (3) we have
\[
\lim_{n \to \infty} \| (I - P)(T - z)^{k-1} \partial^i f_n \|_{2,D_0} = 0
\]
for \( i = 1, \ldots, 4k - 4 \). From (8) and (9), we obtain
\[
\lim_{n \to \infty} \| zP(T - z)^{k-1} \partial^i f_n \|_{2,D_0} = 0
\]
for \( i = 1, \ldots, 4k - 4 \). By [6], there exists a constant \( c > 0 \) such that
\[
c \| P(T - z)^{k-1} \partial^i f_n \|_{2,D_0} \leq \| zP(T - z)^{k-1} \partial^i f_n \|_{2,D_0}
\]
for \( i = 1, \ldots, 4k - 4 \). Hence from (10) we have \( \lim_{n \to \infty} \| (T - z)^{k-1} \partial^i f_n \|_{2,D_0} = 0 \) for \( i = 1, \ldots, 4k - 4 \). Then (9) implies that \( \lim_{n \to \infty} \| (T - z)^{k-1} \partial^i f_n \|_{2,D_0} = 0 \) for \( i = 1, \ldots, 4k - 4 \). Thus \( \lim_{n \to \infty} \| (T - z)^k \partial^i f_n \|_{2,D_0} = 0 \) for \( i = 1, \ldots, 4k - 4 \). In particular, if \( k = 1 \) in (9), we have \( \lim_{n \to \infty} \| (I - P)f_n \|_{2,D_0} = 0 \). So we complete the proof. \( \square \)

Now we are ready to prove our main theorem.

**Theorem 4.2.** Let \( T \in \mathcal{L}(H) \) be a p-hyponormal operator where \( 0 < p \leq 1 \). If \( S \in \text{Helton}(T) \), then \( S \) is a subscalar operator of order \( 4k \).

**Proof.** Consider an arbitrary bounded open disk \( D \) of the complex plane \( C \) containing \( \sigma(S) \) and 0 and the quotient space
\[
H(D) = W^{4k}(D,H)/(z - S)W^{4k}(D,H)
\]
edowed with the Hilbert space norm. The class of a vector \( f \) or an operator \( A \) on \( H(D) \) will be denoted by \( \tilde{f} \), respectively \( \tilde{A} \). Note that \( M \), the multiplication operator with \( z \) on \( W^{4k}(D,H) \), leaves invariant \( \text{ran}(z - S) \); hence \( \tilde{M} \) is well defined. Moreover, the spectral distribution \( \Phi \) of \( M \) commutes with \( z - S \); therefore, \( \tilde{M} \) is still a scalar operator of order \( 4k \), with \( \tilde{\Phi} \) as spectral distribution.

Let \( V \) be the operator \( V(h) = 1 \otimes \tilde{h} \), from \( H(D) \) into \( H(D) \), denoting by \( 1 \otimes h \) the constant function \( h \). Then \( VS = \tilde{M}V \). Indeed, \( VSh = 1 \otimes \tilde{Sh} = z \otimes \tilde{h} = \tilde{M}(1 \otimes \tilde{h}) = \tilde{MV}h \) for any \( h \in H \). Since \( \text{ran}V \) is an invariant subspace for \( \tilde{M} \), it suffices to show that \( V \) is one-to-one and has closed range.

If \( h_n \in H \) and \( f_n \in W^{4k}(D,H) \) are sequences such that
\[
\lim_{n \to \infty} \| (z - S)f_n + 1 \otimes h_n \|_{W^{4k}} = 0,
\]
then we have

\[(12) \quad \lim_{n \to \infty} \|(z-S)\bar{\partial}^i f_n\|_{2,D} = 0 \]

for \(i = 1, 2, \cdots, 4k\). Now for \(i = 1, 2, \cdots, 4k\),

\[
\lim_{n \to \infty} \| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} T^j S^{k-j} \bar{\partial}^j f_n - (T-z)^k \bar{\partial}^j f_n \|_{2,D} \\
= \lim_{n \to \infty} \| \sum_{j=0}^{k} \binom{k}{j} (T-z)^j (z-S)^{k-j} \bar{\partial}^j f_n - (T-z)^k \bar{\partial}^j f_n \|_{2,D} \\
= \lim_{n \to \infty} \| \sum_{j=0}^{k-1} \binom{k}{j} (T-z)^j (z-S)^{k-j} \bar{\partial}^j f_n \|_{2,D} \\
\leq \lim_{n \to \infty} \| \sum_{j=0}^{k-1} \binom{k}{j} (T-z)^j (z-S)^{k-j-1} \| \| (z-S) \bar{\partial}^j f_n \|_{2,D} \\
= 0.
\]

Hence we have \(\lim_{n \to \infty} \|(T-z)^k \bar{\partial}^j f_n\|_{2,D} = 0\) for \(i = 1, 2, \cdots, 4k\). Then Lemma 4.1 can be applied to get

\[(13) \quad \lim_{n \to \infty} \|(I-P)f_n\|_{2,D_0} = 0,\]

where a disk \(D_0\) is strictly contained in \(D\). Now we apply (13) to (11). Then we get

\[
\lim_{n \to \infty} \|(z-S)Pf_n + 1 \otimes h_n\|_{2,D_0} = 0.
\]

Let \(\Gamma\) be a curve in \(D_0\) surrounding \(\sigma(S)\). Then for \(z \in \Gamma\),

\[
\lim_{n \to \infty} \| Pf_n(z) + (z-S)^{-1}(1 \otimes h_n)\| = 0
\]

uniformly. Hence by the Riesz-Dunford functional calculus we have

\[
\lim_{n \to \infty} \| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z)dz + h_n \| = 0.
\]

But since \(\int_{\Gamma} Pf_n dz = 0\) by Cauchy’s theorem, \(\lim_{n \to \infty} h_n = 0\). Hence \(V\) is one-to-one and has closed range.

**Corollary 4.3.** Every \(p\)-hyponormal operator is subscalar of order 4.

**Proof:** If \(k = 1\) in Theorem 4.2, then \(S = T\). Hence the proof follows from Theorem 4.2. \(\square\)

If \(U\) is a bounded open set in the complex plane \(\mathbb{C}\), recall that a subset \(\Lambda \subset U\) is said to be *dominating* for \(U\) if every function \(h(z)\) analytic and bounded on \(U\) satisfies

\[
\sup_{z \in U} |h(z)| = \sup_{z \in U \cap \Lambda} |h(z)|.
\]

**Corollary 4.4.** Let \(S \in Helton_k(T)\) where \(T \in \mathcal{L}(H)\) is a \(p\)-hyponormal operator with \(0 < p \leq 1\). If there exists a nonempty open set \(U\) in the complex plane \(\mathbb{C}\) such that \(\sigma(S) \cap U\) is dominating for \(U\), then \(S\) has a nontrivial invariant subspace.

**Proof:** This follows from Theorem 4.2 and \(\square\).
From Theorem 4.2 we get the following corollary.

**Corollary 4.5.** If \( S \in \text{Helton}_k(T) \) where \( T \in \mathcal{L}(H) \) is a \( p \)-hyponormal operator with \( 0 < p \leq 1 \), then

(a) \( S \) has the property (\( \beta \));

(b) \( S \) has the single-valued extension property.

**Proof.** Since every scalar operator satisfies the property (\( \beta \)) and the property (\( \beta \)) is transmitted from an operator to its restrictions to closed invariant subspaces, it follows from Theorem 4.2 that \( S \) satisfies the property (\( \beta \)). In particular, it has the single-valued extension property. \( \square \)

We close this paper with the following remark pointed out by the referee.

**Remark.** \( T^* \) being in the Helton class of order \( k \) of \( T \) translates to (in Helton’s terminology) that \( T^* \) is coadjoint of order \( k-1 \), from which it follows (as in [9]) that \( \|e^{i\alpha T^*}\| = O(|s|^{k-1}) \). Hence \( T^* \) (and also \( T \)) possesses a continuous \( C^k(\mathbb{R}) \)-functional calculus. But then it follows that \( T \) is decomposable (see [5]). By definition any decomposable operator having spectrum with more than one point has nontrivial invariant subspaces. Helton conjectured and proved in some special case while Agler provided a proof for the general case (at least for \( k = 3 \)) (see also the paper of Ball and Helton, [3]) an alternative characterization of coadjoint operators as **sub-Jordan operators**. Sub-Jordan operators appear to be related to but a priori different from subscalar operators.

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