PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 7, July 2007, Pages 2177–2185
S 0002-9939(07)08718-7
Article electronically published on March 1, 2007

FORM ESTIMATES FOR THE $p(x)$-LAPLACEAN

W. ALLEGRETTO

(Communicated by David S. Tartakoff)

Abstract. We consider the problem of establishing conditions on $p(x)$ that ensure that the form associated with the $p(x)$-Laplacean is positive bounded below. It was shown recently by Fan, Zhang and Zhao that – unlike the $p =$ constant case – this is not possible if p has a strict extrema in the domain. They also considered the closely related problem of eigenvalue existence and estimates. Our main tool is the adaptation of a technique, employed by Protter for $p = 2$, involving arbitrary vector fields. We also examine related results obtained by a variant of Picone Identity arguments. We directly consider problems in $\Omega \subset R^n$ with $n \geq 1$, and while we focus on Dirichlet boundary conditions we also indicate how our approach can be used in cases of mixed boundary conditions, of unbounded domains and of discontinuous $p(x)$. Our basic criteria involve restrictions on $p(x)$ and its gradient.

1. Introduction

While problems for the p- Laplacean – with constant $p > 1$ – have a long history, there has recently been considerable interest in $p(x)$-growth elliptic equations. A prototypical model is formally given by the expression

$$\mathcal{L}(u) = -\vec{\nabla}\Big[(\alpha + |\vec{\nabla}u|^2)^{\frac{p(x)-2}{2}}\vec{\nabla}u\Big]$$

subject to suitable boundary conditions. If $\alpha = 0$, $\mathcal{L}$ is termed the $p(x)$-Laplacean. These equations arise in such fields as electrorheological fluids, and the interested readers may find results and applications for such problems, as well as properties of the associated $L^{p(x)}$ and $W^{k,p(x)}$ spaces in the articles [1], [2], [5], [7], [8], [9], [10], [12] and the references therein.

In this paper we consider form estimates for the $p(x)$-Laplacean. These questions and their consequent applications to the eigenvalue problems: $-\vec{\nabla}[\,|\vec{\nabla}u|^{p(x)-2}\vec{\nabla}u] = \lambda|u|^{p(x)-2}u$, have been investigated recently by Fan, Zhang and Zhao in [7]. In this reference the authors consider the case of Dirichlet boundary conditions in a domain Ω and show the existence of infinitely many eigenvalues. Despite the obvious formal resemblance of this equation to the p-Laplacean, it is pointed out in [7] that there

Received by the editors December 14, 2005 and, in revised form, March 21, 2006.
2000 *Mathematics Subject Classification.* Primary 35P15; Secondary 35J60, 35J25.
Key words and phrases. $p(x)$-Laplacean, arbitrary vector field, Picone, positive forms.
Research supported by NSERC Canada.

are several striking differences. Observe that if we set Q to be the Rayleigh quotient:

$$Q(\varphi) = \frac{\int_\Omega \frac{|\vec{\nabla}\varphi|^p}{p}}{\int_\Omega \frac{|\varphi|^p}{p}}$$

for any $0 \underset{\neq}{<} \varphi \in C_0^\infty(\Omega)$, then the eigenvalues of the $p(x)$-Laplacean are bounded away from zero precisely when $\mu \overset{\triangle}{=} \inf_{\varphi \neq 0} [Q(\varphi)] > 0$. In [7], the authors in particular show the interesting result that if $1 < p(x)$ has a strict interior maximum or minimum in Ω, then $\mu = 0$, but it is also shown that this is not the case if $p(x)$ has suitable monotonicity properties. In the latter case, positive lower estimates are obtained for μ. The approach used relies on one dimensional estimates and implies that these bounds depend on the values of p and the diameter of Ω. We remark that the arguments in [7] also implicitly show that if $\Omega = (a, b)$ and the boundary conditions are assumed mixed, $\varphi \in C_0^\infty(\Omega \cup \{a\})$ say, then $\mu = 0$ even if p is strictly monotone in Ω : one need only replace the internal calculations by arguments at $x = a$, keeping in mind that φ now need not vanish at $x = a$. As mentioned above, the positive lower bound results for μ obtained in [7] are based, even in higher dimension, on delicate one dimensional considerations applied to a restricted set of test functions φ, in which the minimum of Q must still be found, and suitable decompositions of the interval (a, b). It thus appears difficult to extend this technique to non-Dirichlet higher dimensional problems. This is one of the motivations for this work.

In the present paper we also estimate μ, but the approach is different: we introduce and apply a variant of classical arguments of Protter, [11], originally established for the case $p = 2$. It involves the introduction of an arbitrary vector field $\vec{b}$, which then must be suitably chosen. This approach is related to Picone Identity arguments (see [3], [4], [6] and elsewhere) if p is a constant, since the "best" fields $\vec{b}$ usually involve at least in part the gradient of the eigenfunction associated with the least eigenvalue. But in the present context, where terms involving $\vec{\nabla} p$ arise, the former procedure appears more advantageous. Thus, while we briefly discuss the Picone Identity approach and some of its consequence, we focus our attention on the vector field approach. It is also interesting to note that it will be important for us to take vector fields $\vec{b}$ that only have distributional derivatives.

We consider smooth, possibly unbounded, domains $\Omega \subset R^n$, and the flavour of our results can be seen from the following examples whose proof is given below. The first corollary recovers a result of [7].

Corollary 1. *Let $n = 1$, Ω bounded, Dirichlet conditions. If $p' \geq 0$, then $\mu > 0$. A similar result holds for $p' \leq 0$.*

Corollary 2. *Let $n = 1$, Ω bounded, Dirichlet conditions. If $|\vec{\nabla} p| > 0$ in $\overline{\Omega}$, then μ is positive.*

We observe that μ can be explicitly estimated in both cases from the arguments used.

We also obtain positive μ results for $n > 1$ under mixed boundary conditions. In this case, the form of the domain Ω is important. The paper concludes with some considerations on the cases of Ω unbounded and/or p discontinuous.

In conclusion, we note that our results cover cases which allow $\vec{\nabla} p = 0$ at some points, but only if p can be suitably perturbed to a function ξ with $|\vec{\nabla}\xi| > 0$. Such situations include $p =$ constant or, if $n = 1$, $p' \geq 0$ (resp. $p' \leq 0$).

In keeping with the situation considered in [7], we give our results only for the functional Q mentioned above. But our main considerations apply without change if $Q(\varphi)$ is replaced by more general cases, for example:

$$Q(\varphi) = \frac{\int_\Omega a(x)\, \frac{|\vec{\nabla}\varphi|^p}{p}}{\int_\Omega b(x)\, \frac{|\varphi|^p}{p}}$$

with $a(x)$, $b(x)$ smooth and positive.

Furthermore, we observe that estimates for μ lead directly to estimates on the embedding constant C where $\|\vec{\nabla}\varphi\|_{L^{p(x)}} \geq C\|\varphi\|_{L^{p(x)}}$. Indeed a direct calculation shows

$$C \geq \Big(\frac{\mu \inf\limits_x p(x)}{\sup\limits_x p(x)}\Big)^{1/p^*}$$

where $p^* = \begin{cases} \sup\limits_x p(x) & \text{if} \quad \mu \inf\limits_x p(x) \geq \sup\limits_x p(x), \\ \inf\limits_x p(x) & \text{if} \quad \mu \inf\limits_x p(x) < \inf\limits_x p(x). \end{cases}$

2. Preliminary results

Our criteria for positive μ depend on the following lemmas, which extend to the present situation some of the arguments of [3], [4], [11]:

Lemma 0. *Let $p > 1$ be continuously differentiable and $\vec{b}(x) \in L^\infty(\Omega)$. For any $0 \leq \varphi \in C_0^\infty(\Omega)$ we have*

$$(1)\quad -\vec{\nabla}\cdot\vec{b}\,\Big(\frac{\varphi^p}{p}\Big) - \int_\Omega \frac{\varphi^p}{p}\,\Big\{(p-1)|\vec{b}\,|^{p/p-1} + \vec{\nabla}p\cdot\vec{b}\Big(-\frac{1}{p} + \ell\mathrm{n}\,\varphi\Big)\Big\} \leq \int_\Omega \frac{|\vec{\nabla}\varphi|^p}{p}$$

where: $\vec{\nabla}\cdot\vec{b}$ is understood in the sense of distributions, and the last term in the integrand on the left hand side is taken to vanish wherever $\varphi(x) = 0$. If φ vanishes only on the part $\partial\Omega_D$ of $\partial\Omega$ and $\vec{b}$ is smooth near $\partial\Omega$, then the same estimate holds if $\vec{b}\cdot\vec{n} \geq 0$ on $\partial\Omega - \partial\Omega_D$ where $\vec{n}$ is the unit outward normal.

Proof. We simply note by direct calculation:

$$\int_\Omega \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big)\cdot\vec{b} = \int_\Omega \varphi^{p-1}\vec{\nabla}\varphi\cdot\vec{b} + \int_\Omega \varphi^p\,\Big\{\frac{\ell\mathrm{n}\,\varphi}{p} - \frac{1}{p^2}\Big\}\vec{\nabla}p\cdot\vec{b}\,.$$

We apply Minkowski's Inequality

$$\int_\Omega \varphi^{p-1}\vec{\nabla}\varphi\cdot\vec{b} \leq \int_\Omega \frac{|\vec{\nabla}\varphi|^p}{p} + \int_\Omega \frac{\varphi^p}{p}\,|\vec{b}\,|^{p/p-1}(p-1)$$

and the result follows from the Divergence Theorem. □

It is the choice of $\vec{b}$ that generates the results. However, we observe first that we also have the closely related Picone estimates mentioned above. Specifically:

Lemma 1. *Let $p > 1$ be continuously differentiable and $0 < v$ a smooth function in $\overline{\Omega}$. Then for any $0 \le \varphi \in C_0^\infty(\Omega)$:*

$$\int_\Omega \frac{|\vec{\nabla}\varphi|^p}{p} - \int_\Omega \vec{\nabla}\Big[\frac{\varphi^p}{pv^{p-1}}\Big]|\vec{\nabla}v|^{p-2}\nabla v + \int_\Omega \frac{\varphi^p}{v^{p-1}}\,\frac{|\vec{\nabla}v|^{p-2}}{p}\Big[\ell\mathrm{n}\Big(\frac{\varphi}{v}\Big) - \frac{1}{p}\Big]\vec{\nabla}v\cdot\vec{\nabla}p \ge 0. \tag{2}$$

Proof. (a) Direct calculation gives:

$$\begin{aligned}\frac{|\vec{\nabla}\varphi|^p}{p} - \vec{\nabla}\Big[\frac{\varphi^p}{pv^{p-1}}\Big]|\vec{\nabla}v|^{p-2}\vec{\nabla}v &= \frac{|\vec{\nabla}\varphi|^p}{p} - \Big(\frac{\varphi}{v}\Big)^{p-1}|\vec{\nabla}v|^{p-2}\vec{\nabla}\varphi\vec{\nabla}v \\ &\quad + \frac{1}{p}\Big(\frac{\varphi}{v}\Big)^p(p-1)|\vec{\nabla}v|^p \\ &\quad + \Big(\frac{\varphi}{v}\Big)^{p-1}\frac{\varphi}{p}\,|\vec{\nabla}v|^{p-2}\vec{\nabla}v\cdot\vec{\nabla}p\Big[\frac{1}{p} - \ell\mathrm{n}\Big(\frac{\varphi}{v}\Big)\Big].\end{aligned}$$

Again by Minkowski's Inequality we note

$$\Big(\frac{\varphi}{v}\Big)^{p-1}|\vec{\nabla}v|^{p-1}|\vec{\nabla}\varphi| \le \frac{(p-1)}{p}\Big(\frac{\varphi}{v}\Big)^p|\vec{\nabla}v|^p + \frac{|\vec{\nabla}\varphi|^p}{p}.$$

Hence:

$$\frac{|\vec{\nabla}\varphi|^p}{p} - \vec{\nabla}\Big[\frac{\varphi^p}{pv^{p-1}}\Big]|\vec{\nabla}v|^{p-2}\vec{\nabla}v \ge \Big(\frac{\varphi}{v}\Big)^{p-1}\frac{\varphi}{p}\,|\vec{\nabla}v|^{p-2}\vec{\nabla}v\cdot\vec{\nabla}p\Big[\frac{1}{p} - \ell\mathrm{n}\Big(\frac{\varphi}{v}\Big)\Big]$$

and the result. □

3. Results for Dirichlet conditions

We first note some immediate consequence of Lemmas 0 and 1:

Corollary 3. (a) *If $\vec{\nabla}p\cdot\vec{b} \equiv 0$ (resp.: $\vec{\nabla}v\cdot\vec{\nabla}p \equiv 0$) in Lemma 0 (resp.: in Lemma 1), then the associated terms in inequalities (1) and (2) vanish. In this case, we have from Lemma 1 by the Divergence Theorem:*

$$\mu \ge \inf_{x\in\Omega}\Big[\frac{\mathcal{L}_p(v)}{v^{p-1}}\Big]$$

where $\mathcal{L}_p$ is formally defined by $-\vec{\nabla}\cdot[\,|\vec{\nabla}v|^{p-2}\nabla v]$.

(b) *If $\Omega = \Omega_1 \cup \Omega_2 \cup [(\partial\Omega_1 \cap \partial\Omega_2) \cap \Omega]$ and p is smooth in the subdomains $\overline{\Omega}_1$, $\overline{\Omega}_2$ but has a discontinuity on the smooth surface $\partial\Omega_1 \cap \partial\Omega_2$, then in this case,*

$$\mu \ge \min\Big\{\inf_{x\in\Omega_1}\Big[\frac{\mathcal{L}_p(v_1)}{v_1^p}\Big],\quad \inf_{x\in\Omega_2}\Big[\frac{\mathcal{L}_p(v_2)}{v_2^p}\Big]\Big\}$$

where $0 < v_i$ is smooth in $\overline{\Omega}_i$, $\frac{\partial v_i}{\partial n_i} \ge 0$ on $\partial\Omega_1 \cap \partial\Omega_2$ for $i = 1, 2$, and n_i denotes the outward normal to Ω_i, $\nabla v_i \cdot \nabla p = 0$ in Ω_i.

Once again the proof of (b) follows from Lemma 1 by the Divergence Theorem applied separately to Ω_1 and Ω_2.

We observe that a variant of Corollary 2 – with the obvious changes – also follows from Lemma 0. Corollary 3 shows in particular that some of the earlier $p =$ constant results have direct analogies to the present case if p is smooth and the "test field"

$\vec{b}$ (resp. the "test function" v) satisfy $\vec{b}\cdot\vec{\nabla}p\equiv 0$ (resp.: $\vec{\nabla}v\cdot\vec{\nabla}p\equiv 0$ in Ω). This is quite restrictive, but some results can be obtained easily. As an example for later use we give:

Lemma 2. *Suppose $n=2$, Ω is bounded and simply connected, $-\Delta p=0$, $|\vec{\nabla}p|\neq 0$ in $\overline{\Omega}$. Then $\mu>0$ (and can be estimated).*

Proof. Choose $\omega>0$ by: $\frac{\partial\omega}{\partial x_1}=-\frac{\partial p}{\partial x_2}$, $\frac{\partial\omega}{\partial x_2}=\frac{\partial p}{\partial x_1}$ and $\vec{b}$ by $\vec{b}=-\varepsilon\omega\vec{\nabla}\omega$, for some constant $\varepsilon>0$ chosen below. Obviously $\vec{b}\cdot\vec{\nabla}p=0$ and Lemma 0 yields:

$$\mu\geq\inf_x\Big\{-\vec{\nabla}\cdot\vec{b}-(p-1)|\vec{b}|^{p/p-1}\Big\}.$$

But $-\vec{\nabla}\cdot\vec{b}-(p-1)|\vec{b}|^{p/p-1}=\varepsilon|\vec{\nabla}\omega|^2-(p-1)\varepsilon^{p/p-1}|\omega\vec{\nabla}\omega|^{p/p-1}$. Since $|\vec{\nabla}\omega|=|\vec{\nabla}p|\neq 0$ and $\frac{p}{p-1}>\delta>1$ in Ω, the result follows by choosing ε positive and small. □

A result that depends on the second derivatives of p can also be obtained.

Lemma 3. *Let n, Ω, p, ω be as in Lemma 2. Then:*

$$\mu\geq\sup_{0<\alpha\leq 1}\inf_{(x,y)\in\Omega}\Big\{\frac{|\vec{\nabla}w|^p}{w^p}\,\alpha^{p-1}\Big[(1-\alpha)(p-1)-\frac{w(p-2)}{|\vec{\nabla}w|^2}\,(H\eta,\eta)\Big]\Big\}$$

where $\eta=\vec{\nabla}w/|\vec{\nabla}w|$, $H=$ Hessian matrix of $w=\begin{pmatrix}w_{x_1x_1} & w_{x_1x_2}\\ w_{x_1x_2} & w_{x_2x_2}\end{pmatrix}$.

Proof. Since clearly $\vec{\nabla}w\cdot\vec{\nabla}p=0$ and $-\Delta w=0$, we need only calculate $\frac{\mathcal{L}_p(v)}{v^{p-1}}$ from Corollary 3(a), where we choose $v\triangleq w^\alpha$, for some constant α with $0<\alpha\leq 1$. We have: $\vec{\nabla}v=\alpha w^{\alpha-1}\vec{\nabla}w$, i.e. $|\vec{\nabla}v|^{p-2}\vec{\nabla}v=\alpha^{p-1}w^{(\alpha-1)(p-1)}|\vec{\nabla}w|^{p-2}\vec{\nabla}w$. Hence:

$$\begin{aligned}\mathcal{L}_p(v)&=-\vec{\nabla}[|\vec{\nabla}v|^{p-2}\vec{\nabla}v]\\&=-\alpha^{p-1}|\vec{\nabla}w|^p w^{(\alpha-1)(p-1)-1}\Big[(\alpha-1)(p-1)+w\,\frac{(p-2)}{|\vec{\nabla}w|^2}\Big(\frac{\vec{\nabla}w}{|\vec{\nabla}w|},\,H\,\frac{\vec{\nabla}w}{|\vec{\nabla}w|}\Big)\Big]\end{aligned}$$

and

$$\frac{\mathcal{L}_p(v)}{v^{p-1}}=\frac{|\vec{\nabla}w|^p\alpha^{p-1}}{w^p}\,\Big[(1-\alpha)(p-1)-\frac{w(p-2)}{|\vec{\nabla}w|^2}\,\Big(\frac{\vec{\nabla}w}{|\vec{\nabla}w|}\,,\,H\,\frac{\vec{\nabla}w}{|\vec{\nabla}w|}\Big)\Big]$$

which gives the result. □

Observe that if for example p is near 2 and/or $|H|$ is small, then this gives a meaningful result. Whether it is better than the one obtained in Lemma 2 depends on the properties of p and Ω. Some brief considerations on similar results for $n>2$ will be given at the end of Section 4.

We now obtain a more general result, by choosing $\vec{b}$ with distributional derivatives.

Theorem 1. *Assume there exists a function $\xi\geq 0$ such that: $\vec{\nabla}p\cdot\vec{\nabla}\xi\geq 0$, $|\vec{\nabla}\xi|\neq 0$ in $\overline{\Omega}$. Then $\mu>0$ and can be estimated in terms of p,ξ.*

Proof. Suppose, without loss of generality, that $\varphi > 1$ somewhere in Ω, and set $O_n = \{x | \ell\mathrm{n}\,\varphi(x) > \frac{1}{p} + \varepsilon(n)\}$. Here $0 < \varepsilon(n)$ is a sequence chosen in accordance with Sard's Theorem, such that $\varepsilon(n) \to 0$ and the Divergence Theorem may be applied to O_n and $\Omega - O_n$. Note that $O_n \subset\subset \Omega$. Choose $\vec{b} = \eta\beta\vec{\nabla}\xi$ with:

$$\beta = \begin{cases} h(x) - k(x), & x \in O_n, \\ h(x), & x \notin O_n, \end{cases}$$

where $k \triangleq e^{M\xi}$, $h \triangleq e^{-M\xi}$ and M is a fixed constant such that $M|\vec{\nabla}\xi|^2 > |\Delta\xi|$, while η is a positive constant. We now estimate the various terms in inequality (1). Since $k \geq h$ by construction, we have

$$\begin{aligned}
\int_\Omega \frac{\varphi^p}{p}\Big(\ell\mathrm{n}\,\varphi - \frac{1}{p}\Big)\vec{\nabla}p \cdot \vec{b} &= \eta \int_\Omega \frac{\varphi^p}{p}\Big(\ell\mathrm{n}\,\varphi - \frac{1}{p}\Big)\beta\vec{\nabla}p \cdot \vec{\nabla}\xi \\
&= \eta\Big[\int_{O_n} \frac{\varphi^p}{p}\Big(\ell\mathrm{n}\,\varphi - \frac{1}{p}\Big)\beta\vec{\nabla}p \cdot \vec{\nabla}\xi \\
&\quad + \int_{\Omega - O_n} \frac{\varphi^p}{p}\Big(\ell\mathrm{n}\,\varphi - \frac{1}{p}\Big)\beta\vec{\nabla}p \cdot \vec{\nabla}\xi\Big] \\
&\leq \eta \int_{\Omega - O_n} \frac{\varphi^p}{p}\Big(\ell\mathrm{n}\,\varphi - \frac{1}{p}\Big)\beta\vec{\nabla}p \cdot \vec{\nabla}\xi \\
&\leq \eta \int_{\Omega - O_n} \frac{\varphi^p}{p}\big(\varepsilon(n)\big)h\vec{\nabla}p \cdot \vec{\nabla}\xi \\
&\leq \eta\varepsilon(n) \int_\Omega \frac{\varphi^p}{p}\, h\vec{\nabla}p \cdot \vec{\nabla}\xi \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_\Omega \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big) \cdot \vec{b} &= \eta \int_\Omega \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big) \cdot \beta\vec{\nabla}\xi \\
&= \eta \int_\Omega \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big) \cdot h\vec{\nabla}\xi - \eta \int_{O_n} \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big) \cdot k\vec{\nabla}\xi \\
&= \eta \int_\Omega \Big(\frac{\varphi^p}{p}\Big)[-\vec{\nabla}(h\vec{\nabla}\xi)] - \eta \int_{\partial O_n} \frac{\varphi^p}{p}\, k\vec{\nabla}\xi \cdot \vec{n} \\
&\quad + \eta \int_{O_n} \frac{\varphi^p}{p}\, \vec{\nabla}[k\vec{\nabla}\xi] \\
&= \eta \int_\Omega \frac{\varphi^p}{p}\,[-\vec{\nabla}(h\vec{\nabla}\xi)] - \eta \int_{\partial O_n} \frac{e^{\frac{1}{p}+\varepsilon(n)}}{p}\, k\vec{\nabla}\xi \cdot n \\
&\quad + \eta \int_{O_n} \frac{\varphi^p}{p}\, \vec{\nabla}(k\vec{\nabla}\xi) \\
&= \eta \int_\Omega \frac{\varphi^p}{p}\,[-\vec{\nabla}(h\vec{\nabla}\xi)] + \eta \int_{O_n} \Big\{\frac{\varphi^p}{p}\,\vec{\nabla}(k\nabla\xi) - \eta\nabla\Big[\frac{e^{1+\varepsilon(n)p}}{p}\, k\vec{\nabla}\xi\Big]\Big\} \\
&= \eta \int_\Omega \frac{\varphi^p}{p}[-\vec{\nabla}(h\vec{\nabla}\xi)] + \eta \int_{O_n} \Big[\frac{\varphi^p}{p} - \frac{e^{[1+\varepsilon(n)]p}}{p}\Big]\vec{\nabla}[k\vec{\nabla}\xi] \\
&\quad - \eta \int_{O_n} k\vec{\nabla}\xi \cdot \vec{\nabla}\Big[\frac{e^{1+\varepsilon(n)p}}{p}\Big].
\end{aligned}$$

But

$$-\int_{O_n} k\vec{\nabla}\xi\cdot\vec{\nabla}\Big[\frac{e^{1+\varepsilon(n)p}}{p}\Big] = -\int_{O_n}\frac{k\vec{\nabla}\xi}{p^2}\cdot[e^{1+\varepsilon(n)p}\varepsilon(n)p\vec{\nabla}p - e^{1+\varepsilon(n)p}\vec{\nabla}p].$$

The first term vanishes as $n\to 0$, while the second term is positive by definition of ξ. Finally, we note that in O_n we have $\varphi^p \geq e^{(1+\varepsilon(n))p}$ and $k = e^{M\xi}$ implies $\vec{\nabla}(k\vec{\nabla}\xi)\geq 0$.

We conclude:

$$\begin{aligned}\int_\Omega \vec{\nabla}\Big(\frac{\varphi^p}{p}\Big)\vec{b} &\geq \eta\int_\Omega \frac{\varphi^p}{p}\,[-\vec{\nabla}(h\vec{\nabla}\xi)\,]\\ &\geq \int_\Omega \eta\,\frac{\varphi^p}{p}\,h[M|\vec{\nabla}\xi|^2 - |\Delta\xi|].\end{aligned}$$

Inequality (1) thus implies:

$$\mu \geq \sup_{\eta\geq 0}\Big\{\eta h[M|\nabla\xi|^2 - |\Delta\xi|\,] - (p-1)\eta^{\frac{p}{p-1}}\,|\beta|^{\frac{p}{p-1}}\,|\nabla\xi|^{\frac{p}{p-1}}\Big\}.$$

Since $p/(p-1)>1$, we note once again that this gives a positive lower bound for μ by choosing $0<\eta$ small enough. □

Proof of Corollary 1. If $p'\geq 0$, we simply choose $\xi = A + x$, for some convenient A. If $p'\leq 0$, the choice is $\xi = A - x$. □

Proof of Corollary 2. In this case we merely choose $\xi = p$. □

4. Results for non-Dirichlet problems

Assume now that $\varphi\in C_0^\infty(\Omega\cup\partial\Omega_N)$ where $\partial\Omega_N$ is (a part of) $\partial\Omega$. The proof of Theorem 1 fails, since we no longer can conclude – in the notation of Theorem 1 – that $\varphi = e^{1/p+\varepsilon(n)}$ on ∂O_n. Nevertheless, as mentioned above, some results can still be obtained if we assume that $\vec{b}\cdot\vec{n}\geq 0$ on $\partial\Omega_N$ and $\vec{b}\cdot\vec{\nabla}p = 0$ and restrict the domain. We recall that if $n=1$ we have $\mu = 0$ in this case even if p is monotone, so it appears natural to formulate these problems for $n>1$. We illustrate these comments by establishing a result based on Lemma 1 and Corollary 2.

Theorem 2. *Let $n=2$ and Ω be a simply connected domain. Put $\partial\Omega = \partial\Omega_D\cup\partial\Omega_N$ with $\partial\Omega_D\cap\partial\Omega_N = \Phi$ and $\partial\Omega_D$ closed in $\partial\Omega$. Using the notation of Lemma 2, suppose further and $\Delta p = 0$, $|\vec{\nabla}p|\neq 0$ in $\overline{\Omega}$, $\frac{\partial w}{\partial n}$ does not change sign on $\partial\Omega_N$. Then $\mu>0$ if Q is now calculated for the wider class $0\leq\varphi\in C_0^\infty\ (\Omega\cup\partial\Omega_N)$.*

Proof. If $\frac{\partial w}{\partial n}\leq 0$, the proof of Lemmas 0 and 2 still hold and we choose $\vec{b}$ accordingly. Otherwise, replace w by $A-w$ for some large constant A. □

Observe that the condition that Ω be simply connected cannot be removed from Theorem 2. If, for example, $p = p(r)$ monotone, $\Omega = \{x|\,0<a<|x|<b\}$ for some constants a,b with $\partial\Omega_D = \{x\big|\,|x| = a\}$, and $\partial\Omega_N = \{x\big|\,|x| = b\}$, then the one dimensional results applied to the radial functions show that $\mu = 0$ in this case. We also note that the condition $\frac{\partial w}{\partial n}\neq 0$ on $\partial\Omega_N$ is satisfied if $p\equiv$ constant on $\partial\Omega_N$.

We comment briefly that such results also hold for $n>2$, although the case $n=2$ is the simplest. Suppose for example that $n=3$ and $\Delta p = 0$. We apply

Helmoltz's Theorem and conclude that $\vec{\nabla} p$ has a vector potential $\vec{A} : \vec{\nabla} p = \vec{\nabla} \times \vec{A}$. Now we choose $\vec{b} = p(\vec{\nabla} p \times \vec{A})$, whence

$$(*) \qquad -\vec{\nabla} \cdot \vec{b} = p\vec{\nabla} p \cdot (\vec{\nabla} \times \vec{A}) = p|\vec{\nabla} p|^2 > 0$$

if $|\vec{\nabla} p| \neq 0$. If $\varphi \in C_0^\infty(\Omega)$ we immediately have $\mu > 0$ by the earlier scaling argument. If $\varphi \in C_0^\infty(\Omega \cup \partial\Omega_N)$, then we also require $(\vec{\nabla} p \times \vec{A}) \cdot \vec{n} \geq 0$, i.e.: $[(\vec{\nabla} \times \vec{A}) \times \vec{A}] \cdot \vec{n} \geq 0$ on $\partial\Omega_N$ in order to conclude $\mu > 0$.

5. Results for p discontinuous and unbounded domains

The previous results can be used to obtain $\mu > 0$ criteria in these cases, where if Ω is unbounded we may need to replace the constant μ by a function.

As an example we state:

Corollary 4. *Assume $1 < p < 2$ and for some constants N, M we have*

$$N \geq (M-1)|\vec{\nabla} p|^2 - |\Delta p| > 0$$

in $\overline{\Omega}$, an unbounded domain in R^n. Then for a (calculable) constant $C = C(p)$ we have

$$\int_\Omega \frac{|\vec{\nabla}\varphi|^p}{p} \geq C(p) \int_\Omega |\nabla p|^2 \, \frac{|\varphi|^p}{p} \, .$$

Proof. We apply the Proof of Theorem 1, noting that p, $|\vec{\nabla} p|$ are bounded and $p/(p-1) > 2$. The result follows again by choosing $0 < \eta$ small enough. □

A simple case where Corollary 4 applies is given by $p = 1 + r^{1-n}$ with $n \geq 3$ and $\Omega \subset \{x \big| \, |x| > 1\}$.

We conclude with a brief discussion of applications of Corollary 3 and the constant p results to the situation of p piecewise constant. It is our intention to obtain an explicit μ estimate for this situation as well as give another result for Ω unbounded. Suppose, to be specific, that $p_2 < n < p_1$ with: $p = p_1$ in Ω_1 and $p = p_2$ in Ω_2 for positive constants p_1, p_2. Here $\Omega, \Omega_1, \Omega_2$ are as given in Corollary 3(b). Suppose $\Omega \subset \{x \big| 0 < a < |x| < b\}$ and $\Omega_1 = \Omega \cap \{x \big| \, |x| < c\} \neq \emptyset$, $\Omega_2 = \Omega \cap \{x \big| \, |x| > c\} \neq \emptyset$ for some c with $a < c < b$. Then for any v_i, $i = 1, 2$, we have $\vec{\nabla} v_i \cdot \vec{\nabla} p = 0$ in Ω_i and, choosing $v_i = v_i(|x|)$, we need only ensure that $\frac{\partial v_1}{\partial r} \geq 0$ and $\frac{\partial v_2}{\partial r} \leq 0$ on $|x| = c$. For v_2 we can make the standard choice: $v_2 = r^\alpha$, $\alpha < 0$, and recall the consequential constant p estimate, [3], [4],

$$\int_{\Omega_2} |\vec{\nabla}\varphi|^p \geq \int_{\Omega_2} \mu_2(|x|)|\varphi|^p$$

valid for $\varphi = 0$ just at infinity but not necessarily on the "inner boundary" $|x| = c$, where

$$\mu_2(|x|) = \Big(\frac{n - p_2}{p_2}\Big)^{p_2} \frac{1}{|x|^{p_2}} \, .$$

For future use, we note that this estimate is independent of b, but conclude in the present case:

$$\inf_{x \in \Omega_2} \Big[\frac{\mathcal{L}_p(v_2)}{v_2^{p-1}}\Big] \geq \mu_2(b).$$

Next, we construct $v_1 = v_1(|x|)$ with $\frac{\partial v_1}{\partial r} \geq 0$ on $|x| = c$ by choosing the function

$$v_1 = |x|^\alpha$$

with $\alpha = \frac{(p_1 - n)}{p_1}$, and find that

$$\inf_{x \in \Omega_1} \left[\frac{\mathcal{L}_p(v_1)}{v_1^{p-1}} \right] \geq \mu_1(c)$$

where

$$\mu_1(c) = \frac{(p_1 - n)^{p_1}}{p_1^{p_1}} \, \frac{1}{c^{p_1}} \, .$$

Combining the two estimates we obtain:

$$\mu \geq \frac{p_2}{p_1} \min \left[\mu_2(b), \mu_1(c) \right].$$

Clearly the same results also show that in the exterior domain case $\Omega \subset \{x \big| a < |x|\}$, $\Omega_1 = \Omega \cap \{x \big| \, |x| < c\} \neq \emptyset$, $\Omega_2 = \Omega \cap \{x \big| \, |x| > c\} \neq \emptyset$; then

$$\int_\Omega \frac{|\vec{\nabla} \varphi|^p}{p} \geq \frac{p_2}{p_1} \int_\Omega w(|x|) \, \frac{\varphi^p}{p}$$

for any $\varphi \in C_0^\infty(\Omega)$, where:

$$w(|x|) = \begin{cases} \mu_1(c), & a < |x| \leq c, \\ \\ \mu_2(|x|), & |x| > c. \end{cases}$$

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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

E-mail address: wallegre@math.ualberta.ca