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ON THE POLES OF THE RESOLVENT IN CALKIN ALGEBRA

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Abstract. In the present note, we study the problem of lifting poles in Calkin algebra on a separable infinite-dimensional complex Hilbert space H. We show by an example that such lifting is not possible in general, and we prove that if zero is a pole of the resolvent of the image of an operator T in the Calkin algebra, then there exists a compact operator K for which zero is a pole of $T+K$ if and only if the index of $T-\lambda$ is zero on a punctured neighbourhood of zero. Further, a useful characterization of poles in Calkin algebra in terms of essential ascent and descent is provided.

1. Introduction

Throughout this note, $\mathcal{L}(H)$ will denote the algebra of all bounded operators on a separable infinite-dimensional complex Hilbert space H and $\mathcal{K}(H)$ its ideal of compact operators. For $T \in \mathcal{L}(H)$, write T^* for its adjoint, $\ker(T)$ for its kernel and $\operatorname{ran}(T)$ for its range.

Let A be a unital complex Banach algebra. The spectrum of an element $x \in A$ is denoted by $\sigma(x, A)$, or simply $\sigma(x)$ when no confusion is possible. It is a classical fact that zero is an isolated point of $\sigma(x)$ if and only if there exists an idempotent e that commutes with x and for which $\sigma(exe; eAe) = \{0\}$ and

$$\sigma((1-e)x(1-e); (1-e)A(1-e)) = \sigma(x) \setminus \{0\};$$

see for instance [4].

For a bounded linear operator $T \in \mathcal{L}(H)$, the *ascent*, $\mathrm{a}(T)$, and the *descent*, $\mathrm{d}(T)$, are defined by $\mathrm{a}(T) = \inf\{n \geq 0 : \ker(T^n) = \ker(T^{n+1})\}$ and $\mathrm{d}(T) = \inf\{n \geq 0 : \operatorname{ran}(T^n) = \operatorname{ran}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be ∞. In the case of a Banach algebra A, the ascent and the descent of an element x are defined to be respectively the ascent and the descent of the corresponding left multiplication operator L_x given by $L_x(y) = xy$. It is well known that x is of finite ascent and descent if and only if zero is a pole of the resolvent of x, and in this case, the order of the pole is $d := \mathrm{a}(x) = \mathrm{d}(x)$; see[12, Theorem 2.1].

From [7] we recall that an operator $T \in \mathcal{L}(H)$ is said to be *Fredholm* if $\dim\ker(T)$ and $\operatorname{codim}\operatorname{ran}(T)$ are both finite, or equivalently $\pi(T)$ is invertible in the Calkin algebra $\mathcal{C}(H) := \mathcal{L}(H)/\mathcal{K}(H)$ where $\pi : \mathcal{L}(H) \to \mathcal{C}(H)$ is the quotient map. Notice that the range of such operators is closed; see [7].

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In [14], Olsen proved that if T is such that $p(\pi(T)) = 0$, where p is a non-zero complex polynomial, then there exists a compact operator K satisfying $p(T+K) = 0$. The problem of lifting an element in a class of $\mathcal{C}(H)$ to an element in the same class of $\mathcal{L}(H)$ has interested many mathematicians ([14], [2], [1], [8]), and several results are known in this direction. For instance, we mention that for the class of idempotents, it is established that $\pi(E(\mathcal{L}(X))) = E(\mathcal{C}(X))$ where X is a Banach space and $E(\mathcal{L}(X))$ (resp. $E(\mathcal{C}(X))$) denotes the set of idempotent elements of $\mathcal{L}(X)$ (resp. $\mathcal{C}(X)$), [2]. In this note we are motivated by the following question: Let T be an operator such that zero is a pole of the resolvent of $\pi(T)$. Is there a compact operator K for which zero is a pole of the resolvent of $T+K$? As will be showed by an example, the answer to this question is generally negative. However, we prove that for such an operator T, there exists a compact operator K such that $T+K = A \oplus B$ where A is nilpotent and B is Fredholm. Furthermore, we give a necessary and sufficient condition such that $K = 0$.

2. Poles in Calkin algebra

As mentioned above, the following example shows that lifting of poles in the Calkin algebra is not possible in general.

Example 2.1. Let S be the unilateral left shift operator given on the Hilbert space $\ell^2(\mathbb{N})$ by $Se_{i+1} = e_i$ and $Se_1 = 0$ where $\{e_i : i \geq 1\}$ denotes the canonical basis. Consider the operator $T = 0 \oplus S$ defined on $H = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then it is clear that $T - \lambda$ is Fredholm and $\operatorname{ind}(T-\lambda) = 1$ whenever λ is in the punctured open unit disk ($0 <| \lambda |< 1$). Also, because T has finite descent then so does $\pi(T)$; see [5]. Consequently, zero is a pole of the resolvent of $\pi(T)$. However, if there exists a compact operator K such that zero is a pole of $T+K$, then $\operatorname{ind}(T+K-\lambda) = \operatorname{ind}(T-\lambda) = 0$ for $0 <| \lambda |< 1$, a contradiction.

Theorem 2.2. *Let T be a bounded operator on H. Then zero is a pole of the resolvent of $\pi(T)$ if and only if there exists a compact operator K such that $T+K = A \oplus B$ where A is nilpotent and B is Fredholm.*

Proof. Suppose that zero is a pole of the resolvent of $\tilde{T} = \pi(T)$ of order d. Then there exists an idempotent $\tilde{R}$ such that $\tilde{R}\tilde{T} = \tilde{T}\tilde{R}$, $\sigma(\tilde{R}\tilde{T}\tilde{R}, \tilde{R}\mathcal{C}(H)\tilde{R}) = \{0\}$ and $\sigma((1-\tilde{R})\tilde{T}(1-\tilde{R}), (1-\tilde{R})\mathcal{C}(H)(1-\tilde{R})) = \sigma(\tilde{T}) \setminus \{0\}$, [4]. Let λ be a non-zero complex number; then $\tilde{R}\tilde{T}\tilde{R} - \lambda\tilde{R}$ is invertible in $\tilde{R}\mathcal{C}(H)\tilde{R}$. Moreover, by Corollary 3.3 of [2] we can lift $\tilde{R}$ to an idempotent R in H, hence there exists $S \in \mathcal{L}(H)$ such that $[R(T-\lambda)R][RSR] - R$ and $[RSR][R(T-\lambda)R] - R$ are compacts. Therefore it follows easily that $RTR_{|\operatorname{ran}(R)} - \lambda$ is Fredholm and so $\sigma_{\mathrm{e}}(RTR_{|\operatorname{ran}(R)}) = \{0\}$. In the same way we get that $(I-R)T(I-R)_{|\ker(R)}$ is Fredholm. On the other hand, because $\tilde{T}$ has finite descent, we obtain the existence of $U \in \mathcal{L}(H)$ for which $\tilde{T}^d = \tilde{T}^{d+1}\tilde{U}$ where $\tilde{U} = \pi(U)$. Therefore,

$$(\tilde{R}\tilde{T}\tilde{R})^d = \tilde{R}\tilde{T}^d\tilde{R} = \tilde{R}\tilde{T}^{d+1}\tilde{U}\tilde{R} = [\tilde{R}\tilde{T}^{d+1}\tilde{R}][\tilde{R}\tilde{U}\tilde{R}] = (\tilde{R}\tilde{T}\tilde{R})^{d+1}(\tilde{R}\tilde{U}\tilde{R}).$$

Consequently $(RTR)^d - (RTR)^{d+1}(RUR)$ is compact and hence so is its restriction to the invariant subspace $\operatorname{ran}(R)$. This implies that $\pi_{\operatorname{ran}(R)}(RTR_{|\operatorname{ran}(R)})$ has finite descent where $\pi_{\operatorname{ran}(R)} : \mathcal{L}(\operatorname{ran}(R)) \to \mathcal{C}(\operatorname{ran}(R))$ denotes the quotient map, and since it is quasi-nilpotent in the Calkin algebra $\mathcal{C}(\operatorname{ran}(R))$, we obtain by [5, Theorem 2.4] that $\pi_{\operatorname{ran}(R)}(RTR_{|\operatorname{ran}(R)})$ is nilpotent. Hence, [14, Theorem 2.4] ensures the existence of a compact operator K_1 on $\operatorname{ran}(R)$ such that $RTR_{|\operatorname{ran}(R)} + K_1$ is

nilpotent. Now if we put $K = RK_1R - RT(I-R) - (I-R)TR$, then it is easy to verify that K is compact. Finally $T + K = A \oplus B$ where $A = (RTR)_{\mathrm{ran}(R)} + K_1$ is nilpotent and $B = (I-R)T(I-R)_{|\ker(R)}$ is Fredholm.

Conversely, suppose that there exists a compact operator K that satisfies $T+K = A \oplus B$ where $A^n = 0$ for some positive integer n and B is Fredholm. Then it follows that zero is an isolated point of $\sigma(\pi(T))$, and if we let P be the projection such that $P(T+K)P = A$ and $(I-P)(T+K)(I-P) = B$, there exists an operator $S \in \mathcal{L}(H)$ such that $(I-P) - B(I-P)S(I-P)$ is compact. Hence, $T^n - T^{n+1}(I-P)S(I-P)$ is compact and so $\mathrm{d}(\pi(T))$ is finite, which implies that zero is a pole of the resolvent of $\pi(T)$ of order $d \leq n$. □

In the sequel, we shall denote by $\mathrm{G_o}$ the connected component of the Calkin algebra $\mathcal{C}(H)$ formed by invertible elements and which contains the identity. It is well known that for every $\pi(T) \in \mathrm{G_o}$ there exists a compact operator K such that $T + K$ is invertible, [9].

Remark. As we have showed in Example 2.1, the lifting of poles in the Calkin algebra is generally not possible. Furthermore, if such a lifting is possible for some pole of $\pi(T)$, then the index of $T - \lambda$ is zero in a punctured neighbourhood of the pole. Hence non-zero index constitutes an obstruction for lifting.

Corollary 2.3. *Let T be a bounded operator on H. If $0 \in \sigma_{\mathrm{e}}(T)$ is a pole of the resolvent of $\pi(T)$, then the following assertions are equivalent:*

(i) *there exists a compact operator $K \in \mathcal{L}(H)$ such that zero is a pole of $T + K$,*

(ii) *there exists a $\delta > 0$ such that $\pi(T) - \lambda \in \mathrm{G_o}$ for $0 <| \lambda |< \delta$,*

(iii) *there exists a $\delta > 0$ such that $T - \lambda$ is Fredholm of index zero for $0 <| \lambda |< \delta$.*

Proof. (i) $\Rightarrow$ (ii). Because zero is a pole of the resolvent of $T+K$, there exists $\delta > 0$ such that for $0 <| \lambda |< \delta$, $T + K - \lambda$ is invertible and consequently $\mathrm{ind}(T - \lambda) = \mathrm{ind}(T + K - \lambda) = 0$.

(ii) $\Rightarrow$ (i). Since zero is a pole of $\pi(T)$, Theorem 2.2 ensures the existence of a compact operator K_1 such that $T + K_1 = A \oplus B$ where A is nilpotent and B is Fredholm. Furthermore, from the fact that $\mathrm{ind}(T - \lambda) = 0$ for $0 <| \lambda |< \delta$ and the continuity of the index, we get that $\mathrm{ind}(B) = 0$. Hence $B + K_2$ is invertible for some compact operator K_2. Now it suffices to put $K = K_1 + (0 \oplus K_2)$.

For (iii) $\Leftrightarrow$ (ii) see [9]. □

3. Poles in Calkin algebra and essential ascent and descent

Before outlining the statement of our results, we need to introduce the following notions.

Following S. Grabiner [10], we associate to every operator T the following two sequences $c_n(T) := \dim \ker(T^{n+1})/\ker(T^n)$ and $c'_n(T) := \dim \mathrm{ran}(T^n)/\mathrm{ran}(T^{n+1})$. The *essential ascent* and the *essential descent* are then defined by

$$\mathrm{a_e}(T) = \inf\{n \geq 0\colon c_n(T) \text{ is finite }\} \quad \text{and} \quad \mathrm{d_e}(T) = \inf\{n \geq 0\colon c'_n(T) \text{ is finite }\}.$$

We mention the following characterizations due to S. Grabiner and J. Zemánek: [11]

$$\mathrm{a_e}(T) \text{ is finite } \Leftrightarrow \dim(\mathrm{ran}(T^d) \cap \ker(T)) < \infty \text{ for some } d \geq 0 \tag{3.1}$$

and

$$\mathrm{d_e}(T) \text{ is finite } \Leftrightarrow \mathrm{codim}\,(\mathrm{ran}(T) + \ker(T^d)) < \infty \text{ for some } d \geq 0. \tag{3.2}$$

Theorem 3.1. *Let T be a bounded operator on H. Then zero is a pole of the resolvent of $\pi(T)$ if and only if there exists a compact operator K such that $\mathrm{a_e}(T+K)$ and $\mathrm{d_e}(T+K)$ are both finite.*

Proof. The direct implication follows immediately from Theorem 2.2. For the converse, assume that $\mathrm{a_e}(T+K)$ and $\mathrm{d_e}(T+K)$ are finite; then [13, Chapter III §22, Theorem 12] provide the existence of two closed subspaces M, N invariant by $T+K$ and such that $H = M \oplus N$ and $T+K = T_1 \oplus T_2$ where T_1 nilpotent and T_2 Fredholm, as desired. □

Remark. In the preceding result, the compact operator K cannot be chosen to be zero. Indeed, consider the compact operator K defined on the Hilbert space $\ell^2(\mathbb{N})$ by $Ke_n = \frac{1}{n}e_n$; then zero is obviously a pole of the resolvent of $\pi(K) = 0$. However, if K has finite essential ascent and descent, then Theorem 5.3 of [11] will imply that $\mathrm{ran}(K^d)$ is closed for $d = \mathrm{a_e}(K) = \mathrm{d_e}(K)$, a contradiction.

Before stating our next result, we have to recall the following notion introduced by B. N. Sadovskii in [15]; see also [13, Chapter III §17]. Let $\ell^\infty(H)$ denote the Banach space of all bounded sequences of H equipped by the sup norm and $m(H)$ its closed subspace consisting of all the sequences $\{x_n\}_n$ such that $\{x_n : n \in \mathbb{N}\}$ is totally bounded or equivalently has compact closure in H. To an operator $T \in \mathcal{L}(H)$ we associate the operator $\mathrm{P}(T)$ defined on the Banach space $\mathrm{P}(H) = \ell^\infty(H)/m(H)$ by $\mathrm{P}(T)(\{x_n\}_n + m(H)) = \{Tx_n\}_n + m(H)$. It is well known that for an operator $T \in \mathcal{L}(H)$, $\mathrm{P}(T) = 0$ if and only if T is compact, and if $\mathrm{ran}(T)$ is closed, then $\ker(\mathrm{P}(T)) = \ell^\infty(\ker(T)) + m(H)$.

Theorem 3.2. *If $T \in \mathcal{L}(H)$ is an operator such that zero is a pole of the resolvent of $\pi(T)$ of order d and $\mathrm{ran}(T^d)$ and $\mathrm{ran}(T^{d+1})$ are closed, then $d = \mathrm{a_e}(T) = \mathrm{d_e}(T)$.*

Proof. Let Q be the orthogonal projection on $\ker(T^{d+1})$. Then we have

$$\pi(T^{d+1})\pi(Q) = 0,$$

and since $d = \mathrm{a}(\pi(T))$, we obtain that $\pi(T^d)\pi(Q) = 0$. On the other hand, since $\mathrm{ran}(T^d)$ is closed then so is $\mathrm{ran}(T^dQ) = \mathrm{ran}(T^d) \cap \ker(T)$, and consequently $\mathrm{ran}(T^d) \cap \ker(T)$ is of finite dimension. Thus, using (3.1), we obtain that $\mathrm{a_e}(T) \leq d$ because $\ker(T^{d+1})/\ker(T^d) \cong \mathrm{ran}(T^d) \cap \ker(T)$. Moreover, we get by duality that $\dim \mathrm{ran}(T^{*d}) \cap \ker(T^*)$ is finite and hence so is $\mathrm{codim}\,(\ker(T^d) + \mathrm{ran}(T))$. Finally, by (3.2), it follows that $\mathrm{d_e}(T)$ is finite. Now suppose that $\mathrm{a_e}(T) = \mathrm{d_e}(T) < d$; then $\dim \ker(T^d)/\ker(T^{d-1})$ is finite. Hence by [13, Chapter III §17, Lemma 2], we have $\ell^\infty(\ker(T^d)) + m(H) = \ell^\infty(\ker(T^{d-1})) + m(H)$, and since $\mathrm{ran}(T^d)$ is closed, we obtain that $\ker(\mathrm{P}(T^d)) \subseteq \ker(\mathrm{P}(T^{d-1}))$. Thus $\ker(\mathrm{P}(T^d)) = \ker(\mathrm{P}(T^{d-1}))$. Finally, if S is an operator such that T^dS is compact, then so is $T^{d-1}S$. This shows that $\mathrm{a}(\pi(T)) < d$, the desired contradiction. □

Notice that the inverse implication of Theorem 3.2 does not hold in general. In fact, if we consider the compact operator K given on the Hilbert space $\ell^2(\mathbb{N})$ by $Ke_{2p} = \frac{1}{p}e_{2p-1}$ and $Ke_{2p-1} = 0$, then $\mathrm{a_e}(K) = \mathrm{d_e}(K) = 2$ while zero is a pole of the resolvent of $\pi(K) = 0$ of order 1. However, if we suppose that $\mathrm{ran}(T^{d-1})$ is closed, we obtain the following result:

Proposition 3.3. *Let T be a bounded operator on H. If $d = \mathrm{a_e}(T) = \mathrm{d_e}(T)$ and $\mathrm{ran}(T^{d-1})$ is closed, then zero is a pole of the resolvent of $\pi(T)$ of order d.*

Proof. Because Theorem 3.1 ensures that zero is a pole of the resolvent of $\pi(T)$, then it suffices to establish that $d = \mathrm{a}(\pi(T))$. First suppose that $n := \mathrm{a}(\pi(T)) < d$, and let Q be the orthogonal projection on $\ker(T^d)$. Then $T^{d-1}Q$ is compact and so $\mathrm{P}(T^{d-1}Q) = \mathrm{P}(T^{d-1})\mathrm{P}(Q) = 0$. We claim that $\ell^\infty(\ker(T^d)) + m(H) = \ell^\infty(\ker(T^{d-1})) + m(H)$. Let $\{x_n\}_n \in \ell^\infty(\ker(T^d))$; then we have

$$\mathrm{P}(T^{d-1})(\{x_n\}_n + m(H)) = \mathrm{P}(T^{d-1})\mathrm{P}(Q)(\{x_n\}_n + m(H)) = 0,$$

and hence $\{x_n\}_n + m(H) \in \ker(\mathrm{P}(T^{d-1})) = \ell^\infty(\ker(T^{d-1})) + m(H)$ because $\mathrm{ran}(T^{d-1})$ is closed. Now, using [13, Chapter III §17, Lemma 2], we get that $\dim \ker(T^d)/\ker(T^{d-1})$ is finite, a contradiction. Therefore $n \geq d$, and consequently, Theorem 5.3 of [11] implies that $\mathrm{ran}(T^n)$ and $\mathrm{ran}(T^{n+1})$ are closed. Finally, $n = d$ by Theorem 3.2. □

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