TRANSVERSALS
FOR STRONGLY ALMOST DISJOINT FAMILIES

PAUL J. SZEPTYCKI

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Abstract. For a family of sets $A$, and a set $X$, $X$ is said to be a transversal of $A$ if $X \subseteq \bigcup A$ and $|a \cap X| = 1$ for each $a \in A$. $X$ is said to be a Bernstein set for $A$ if $\emptyset \neq a \cap X \neq a$ for each $a \in A$. Erdős and Hajnal first studied when an almost disjoint family admits a set such as a transversal or Bernstein set. In this note we introduce the following notion: a family of sets $A$ is said to admit a $\sigma$-transversal if $A$ can be written as $A = \bigcup \{A_n : n \in \omega\}$ such that each $A_n$ admits a transversal. We study the question of when an almost disjoint family admits a $\sigma$-transversal and related questions.

1. Introduction

Given a family of sets $A$, we can say that a set $X$ splits the sets in $A$ if for every $a \in A$, $X \cap a$ is nonempty but small. It is natural to consider notions of smallness that include cardinality, small in measure, topologically small and others. For example, a Bernstein set $X \subseteq \mathbb{R}$ is a set that splits the family of perfect subsets of $\mathbb{R}$, where small means “is not equal to”. In [2], the question of which almost disjoint families can be split was considered.

Most of our notation and terminology are standard and can be found in [6]. A family of sets $A \subseteq [\lambda]^{<\kappa}$ is almost disjoint if pairwise intersections are of cardinality $< \kappa$. A family of sets is said to be strongly almost disjoint if pairwise intersections are finite, and $r$-almost disjoint if pairwise intersections have cardinality less than $r$. A family of sets $A$ is said to be point-finite (point-countable) if $\{a \in A : x \in a\}$ is finite (countable).

The following theorems were proven in [2].

Theorem 1. Assume GCH. For every strongly almost disjoint family $A$ consisting of sets of size $\geq \aleph_1$, if $|A| \leq \aleph_\omega$, then $A$ has the Bernstein property $B$. I.e., there is $X$ such that $\emptyset \neq a \cap X \neq a$ for each $a \in A$.

Theorem 2. For each $r < \omega$ and each $r$-almost disjoint family $A$ of countable sets, if $|A| = \aleph_n$, then there is $X$ such that $\emptyset \neq |a \cap X| < (r - 1)(n + 1) + 2$ for each $a \in A$. 

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Theorem 3. \( \omega \times \omega \) can be covered by countably many functions and their inverses.

The rows and columns form a 2-almost disjoint family, and each function is a transversal of the columns and each inverse is a transversal of the rows. The content of the theorem is that there are countably many of these partial transversals that cover \( \omega \times \omega \).

In [5] the question of whether certain strongly almost disjoint families of sets may admit a countable family of partial transversals covering the underlying set was central to the analysis of scattered compact spaces of finite Cantor-Bendixson height (e.g., the one-point compactification of \( \omega \times \omega \)). Motivated by those preliminary results we now study more systematically when a strongly almost disjoint family admits a countable family of partial transversals (possible covering the underlying set).

Following the notation of [2], we introduce the following notation.

Definition 4. Suppose that \( m, p, r, s \) are cardinals. \( M(m, p, r) \rightarrow \sigma - B(s) \) means that for every \( r \)-almost disjoint family \( A \) of size \( m \) consisting of infinite sets of size \( \leq p \), there are sets \( \{ X_n : n < \omega \} \) and \( \{ A_n : n < \omega \} \) such that

1. \( A = \bigcup \{ A_n : n < \omega \} \), and
2. \( 0 \neq |X_n \cap a| < s \) for each \( a \in A_n \),
3. \( X_n \subseteq \bigcup A_n \).

Also, \( M(m, p, r) \rightarrow \sigma - B \) means (1), (3), and instead of (2) we require

\[
(2') \quad 0 \neq X_n \cap a \neq a \quad \text{for each } a \in A_n.
\]

In the case that \( s = 2 \), the family of sets \( \{ X_n : n \in \omega \} \) satisfying (1)-(3) will be called a \( \sigma \)-transversal of \( A \). A \( \sigma \)-transversal that covers the underlying set will be called a covering \( \sigma \)-transversal. Even if a family \( A \) admits a transversal, one cannot expect to obtain a covering \( \sigma \)-transversal unless for every \( a \in A \), all but countably many points of \( a \) are covered by \( A \setminus \{ a \} \). For example, any disjoint family of uncountable sets obviously admits a transversal, but there can be no covering \( \sigma \)-transversal.

In general, one cannot expect to prove \( M(m, p, r) \rightarrow \sigma - B(s) \) for \( s \leq \omega \) unless \( r \leq \omega \). Indeed, if \( A = \{ a_{\beta, \alpha} : \beta < \alpha < \omega_1 \} \) is an almost disjoint family of uncountable subsets of \( \omega_1 \) such that \( (\beta, \alpha) \subseteq a_{\beta, \alpha} \subseteq (\beta, \omega_1) \) for each \( \beta < \alpha \), then \( A \) witnesses that \( M(\omega_1, \omega_1, \omega_1) \not\rightarrow \sigma - B(\aleph_0) \); indeed, if \( \{ X_n : n \in \omega \} \) is a family of subsets of \( \omega_1 \). Let \( \beta \) be such that \( X_n \subseteq \beta \) whenever \( X_n \) is countable. Then pick \( \alpha > \beta \) so that \( X_n \cap a \) is infinite whenever \( X_n \) is uncountable. Then \( a_{\beta, \alpha} \) is either disjoint from \( X_n \) or has infinite intersection with \( X_n \) for every \( n \in \omega \).

Thus, we may only expect to obtain positive results about \( \sigma \)-transversals for strongly almost disjoint families or almost disjoint families with additional properties.
Note also that a strongly almost disjoint family may not admit a transversal, even if it is countable and 2-almost disjoint. Indeed, if \( A \) is a maximal 2-almost disjoint family in \( [\omega]^\aleph_0 \), then by maximality, it does not admit a transversal. Even a 2-almost disjoint and point-finite family may not admit a transversal: the collection of rows and columns of \( \omega \times \omega_1 \) has no transversal.

2. \( \sigma \)-Transversals of Strongly Almost Disjoint Families

Any family \( A \) of subsets of \( \omega \) admits a \( \sigma \)-transversal for trivial reasons: the families \( X_n = \{ n \} \) and \( A_n = \{ a \in A : n \in a \} \) form a \( \sigma \)-transversal of \( A \).

almost disjoint family of subsets of \( \omega \) consisting of infinite \( X_n \)'s. However, any almost disjoint family \( A \) of subsets of \( \omega \) admits a \( \sigma - B(\aleph_0) \) family consisting of infinite sets: to see this find \( \{ a_n : n \in \omega \} \subseteq A \) such that

For strongly almost disjoint families of subsets of \( \omega_1 \) the situation is more complicated. In [5] it was proven, assuming \( MA_{\omega_1} \), that certain strongly almost disjoint families admit nice families of partial transversals. In this note, we refine that result to obtain:

**Theorem 5.** Assume \( MA_{\omega_1} \). Then any strongly almost disjoint family \( A \) such that \( |A| = \omega_1 \) consisting of infinite sets of size \( \leq \aleph_1 \) admits a \( \sigma \)-transversal. Moreover, under the additional assumption that each \( a \in A \) is covered by \( A \setminus \{ a \} \), then \( A \) admits a covering \( \sigma \)-transversal.

**Proof.** The proof is similar to the result from [5] mentioned above. For completeness sake we give a full proof. First note that we may assume that \( A \) is a family of subsets of \( \omega_1 \). We need the following lemma about strongly almost disjoint families:

**Lemma 6.** Suppose that \( \{ a_\alpha : \alpha < \omega_1 \} \) is a strongly almost disjoint family of sets. Suppose also that \( \{ p_\alpha : \alpha < \omega_1 \} \) is a sequence of pairwise disjoint finite sets. Then there are \( \alpha < \beta \) such that \( p_\alpha \cap a_\beta = \emptyset \) and \( p_\beta \cap a_\alpha = \emptyset \).

**Proof.** By going to a subsequence, we may assume that there is \( n \in \omega \) such that \( |p_\alpha| = n \) for all \( \alpha \). Let \( M \) be a countable elementary submodel of a suitably large \( H(\theta) \) containing everything relevant and let \( \gamma = M \cap \omega_1 \).

**Claim 7.** There are \( \{ \alpha_i : i < n + 1 \} \subseteq \gamma \) and a \( \beta > \gamma \) such that

\[
\left( \bigcup_{i < n+1} p_{\alpha_i} \right) \cap a_\beta = \emptyset.
\]

**Proof.** If not, then for each \( \beta > \gamma \) there are at most \( n \) \( \alpha \)'s below \( \gamma \) such that \( p_\alpha \cap a_\beta = \emptyset \). Thus, there is a \( \alpha_\beta < \gamma \) such that \( p_\alpha \cap a_\beta \neq \emptyset \) for each \( \alpha \in M \setminus \alpha_\beta \). Choose \( \{ \beta_i : i < n + 1 \} \subseteq \omega_1 \setminus \gamma \). Choose an index \( \alpha \in M \) above \( \{ \alpha_\beta_i : i < n + 1 \} \) such that \( p_\alpha \) is disjoint from the following finite set:

\[
\bigcup_{0 \leq i < j < n+1} (a_\beta_i \cap a_\beta_j) \cap M.
\]

Then for \( i < n + 1 \) we have that \( p_\alpha \cap a_\beta_i \neq \emptyset \) and the family \( \{ p_\alpha \cap a_\beta_i : i < n + 1 \} \) is pairwise disjoint. This contradicts that \( |p_\alpha| = n \).

To complete the proof of the main lemma, note that by elementarity \( a_\alpha \cap a_\alpha \subseteq M \) for each \( i \neq j \) and \( p_\beta \cap M = \emptyset \) since \( \beta \notin M \) and the \( p_\alpha \)’s are pairwise disjoint. By the pigeon-hole argument just presented in the proof of Claim 7 we may conclude that \( p_\beta \cap a_\alpha_i = \emptyset \) for some \( i < n + 1 \).
Returning now to the proof of the theorem we need to find sets \( A_n \) and \( X_n \) satisfying (1)–(3) of Definition \( 4 \) and, assuming that each \( a \in A \) is covered by the rest of \( A \), also

(4) \( \bigcup \{ X_n : n \in \omega \} = \bigcup A \).

Let \( (M_\alpha : \alpha < \omega_1) \) be a continuous \( \in \)-chain of countable elementary submodels of a suitably large \( H(\theta) \) containing everything relevant. For each \( a \in A \), let \( \alpha \) be minimal such that \( a \in M_\alpha \). Let \( a' = a \cap M_\alpha \). (To obtain (1)–(3) the elementary submodels are not needed, and it suffices to take \( a' \in [a]^{\aleph_0} \). Elementarity will be used to obtain (4).)

Now define the poset \( P \) to be the set of all pairs \( p = (x_p, F_p) \), where \( x_p \in [\omega_1]^<\omega \) and \( F_p \in [A]^{<\omega} \) with the property that \( |x_p \cap a'| = |x_p \cap a| = 1 \) for all \( a \in F_p \). We define \( p \leq q \) if \( x_q \subseteq x_p \) and \( F_q \subseteq F_p \).

Claim 8. \( P \) has the ccc.

Proof. Given an uncountable subset \( \{ (y_\alpha, G_\alpha) : \alpha \in \omega_1 \} \), we may assume that the \( y_\alpha \)'s form a \( \Delta \)-system with root \( r \). Let \( x_\alpha = y_\alpha \setminus r \). We may also assume that the \( G_\alpha \)'s form a \( \Delta \)-system with root \( R \). Let \( F_\alpha = G_\alpha \setminus R \). By going to a subsequence we may assume that \( x_\alpha \cap \bigcup \{ a' : a \in R \} = \emptyset \) for each \( \alpha \). Thus, for each \( a \in R \), \( y_\alpha \cap a' = r \cap a' \). Thus, if we let \( a_\alpha = \bigcup F_\alpha \) it follows that \( (y_\alpha, G_\alpha) \) is compatible with some \( (y_\beta, G_\beta) \) if and only if \( x_\alpha \cap a_\beta = \emptyset = x_\beta \cap a_\alpha \). The existence of such a pair is given by Lemma \( \text{[6]} \).

To finish the proof, let \( P^\omega \) denote the finite support product of countably many copies of \( P \). Consider the following subsets of \( P^\omega \):

\[ D_\alpha = \{ p \in P^\omega : \exists n \ a \in F_{p(n)} \} \]

It is easy to show that \( D_\alpha \) is dense for each \( a \in A \). By \( MA_{\omega_1} \), we may let \( G \) be a \( \{ D_\alpha : a \in A \} \)-generic filter. Let \( X_\alpha = \bigcup \{ x_{p(n)} : p \in G \} \) and \( A_\alpha = \bigcup \{ F_{p(n)} : p \in G \} \). Then \( \bigcup A_\alpha = A \) by genericity of \( G \), and the rest of (1)–(3) easily follows from the definition of the partial order.

In order to obtain (4) assume now that each \( a \) is covered by \( A \setminus \{ a \} \). It suffices to show that for each \( \alpha \in \bigcup A \) the following set is dense:

\[ E_\alpha = \{ p \in P^\omega : \exists \alpha' a \in p_{(n)} \} \]

Note that \( E_\alpha \) is dense if \( a \in a' \) for some \( a \in A \). To see this, consider any \( a \in A \) and any \( \alpha \in a \). Let \( \beta \) be such that \( \alpha \in M_{\beta+1} \setminus M_\beta \). If \( a \notin M_\beta \), it follows that \( \alpha \in a' \). So assume that \( a \in M_\beta \). Since \( a \) is covered by \( A \setminus \{ a \} \), there is a \( b \in A \) such that \( b \neq a \) and \( \alpha \in b \). Now it suffices to observe that \( b \notin M_\beta \). For if \( b \in M_\beta \), then, since \( a \cap b \) is finite, by elementarity it would follow that \( a \cap b \subseteq M_\beta \), which is impossible. \( \square \)

Remarks. (1) Note that the natural poset for producing a partial transversal is not necessarily ccc: the natural poset being the collection of pairs \((x, F)\) where \( x \) is a finite subset of \( \bigcup A \) and \( F \) is a finite subset of \( A \) with the property that \( |x \cap a| = 1 \) for each \( a \in F \) with the ordering of reverse inclusion on both coordinates. If \( A \) contains an uncountable element \( a \), then this poset is not ccc. Indeed, the collection of pairs \( \{ (\beta), \{ a \} \} \), where \( \beta \in a \), is an uncountable antichain.

(2) The only place where the elementary submodels played a role in the proof is in order to get the \( \sigma \)-transversal to be covering. Otherwise it would have sufficed to choose \( a' \in [a]^{\aleph_0} \) arbitrary. Thus, we obtain as a corollary to the proof of Theorem \( \text{[5]} \) that \( MA_\alpha \) implies \( M(\kappa, \kappa, \omega) \rightarrow \sigma - B(2) \). The restriction on \( \kappa = \omega_1 \) to get the covering \( \sigma \)-transversal is necessary (see Theorem \( \text{[4]} \) below).
Now we show that the assumption of MA in Theorem 5 is necessary.

**Theorem 9.** Assume $\Diamond$. Then $M(\omega_1, \omega, \omega) \not\models B(\aleph_0)$.

**Proof.** Let $\{D_\alpha : \alpha < \omega_1\}$ be a $\Diamond$-sequence for subsets of $\omega_1 \times \omega$. So each $D_\alpha$ is a subset of $\alpha \times \omega$ and for any countable family $\{X_n : n \in \omega\}$ of subsets of $\omega_1$, there are stationary many $\alpha$ for which $\bigcup\{(X_n \cap \alpha) \times \{n\} : n \in \omega\} = D_\alpha$. For each $n$ let $D_\alpha(n) = \{\beta < \alpha : (\beta, n) \in D_\alpha\}$. For each limit $\alpha \in \omega_1$ choose an $\omega$-sequence $a_\alpha$ cofinal in $\alpha$ such that $a_\alpha \cap D_\alpha(n)$ is infinite whenever $D_\alpha(n)$ is cofinal in $\alpha$. Moreover, if there is a $\beta$ such that $D_\alpha(n) \subseteq \beta$ whenever $D_\alpha(n)$ is boundable in $\alpha$, then take $\beta_\alpha$ to be the least such $\beta$, and choose $a_\alpha$ such that $a_\alpha \cap \beta_\alpha = \emptyset$. Then $A = \{a_\alpha : \alpha \in \omega_1\}$ is an almost disjoint family in $[\omega_1]\omega$.

Suppose that $A$ admits families $\{X_n : n \in \omega\}, \{A_n : n \in \omega\}$ such that for each $a_\alpha \in A_n$, we have that $a_\alpha \cap X_n$ is finite and nonempty. Some of the $X_\alpha$'s may be countable, and we may let $\beta$ be an upper bound for all those countable $X_\alpha$'s. If $X_n$ is uncountable, let $E_n$ be the club set of limit $\alpha$'s for which $X_n$ is cofinal in $\alpha$. We may fix $\alpha > \beta$ in $\bigcap E_n$ such that $X_n \cap \alpha = D_\alpha(n)$ for every $n$. It follows that $D_\alpha(n)$ is cofinal in $\alpha$ whenever $X_n$ is uncountable; otherwise $D_\alpha(n)$ is boundable below $\beta$.

Fix $n$ such that $a_n \in A_n$. Then, if $X_n$ is countable, by choice of $a_n$, we have that $a_\alpha \cap X_n$ is empty. So $X_n$ must be uncountable. Thus, $X_n \cap \alpha$ is cofinal in $\alpha$, and by the choice of $a_\alpha$, we have that $a_\alpha \cap X_n$ is infinite. Contradiction. □

While some extra set-theoretic assumption is needed to obtain $\sigma$-transversals of strongly almost disjoint families of subsets of $\omega_1$, if we make some additional almost-disjointness restrictions on the families, we can obtain ZFC results. Namely, for any $r < \omega$, any $r$-almost disjoint family admits a $\sigma$-transversal. As a corollary to the proof we will also obtain the fact that any point-countable almost disjoint family admits a $\sigma$-transversal (Corollary 11 below).

**Theorem 10.** $M(\kappa, \eta, r) \rightarrow \sigma - B(2)$ for every infinite $\kappa, \eta$ and every $r < \omega$.

**Proof.** We prove by induction on $|A|$ that something stronger than $A$ admits a $\sigma$-transversal. Namely:

(IH)$_\kappa$: For every $r$-almost disjoint family $B$ of cardinality $\kappa$, if $B = \bigcup B_n$, then there is $\{X_n : n, k \in \omega\}$ and $\{B_{nk} : n, k < \omega\}$ such that

(a) $\{X_n : n, k \in \omega\}, \{B_{nk} : n, k < \omega\}$ is a $\sigma$-transversal of $B$,

(b) $B_{nk} \subseteq B_n$ for every $n$ and $k \in \omega$, and

(c) $\{X_n : n, k \in \omega\}$ is a point-finite family.

We will refer to such a family of sets as a point-finite $\sigma$-transversal refining $\{B_n : n \in \omega\}$.

It is easy to see that (IH)$_{\aleph_0}$ holds.

Assume that $A$ is $r$-almost disjoint, $A = \bigcup A_n, |A| = \kappa$ and (IH)$_\lambda$ holds for all $\lambda < \kappa$. First note that there is a set $S$ of cardinality $\kappa$ such that $\{a \setminus S : a \in A\}$ is a disjoint family. Thus, we may assume that $A \subseteq P(\kappa)$. Moreover, it suffices to find the appropriate point-finite $\sigma$-transversal for the subsets $A_0 = \{a \in A : |a| = \kappa\}$ and $A_1 = \{a \in A : |a| < \kappa\}$ (putting them together gives the required point-finite $\sigma$-transversal for $A$). Therefore we may assume that either $|a| = \kappa$ for every $a \in A$ or $|a| < \kappa$ for every $a \in A$.

**Case 1:** $\kappa$ is a successor $\lambda^+$. Let $\{M_\alpha : \alpha < \kappa\}$ be a continuous $\in$-chain of elementary submodels of a suitably large $H(\theta)$ each of cardinality $\lambda$ such that $M_0$ contains everything relevant and such that $A \subseteq \bigcup \{M_\alpha : \alpha < \kappa\}$. 

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Subcase 1: $A \subseteq [\kappa]^<\kappa$. Thus, it follow that

(d) for each $\alpha \in \kappa$ and $a \in A \cap M_\alpha$, $a \subseteq M_\alpha$.

Also, since $A$ is $r$-almost disjoint, we have that

(e) for each $\alpha \in \kappa$ and $a \in A \setminus M_\alpha$ $|a \cap M_\alpha| < \omega$.

Indeed this intersection must be of cardinality less than $r$. This follows by elementarity: For if $x \in [a \cap M_\alpha]^r$, then $x$ is an element of $M_\alpha$ and the set of $a \in A$ such that $x \subseteq a$ is of cardinality $\kappa$. Contradiction.

We construct, by recursion on $\alpha < \kappa$, sets $X^\alpha_{nk} \subseteq \kappa \cap M_\alpha$ and $A^\alpha_{nk} \subseteq A \cap M_\alpha$ for each $n, k < \omega$ so that

(f) $\{X^\alpha_{nk} : n, k \in \omega\}, \{A^\alpha_{nk} : n, k < \omega\}$ is a $\sigma$-transversal of $A \cap M_\alpha$,

(g) $A^\alpha_{nk} \subseteq A \cap M_\alpha$ for every $n$ and $k \in \omega$, and

(h) $\{X^\alpha_{nk} : n, k \in \omega\}$ is a point-finite family.

(i) If $\beta < \alpha$, then $X^\alpha_{nk} \cap M_\beta = X^\beta_{nk}$ and $A^\alpha_{nk} \cap M_\beta = A^\beta_{nk}$ for all $n, k \in \omega$.

First note that if such a sequence is constructed, then $X_{nk} = \bigcup_{\alpha < \kappa} X^\alpha_{nk}$, and $A_{nk} = \bigcup_{\alpha < \kappa} A^\alpha_{nk}$ defines the required point-finite $\sigma$-transversal refining $\{A_n : n \in \omega\}$.

If $\alpha$ is a limit we define $X^\alpha_{nk} = \bigcup_{\beta < \alpha} X^\beta_{nk}$ and $A^\alpha_{nk} = \bigcup_{\beta < \alpha} A^\beta_{nk}$. Then by (i) all the inductive hypotheses are preserved.

So suppose that $\alpha < \kappa$ is given and $X^\alpha_{nk}$ and $A^\alpha_{nk}$ have been constructed satisfying (f), (g), (h) and (i). For each finite $x \subseteq \omega$ let

$$A_{x,n}^{\alpha+1} = \{a \in A_n \cap (M_{\alpha+1} \setminus M_\alpha) : x = \{k \in X^\alpha_{nk} \neq \emptyset\}\}.$$

By (e) and (h) we have that

$$A_n \cap (M_{\alpha+1} \setminus M_\alpha) = \bigcup \{A_{x,n}^{\alpha+1} : x \in [\omega]^\omega\}.$$

By our inductive hypotheses we may find sets $Y_{x,n,i}^{\alpha+1}$ and $A_{x,n,i}^{\alpha+1}$ forming a point-finite $\sigma$-transversal refining $\{A_{x,n}^{\alpha+1} : n \in \omega, x \in [\omega]^\omega\}$. For each $i, x, n$ choose $k_{i,x,n}$ such that

(k) $k_{i,x,n} \notin x$,

(l) $k_{i,x,n} \neq k_{j,y,n}$ whenever $(i, x) \neq (j, y)$.

Now, let

$$A_{nk_{i,x,n}}^{\alpha+1} = A^\alpha_{nk_{i,x,n}} \cup A_{x,n,i}^{\alpha+1}$$

and

$$X_{nk_{i,x,n}}^{\alpha+1} = X^\alpha_{nk_{i,x,n}} \cup Y_{x,n,i}^{\alpha+1}$$

for each $i, n \in \omega$ and each finite $x \subseteq \omega$. If $k$ is not of the form $k_{i,x,n}$, let

$$A_{nk}^{\alpha+1} = A^\alpha_{nk}$$

and

$$X_{nk}^{\alpha+1} = X^\alpha_{nk}.$$

It is straightforward to check that the inductive hypotheses (f), (g), (h) and (i) are preserved: (h) follows from the way the family of $X^\alpha_{nk}^{\alpha+1}$’s are constructed from two point-finite families. (g) and (i) are directly from construction. To see (f), fix $a \in A \cap M_{\alpha+1}$. If $a \in M_\alpha$, then there is $n, k$ such that $a \in A^\alpha_{nk}$. And since $|a \cap X_{nk}^{\alpha} | = 1$ and $a \subseteq M_\alpha$ and $X_{nk}^{\alpha+1} \cap M_\alpha = X_{nk}^{\alpha}$, it follows that $|a \cap X_{nk}^{\alpha+1} | = 1$.

In the case that $a \in M_{\alpha+1} \setminus M_\alpha$, first note by (j) that there is an $x$ such that $a \in A_{x,n,i}^{\alpha+1}$. Also, by choice of the $A_{x,n,i}^{\alpha+1}$’s, there is an $i$ such that $a \in A_{x,n,i}^{\alpha+1}$. Thus,
a ∈ A^α+1_{\kappa^{n}k_{i,x,n}}. Moreover, |a ∩ Y^{\alpha+1}_{nk_{i,x,n}}| = 1 and a ∩ X^{\alpha}_{nk_{i,x,n}} = ∅ by choice of k_{n,x,i}.
Thus, |a ∩ X^{\alpha+1}_{nk_{i,x,n}}| = 1, as required.

This completes the construction in the case that A ⊆ [κ]^<κ.

Subcase 2: A ⊆ [κ]^κ. The proof in this case is almost identical. One builds by recursion on α < κ the same families \{X^{\alpha}_{nk} : n, k ∈ ω\} and \{A^{\alpha}_{nk} : n, k ∈ ω\).
At stage α of the construction, it suffices to note that a' = a \bigcup (A ∩ M_α) has cardinality κ for each a ∈ M_{α+1} \ M_α. Therefore, working with the sets
\[ A^{\alpha+1}_{k,n} = \{a' : a ∈ A_n \cap (M_{α+1} \setminus M_α) \text{ and } x = \{k : a ∩ X^{\alpha}_{nk} = ∅\}\}, \]
the rest of the proof is identical to the previous case.

This completes the case that κ is a successor. □

Limit case cf(κ) = ω. This case is almost trivial. Fix an ε-chain \{M_m : m < ω\} of elementary submodels of a suitably large H(θ) each of size < κ such that M_0 contains everything relevant and A ⊆ ∪\{M_m : m < ω\}. For each m fix A^m_{n}’s and \{X^{m}_{nk}\}’s forming the required point-finite σ-transversal of \{A_n ∩ M_m \setminus M_{m-1} : n ∈ ω\} so that each X^{m}_{nk} ⊆ M_m \ M_{m-1}. Then \{X^{m}_{nk} : m, k, n < ω\} is point-finite and thus the required σ-transversal of \{A_n : n ∈ ω\}.

Limit case cf(κ) is uncountable. Let \{M_α : α < cf(κ)\} be an ε-chain of elementary submodels of a suitably large H(θ) each of cardinality < κ such that M_0 contains everything relevant, for each α |M_α| < |M_{α+1}|, and A ⊆ ∪\{M_α : α < cf(κ)\}. Let A_0 = \{a ∈ A : if a ∈ M_α, then |a| ≤ |M_α|\} and let A_1 = A \ A_0. As in the successor case, we may assume that either A = A_0 or A = A_1. The proof is now almost identical to the successor case. In the case that A = A_0, we have that the key property (d) holds, and we proceed as in Subcase 1. In the case that A = A_1, we proceed as in Subcase 2.

From the proof of Theorem 10 we obtain the following corollary.

Corollary 11. Assume that A is a point-countable strongly-almost disjoint family consisting of sets of size ≥ ω. Then A admits a σ-transversal.

Proof. Following the proof of Theorem 10 the key fact is clause (e). However, in the case that A is point-countable, we even have that a ∩ M_α = ∅ if a ∉ M_α. □

3. σ-T RANSVERSALS COV ERING THE UNDERLYING SET

Towards generalizing Sierpiński’s Theorem 5 we have the following.

Theorem 12. Suppose that A ⊆ [ω]^ω is a 2-almost disjoint, point-countable family of size ℵ_1 with the property that for each a ∈ A, A \ \{a\} covers a. Then there is a σ-transversal \{X_n : n ∈ ω\}, \{A_n : n ∈ ω\} such that \{X_n : n ∈ ω\} covers \bigcup A.

Proof. Let \{M_α : α < ω_1\} be a continuous ε-sequence of countable elementary submodels of a suitably large H(θ) such that A ∈ M_0. Let A ∩ M_0 = \{a^0_n : n ∈ ω\}. For each n let X^\beta_n = a^0_n ∩ M_0 and let A^\beta_n = \{a ∈ M_0 ∩ A : |a ∩ a^0_n| = 1 and a ∉ A^0_k for any k < n\}. By our assumptions and by elementarity we have
(a) \bigcup X^\beta_n = (\bigcup A) ∩ M_0,
(b) for each \xi ∈ X^\beta_n there is a ∈ A^\beta_n such that a ∩ X^\beta_n = \{\xi\}.
Suppose X^\beta_n and A^\beta_n have been constructed for each \beta < α and each n ∈ ω such that the following inductive hypotheses are satisfied:

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(c) \( \bigcup \{ X^\beta_n : n \in \omega \} = \bigcup A \cap M_\beta \),
(d) for each \( a \in A^\alpha_n \), \( |a \cap X^\beta_n| = 1 \),
(e) for each \( \xi \in X^\beta_n \) there is \( a \in A^\alpha_n \) such that \( a \cap X^\beta_n = \{ \xi \} \),
(f) for \( \beta < \gamma < \alpha \), \( X^\beta_n \subseteq X^\gamma_n \) and \( A^\beta_n \subseteq A^\gamma_n \).
(g) \( A \cap M_\beta = \bigcup_n A^\beta_n \).
(h) \( A^\alpha_n \cap A^\alpha_m = \emptyset \) for all \( m \not\in n \).

If \( \alpha \) is a limit, we let \( X^\alpha_n = \bigcup_{\beta < \alpha} X^\beta_n \) and \( A^\alpha_n = \bigcup_{\beta < \alpha} A^\beta_n \). The inductive hypotheses are preserved.

If \( \alpha = \beta + 1 \), enumerate \( M_\alpha \cap A \) as \( \{ a^\alpha_n : n \in \omega \} \) such that if \( a^\alpha_n \in M_\beta \cap A^\beta_k \), then \( n \neq k \). For each \( n \) let
\[
(a^\alpha_n)' = \left[ a^\alpha_n \setminus \left( A \cap M_\beta \setminus \{ a^\alpha_n \} \right) \right] \cap M_\alpha \subseteq M_\alpha \setminus M_\beta
\]
and let
\[
X^\alpha_n = X^\beta_n \cup (a^\alpha_n)'.
\]

Finally, let
\[
A^\alpha_n = A^\beta_n \cup \{ a \in A \cap M_\alpha \setminus M_\beta : |a \cap X^\alpha_n| = 1 \text{ and } a \not\in A^\alpha_k \text{ for any } k < n \}.
\]

It remains to verify the inductive hypotheses:

(c) is easy but the only nontrivial case: if \( \xi \in \bigcup A \cap M_\alpha \setminus M_\beta \), fix \( a^\alpha_n \in A \) such that \( \xi \in a^\alpha_n \). If \( a^\alpha_n \in M_\beta \), then since \( A \) is strongly almost disjoint, by elementarity, \( a^\alpha_n \) is the only element of \( M_\beta \) containing \( \xi \). Thus, \( \xi \in (a^\alpha_n)' \). If \( a^\alpha_n \not\in M_\beta \) and \( \xi \not\in (a^\alpha_n)' \), then there must be \( a \in A \cap M_\beta \) such that \( \xi \in a \). But this \( a \) is enumerated as some \( a^\alpha_n \) and as in the previous case we would have \( \xi \in (a^\alpha_n)' \).

(d), (f), (g), and (h) follow by construction.

(e) follows from point-countable and the assumption that each \( a \) is covered by \( A \setminus \{ a \} \); to see this, fix \( \xi \in X^\alpha_n \). We may assume that \( \xi \in (a^\alpha_n)' \) and that \( \xi \not\in M_\beta \).
By assumption there is \( a \in A \setminus \{ a^\alpha_n \} \) such that \( \xi \in a \). By elementarity this \( a \) may be taken from \( M_\alpha \), and by definition of \( (a^\alpha_n)' \) we have that \( a \not\in M_\beta \). By point countable, we know that \( a \cap M_\beta = \emptyset \). Thus, \( a \cap X^\beta_n = \emptyset \), and since \( A \) is 2-almost disjoint we have that \( |a \cap (a^\alpha_n)'| = 1 \). Thus, \( |a \cap X^\alpha_n| = 1 \), as required. \( \Box \)

**Corollary 13** (Sierpiński). \( \omega_1 \times \omega_1 \) can be covered by countably many functions and their inverses.

**Proof.** Let \( A \) be the collection of rows and columns of \( \omega_1 \times \omega_1 \). Then \( A \) satisfies the hypotheses of Theorem 12. Let \( \{ X_n, A_n : n \in \omega \} \) be the \( \sigma \)-transversal that covers \( \omega_1 \times \omega_1 \). For each \( n \) let \( f_n : \omega_1 \rightarrow \omega_1 \) be defined so that if \( \{ \alpha \} \times \omega_1 \in A_n \) and \( (\alpha, \gamma) \in X_n \), then \( f_n(\alpha) = \gamma \). Also, let \( g_n : \omega_1 \rightarrow \omega_1 \) be such that if \( \omega_1 \times \{ \alpha \} \in A_n \) and \( (\beta, \alpha) \in X_n \), then \( g_n(\alpha) = \beta \). Since \( \{ X_n : n \in \omega \} \) covers \( \omega_1 \times \omega_1 \) it follows that the functions \( \{ f_n : n \in \omega \} \) and the inverses \( \{ g_n^{-1} : n \in \omega \} \) cover \( \omega_1 \times \omega_1 \) as well. \( \Box \)

**Theorem 14** (Kuratowski). The family of rows and columns of \( \omega_2 \times \omega_2 \) does not admit a covering \( \sigma \)-transversal. Moreover, no \( \sigma - B(\mathbb{N}_1) \) family can be covering.

**Proof.** Recall that \( \omega_2 \times \omega_2 \) cannot be covered by countably many functions and inverses of functions (see [7]). If \( \{ X_n : n \in \omega \} \) were a \( \sigma - B(\mathbb{N}_1) \) family, wlog we can assume that for each \( n \), \( X_n \) either intersects every column in a countable set, or intersects every row in a countable set. So, each \( X_n \) can either be covered by a countable family of functions, or a countable family of inverses of functions. Contradiction. \( \Box \)
Example. Consider the $\Sigma$-product of $\omega_1$ many copies of $\omega_1$ with the discrete topology. For each $f \in X$ and each $\alpha \in \omega_1$, let

$$A_{f\alpha} = \{g \in X : g(\alpha) \neq 0 \text{ and } g(\beta) = f(\beta) \text{ for } \beta \neq \alpha\}.$$ 

It is easy to see that the $A_{f\alpha}$’s form a point-countable 2-almost disjoint family. Thus, by Theorem 12 for any family $Y$ of $\aleph_1$-many $f$’s in $X$, there is a covering $\sigma$-transversal of the family $\{A_{f\alpha} : f \in Y, \alpha \in \omega_1\}$. In particular, if CH is assumed, then the whole space $X$ admits a covering $\sigma$-transversal.

4. Open problems

It follows from Kuratowski’s example that the assumption that $|A| \leq \omega_1$ in Theorem 12 (as well as Theorem 5) is necessary. However, we do not know if the other assumptions are also necessary.

Question 1. Given a point-countable strongly almost disjoint family $A \subseteq [\omega_1]^{<\aleph_1}$ of size $\aleph_1$ such that for each $a \in A$, $A \setminus \{a\}$ covers $a$, is there a $\sigma$-transversal $\{X_n : n \in \omega\}$ covering $\bigcup A$?

Question 2. Given a 2-almost disjoint family $A \subseteq [\omega_1]^{<\aleph_1}$ of size $\aleph_1$ such that for each $a \in A$, $A \setminus \{a\}$ covers $a$, is there a $\sigma$-transversal $\{X_n : n \in \omega\}$ covering $\bigcup A$?

Another basic question left open is what can be said about strongly almost disjoint families of size $2^{\aleph_0}$. Theorem 9 implies that $M(2^{\aleph_0}, \omega, \omega) \not\rightarrow \sigma - \Sigma B(\aleph_0)$ is consistent.

Question 3. Is it consistent that $M(2^{\aleph_0}, \omega, \omega) \rightarrow \sigma - B(\aleph_0)$?

The existence of $\sigma$-Bernstein sets seems significantly weaker than the existence of Bernstein sets. Indeed, for $\kappa$ any cardinal, any countable point-separating family of subsets of $\kappa$ is a $\sigma$-Bernstein set for $\{X \in P(\kappa) : |X| > 1\}$. Thus,

**Proposition 15.** For any $X$ of cardinality at most $2^{\aleph_0}$, the collection of nonsingleton subsets of $X$ admits a $\sigma$-Bernstein set. Thus, $M(2^{\aleph_0}, \omega, \omega) \rightarrow \sigma - B$.

There are families of sets with no $\sigma$-Bernstein set. An interesting example comes from a measurable cardinal.

**Example 16.** Let $\kappa$ be a measurable cardinal. If $u$ is a countably complete ultrafilter on $\kappa$, then $u$ does not admit a $\sigma$-Bernstein set.

**Proof.** Indeed if $u$ is such an ultrafilter, and $\{X_n : n \in \omega\}$ is any family of subsets of $\kappa$, then for each $n$ either $X_n$ or its complement is in $u$. By countable completeness, there is an $X \in u$ such that $X \subseteq X_n$ or $X \cap X_n = \emptyset$ for every $n$. I.e., $u$ has no $\sigma$-Bernstein set.

However, large cardinals are not needed to get an example.

**Proposition 17.** If $cf(\kappa) > 2^{\aleph_0}$, then $[\kappa]^\kappa$ has no $\sigma$-Bernstein set. If in addition $2^\kappa = \kappa^+$, then there is an almost disjoint family $A \subseteq [\kappa]^\kappa$ with no $\sigma$-Bernstein set.

**Proof.** Consider a countable collection $X = \{X_n : n \in \omega\}$ of subsets of $\kappa$. Let $B \in [\kappa]^\kappa$. For each $a \in B$, let $a_n = \{n : a \in X_n\}$. Since $cf(\kappa) > 2^{\aleph_0}$, there is $a \subseteq \omega$ and $A \in |B|^\kappa$ such that, for each $a \in A$, $a \in X_n$ if and only if $n \in a$. Therefore $A$ is either contained in or disjoint from each $X_n$. Thus, $[\kappa]^\kappa$ has no $\sigma$-Bernstein set.

Moreover, if we assume $2^\kappa = \kappa^+$, we may first enumerate all countable collections...
of subsets of $\kappa$ as $\{X_\alpha : \alpha \in \kappa^+\}$. Then let $\{B_\alpha : \alpha \in \kappa^+\} \subseteq [\kappa]^\kappa$ be an almost disjoint family. By the above argument, we may find $A_\alpha \in [B_\alpha]^\kappa$ such that $A_\alpha$ is not split by any member of $X_\alpha$. Thus, $\{A_\alpha : \alpha \in \kappa^+\}$ has no $\sigma$-Bernstein set.

We can contrast this with the result from [3] that if $S$ is a stationary subset of some $\lambda$ and $\Diamond(S)$ holds, then for any family $\{A_\alpha : \alpha \in S\}$ such that each $A_\alpha \subseteq \omega$ is infinite, there are $B_\alpha \subseteq A_\alpha$ such that the family $\{B_\alpha : \alpha \in S\}$ has no Bernstein set. Moreover, it was shown consistent that there is a strongly almost disjoint family $A \subseteq [\omega_{\omega+1}]^\omega$ with no Bernstein set. However, it is possible that every strongly almost disjoint family admits a $\sigma$-Bernstein set.

**Question 4.** Does $M(\kappa, \lambda, \omega) \rightarrow \sigma - B$ for all $\kappa$ and $\lambda$?  

Given a topological space $X$, a Bernstein set for $X$ is a $B \subseteq X$ that intersects each copy of the Cantor set in $X$ but does not contain any copies of the Cantor set. It was proven by Weiss [11] and Bregman, Šapirovskij and Sostak [1] that it is consistent that every topological space admits a Bernstein set. Whether any extra assumptions are needed to prove this was open until Shelah’s recent construction [9]. It would be of interest to see if one can prove under weaker set-theoretic assumptions (or even ZFC) that every topological space admits a $\sigma$-Bernstein set.

**References**


1 Neil Rogers has shown that $\Diamond(\omega_2)$ implies that there is an almost disjoint family $A \subseteq [\omega_2]^{\aleph_0}$ which does not admit a $\sigma$-Bernstein set [8]. So the answer to Question 4 is consistently no for $\kappa = \omega_2$ and $\lambda = \omega_0$.