ON THE POLES OF THE RESOLVENT IN CALKIN ALGEBRA

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Abstract. In the present note, we study the problem of lifting poles in Calkin algebra on a separable infinite-dimensional complex Hilbert space $H$. We show by an example that such lifting is not possible in general, and we prove that if zero is a pole of the resolvent of the image of an operator $T$ in the Calkin algebra, then there exists a compact operator $K$ for which zero is a pole of $T + K$ if and only if the index of $T - \lambda$ is zero on a punctured neighbourhood of zero. Further, a useful characterization of poles in Calkin algebra in terms of essential ascent and descent is provided.

1. Introduction

Throughout this note, $\mathcal{L}(H)$ will denote the algebra of all bounded operators on a separable infinite-dimensional complex Hilbert space $H$ and $\mathcal{K}(H)$ its ideal of compact operators. For $T \in \mathcal{L}(H)$, write $T^*$ for its adjoint, ker$(T)$ for its kernel and ran$(T)$ for its range.

Let $A$ be a unital complex Banach algebra. The spectrum of an element $x \in A$ is denoted by $\sigma(x, A)$, or simply $\sigma(x)$ when no confusion is possible. It is a classical fact that zero is an isolated point of $\sigma(x)$ if and only if there exists an idempotent $e$ that commutes with $x$ and for which $\sigma(exe; eAe) = \{0\}$ and $\sigma((1 - e)x(1 - e); (1 - e)A(1 - e)) = \sigma(x) \setminus \{0\}$; see for instance [4].

For a bounded linear operator $T \in \mathcal{L}(H)$, the ascent, $a(T)$, and the descent, $d(T)$, are defined by $a(T) = \inf\{n \geq 0 : \ker(T^n) = \ker(T^{n+1})\}$ and $d(T) = \inf\{n \geq 0 : \ran(T^n) = \ran(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be $\infty$. In the case of a Banach algebra $A$, the ascent and the descent of an element $x$ are defined to be respectively the ascent and the descent of the corresponding left multiplication operator $L_x$ given by $L_x(y) = xy$. It is well known that $x$ is of finite ascent and descent if and only if zero is a pole of the resolvent of $x$, and in this case, the order of the pole is $d := a(x) = d(x)$; see [12, Theorem 2.1].

From [7] we recall that an operator $T \in \mathcal{L}(H)$ is said to be Fredholm if $\dim \ker(T)$ and $\dim \ran(T)$ are both finite, or equivalently $\pi(T)$ is invertible in the Calkin algebra $\mathcal{C}(H) := \mathcal{L}(H)/\mathcal{K}(H)$ where $\pi : \mathcal{L}(H) \to \mathcal{C}(H)$ is the quotient map. Notice that the range of such operators is closed; see [7].
In [14], Olsen proved that if \( T \) is such that \( p(\pi(T)) = 0 \), where \( p \) is a non-zero complex polynomial, then there exists a compact operator \( K \) satisfying \( p(T + K) = 0 \). The problem of lifting an element in a class of \( \mathcal{L}(H) \) to an element in the same class of \( \mathcal{L}(H) \) has interested many mathematicians ([14], [2], [1], [8]), and several results are known in this direction. For instance, we mention that for the class of idempotents, it is established that \( \pi(E(\mathcal{L}(X))) = E(\mathcal{C}(X)) \) where \( X \) is a Banach space and \( E(\mathcal{L}(X)) \) (resp. \( E(\mathcal{C}(X)) \)) denotes the set of idempotent elements of \( \mathcal{L}(X) \) (resp. \( \mathcal{C}(X) \)), [2]. In this note we are motivated by the following question: Let \( T \) be an operator such that zero is a pole of the resolvent of \( \pi(T) \). Is there a compact operator \( K \) for which zero is a pole of the resolvent of \( T + K \)? As will be showed by an example, the answer to this question is generally negative. However, we prove that for such an operator \( T \), there exists a compact operator \( K \) such that \( T + K = A \oplus B \) where \( A \) is nilpotent and \( B \) is Fredholm. Furthermore, we give a necessary and sufficient condition such that \( K = 0 \).

2. Poles in Calkin algebra

As mentioned above, the following example shows that lifting of poles in the Calkin algebra is not possible in general.

**Example 2.1.** Let \( S \) be the unilateral left shift operator given on the Hilbert space \( \ell^2(\mathbb{N}) \) by \( S\epsilon_{i+1} = \epsilon_i \) and \( S\epsilon_1 = 0 \) where \( \{\epsilon_i : i \geq 1\} \) denotes the canonical basis. Consider the operator \( T = 0 \oplus S \) defined on \( H = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \). Then it is clear that \( T - \lambda \) is Fredholm and \( \text{ind}(T - \lambda) = 1 \) whenever \( \lambda \) is in the punctured open unit disk \((0 < |\lambda| < 1)\). Also, because \( T \) has finite descent then so does \( \pi(T) \); see [5]. Consequently, zero is a pole of the resolvent of \( \pi(T) \). However, if there exists a compact operator \( K \) such that zero is a pole of \( T + K \), then \( \text{ind}(T + K - \lambda) = \text{ind}(T - \lambda) = 0 \) for \( 0 < |\lambda| < 1 \), a contradiction.

**Theorem 2.2.** Let \( T \) be a bounded operator on \( H \). Then zero is a pole of the resolvent of \( \pi(T) \) if and only if there exists a compact operator \( K \) such that \( T + K = A \oplus B \) where \( A \) is nilpotent and \( B \) is Fredholm.

**Proof.** Suppose that zero is a pole of the resolvent of \( \tilde{T} = \pi(T) \) of order \( d \). Then there exists an idempotent \( \tilde{R} \) such that \( \tilde{R}\tilde{T} = \tilde{T}\tilde{R}, \sigma(\tilde{R}\tilde{T}\tilde{R}, \mathcal{R}(H)\tilde{R}) = \{0\} \) and \( \sigma((1 - \tilde{R})\tilde{T}(1 - \tilde{R}), (1 - \tilde{R})\mathcal{C}(H)(1 - \tilde{R})) = \sigma(\tilde{T}) \setminus \{0\} \), [4]. Let \( \lambda \) be a non-zero complex number; then \( \tilde{R}\tilde{T}\tilde{R} - \lambda\tilde{R} \) is invertible in \( \mathcal{R}(H)\tilde{R} \). Moreover, by Corollary 3.3 of [2] we can lift \( \tilde{R} \) to an idempotent \( R \) in \( H \), hence there exists \( S \in \mathcal{L}(H) \) such that \( [R(T - \lambda)R][RSR] - R \) and \( [RSR][R(T - \lambda)R] - R \) are compact. Therefore it follows easily that \( RTR\mathcal{ran}(R) - \lambda \) is Fredholm and so \( \sigma_e(RTR\mathcal{ran}(R)) = \{0\} \). In the same way we get that \( (I - R)T(I - R)\mathcal{ker}(R) \) is Fredholm. On the other hand, because \( \tilde{T} \) has finite descent, we obtain the existence of \( U \in \mathcal{L}(H) \) for which \( \tilde{T}^d = \tilde{T}^{d+1}U \) where \( U = \pi(U) \). Therefore,

\[
(\tilde{R}\tilde{T}\tilde{R})^d = \tilde{R}\tilde{T}^{d+1}\tilde{R} = \tilde{R}\tilde{T}^{d+1}\tilde{R}U\tilde{R} = (\tilde{R}\tilde{T}\tilde{R})^{d+1}(\tilde{R}U\tilde{R}).
\]

Consequently \( (RTR)^d - (RTR)^{d+1}(RUR) \) is compact and hence so is its restriction to the invariant subspace \( \mathcal{ran}(R) \). This implies that \( \pi_{\mathcal{ran}(R)}(RTR_{\mathcal{ran}(R)}) \) has finite descent where \( \pi_{\mathcal{ran}(R)} : \mathcal{L}(\mathcal{ran}(R)) \to \mathcal{C}(\mathcal{ran}(R)) \) denotes the quotient map, and since it is quasi-nilpotent in the Calkin algebra \( \mathcal{C}(\mathcal{ran}(R)) \), we obtain by [5, Theorem 2.4] that \( \pi_{\mathcal{ran}(R)}(RTR_{\mathcal{ran}(R)}) \) is nilpotent. Hence, [14, Theorem 2.4] ensures the existence of a compact operator \( K_1 \) on \( \mathcal{ran}(R) \) such that \( RTR_{\mathcal{ran}(R)} + K_1 \) is
nilpotent. Now if we put \( K = RK_1 R - RT(I - R) - (I - R)TR \), then it is easy to verify that \( K \) is compact. Finally \( T + K = A \oplus B \) where \( A = (RTR)_{\text{ran}(R)} + K_1 \) is nilpotent and \( B = (I - R)T(I - R)_{\text{ker}(R)} \) is Fredholm.

Conversely, suppose that there exists a compact operator \( K \) that satisfies \( T + K = A \oplus B \) where \( A^n = 0 \) for some positive integer \( n \) and \( B \) is Fredholm. Then it follows that zero is an isolated point of \( \sigma(T) \), and if we let \( P \) be the projection such that \( P(T + K)P \) is a projection such that \( (I - P)(T + K)(I - P) = B \), there exists an operator \( S \in \mathcal{L}(H) \) such that \( (I - P) - B(I - P)S(I - P) \) is compact. Hence, \( T^n - T^{n+1}I \) is compact and so \( d(\pi(T)) \) is finite, which implies that zero is a pole of the resolvent of \( \pi(T) \) of order \( d \leq n \).

In the sequel, we shall denote by \( G_0 \) the connected component of the Calkin algebra \( \mathcal{C}(H) \) formed by invertible elements and which contains the identity. It is well known that for every \( \pi(T) \in G_0 \) there exists a compact operator \( K \) such that \( T + K \) is invertible, \([9]\).

**Remark.** As we have showed in Example 2.1, the lifting of poles in the Calkin algebra is generally not possible. Furthermore, if such a lifting is possible for some pole of \( \pi(T) \), then the index of \( T - \lambda \) is zero in a punctured neighbourhood of the pole. Hence non-zero index constitutes an obstruction for lifting.

**Corollary 2.3.** Let \( T \) be a bounded operator on \( H \). If \( 0 \in \sigma_c(T) \) is a pole of the resolvent of \( \pi(T) \), then the following assertions are equivalent:

(i) there exists a compact operator \( K \in \mathcal{L}(H) \) such that zero is a pole of \( T + K \),

(ii) there exists a \( \delta > 0 \) such that \( \pi(T) - \lambda \in G_0 \) for \( 0 < |\lambda| < \delta \),

(iii) there exists a \( \delta > 0 \) such that \( T - \lambda \) is Fredholm of index zero for \( 0 < |\lambda| < \delta \).

**Proof.** (i) \( \Rightarrow \) (ii). Because zero is a pole of the resolvent of \( T + K \), there exists \( \delta > 0 \) such that for \( 0 < |\lambda| < \delta \), \( T + K - \lambda \) is invertible and consequently \( \text{ind}(T - \lambda) = \text{ind}(T + K - \lambda) = 0 \).

(ii) \( \Rightarrow \) (i). Since zero is a pole of \( \pi(T) \), Theorem 2.2 ensures the existence of a compact operator \( K_1 \) such that \( T + K_1 = A \oplus B \) where \( A \) is nilpotent and \( B \) is Fredholm. Furthermore, from the fact that \( \text{ind}(T - \lambda) = 0 \) for \( 0 < |\lambda| < \delta \) and the continuity of the index, we get that \( \text{ind}(B) = 0 \). Hence \( B + K_2 \) is invertible for some compact operator \( K_2 \). Now it suffices to put \( K = K_1 + (0 \oplus K_2) \).

For (iii) \( \Leftrightarrow \) (ii) see \([9]\).

### 3. Poles in Calkin Algebra and Essential Ascent and Descent

Before outlining the statement of our results, we need to introduce the following notions.

Following S. Grabiner \([10]\), we associate to every operator \( T \) the following two sequences \( c_n(T) := \dim \ker(T^{n+1})/\ker(T^n) \) and \( c'_n(T) := \dim \text{ran}(T^n)/\text{ran}(T^{n+1}) \).

The **essential ascent** and the **essential descent** are then defined by

\[
a_n(T) = \inf\{n \geq 0 : c_n(T) \text{ is finite}\} \quad \text{and} \quad d_n(T) = \inf\{n \geq 0 : c'_n(T) \text{ is finite}\}.
\]

We mention the following characterizations due to S. Grabiner and J. Zemánek: \([11]\)

\[
(3.1) \quad a_n(T) \text{ is finite } \iff \dim(\text{ran}(T^d) \cap \ker(T)) < \infty \text{ for some } d \geq 0
\]

and

\[
(3.2) \quad d_n(T) \text{ is finite } \iff \text{codim } (\text{ran}(T) + \ker(T^d)) < \infty \text{ for some } d \geq 0.
\]
Theorem 3.1. Let \( T \) be a bounded operator on \( H \). Then zero is a pole of the resolvent of \( \pi(T) \) if and only if there exists a compact operator \( K \) such that \( a_c(T+K) \) and \( d_c(T+K) \) are both finite.

Proof. The direct implication follows immediately from Theorem 2.2. For the converse, assume that \( a_c(T+K) \) and \( d_c(T+K) \) are finite; then Chapter III §22, Theorem 12 provide the existence of two closed subspaces \( M, N \) invariant by \( T+K \) and such that \( H = M \oplus N \) and \( T+K = T_1 \oplus T_2 \) where \( T_1 \) nilpotent and \( T_2 \) Fredholm, as desired.

Remark. In the preceding result, the compact operator \( K \) cannot be chosen to be zero. Indeed, consider the compact operator \( K \) we associate the operator \( P(\pi) \) defined on the Banach space \( \ell^2(\mathbb{N}) \) by \( Ke_n = \frac{1}{n} e_n \); then zero is obviously a pole of the resolvent of \( \pi(K) = 0 \). However, if \( K \) has finite essential ascent and descent, then Theorem 5.3 of [11] will imply that \( \text{ran}(K^d) \) is closed for \( d = a_c(K) = d_c(K) \), a contradiction.

Before stating our next result, we have to recall the following notion introduced by B. N. Sadovskii in [15]; see also Chapter III §17. Let \( \ell^\infty(H) \) denote the Banach space of all bounded sequences of \( H \) equipped by the sup norm and \( m(H) \) its closed subspace consisting of all the sequences \( \{x_n\}_n \) such that \( \{x_n : n \in \mathbb{N}\} \) is totally bounded or equivalently has compact closure in \( H \). To an operator \( T \in \mathcal{L}(H) \) we associate the operator \( P(T) \) defined on the Banach space \( P(H) = \ell^\infty(H) \oplus m(H) \) by \( P(T)(\{x_n\}_n + m(H)) = \{Tx_n\}_n + m(H) \). It is well known that for an operator \( T \in \mathcal{L}(H) \), \( P(T) = 0 \) if and only if \( T \) is compact, and if \( \text{ran}(T) \) is closed, then \( \ker(P(T)) = \ell^\infty(\ker(T)) + m(H) \).

Theorem 3.2. If \( T \in \mathcal{L}(H) \) is an operator such that zero is a pole of the resolvent of \( \pi(T) \) of order \( d \) and \( \text{ran}(T^d) \) and \( \text{ran}(T^{d+1}) \) are closed, then \( d = a_c(T) = d_c(T) \).

Proof. Let \( Q \) be the orthogonal projection on \( \ker(T^{d+1}) \). Then we have
\[
\pi(T^{d+1})\pi(Q) = 0,
\]
and since \( d = a(\pi(T)) \), we obtain that \( \pi(T^d)\pi(Q) = 0 \). On the other hand, since \( \text{ran}(T^d) \) is closed then so is \( \text{ran}(T^dQ) = \text{ran}(T^d) \cap \ker(T) \), and consequently \( \text{ran}(T^d) \cap \ker(T) \) is of finite dimension. Thus, using (3.1), we obtain that \( a_c(T) \leq d \) because \( \ker(T^{d+1})/\ker(T^d) \cong \text{ran}(T^d) \cap \ker(T) \). Moreover, we get by duality that \( \dim \text{ran}(T^d) \cap \ker(T^*) \) is finite and hence so is \( \text{codim} \ (\ker(T^d) + \text{ran}(T)) \). Finally, by (3.2), it follows that \( d_c(T) \) is finite. Now suppose that \( a_c(T) = d_c(T) < d \); then \( \dim \ker(T^d)/\ker(T^{d-1}) \) is finite. Hence by Chapter III §17, Lemma 2, we have \( \ell^\infty(\ker(T^d)) + m(H) = \ell^\infty(\ker(T^{d-1})) + m(H) \), and since \( \text{ran}(T^d) \) is closed, we obtain that \( \ker(P(T^d)) \subseteq \ker(P(T^{d-1})) \). Thus \( \ker(P(T^d)) = \ker(P(T^{d-1})) \). Finally, if \( S \) is an operator such that \( T^dS \) is compact, then so is \( T^{d-1}S \). This shows that \( a(\pi(T)) < d \), the desired contradiction.

Notice that the inverse implication of Theorem 3.2 does not hold in general. In fact, if we consider the compact operator \( K \) given on the Hilbert space \( \ell^2(\mathbb{N}) \) by \( Ke_{2p} = \frac{1}{p} e_{2p-1} \) and \( Ke_{2p-1} = 0 \), then \( a_c(K) = d_c(K) = 2 \) while zero is a pole of the resolvent of \( \pi(K) = 0 \) of order 1. However, if we suppose that \( \text{ran}(T^{d-1}) \) is closed, we obtain the following result:

Proposition 3.3. Let \( T \) be a bounded operator on \( H \). If \( d = a_c(T) = d_c(T) \) and \( \text{ran}(T^{d-1}) \) is closed, then zero is a pole of the resolvent of \( \pi(T) \) of order \( d \).
Proof. Because Theorem 3.1 ensures that zero is a pole of the resolvent of \( \pi(T) \), then it suffices to establish that \( d = a(\pi(T)) \). First suppose that \( n := a(\pi(T)) < d \), and let \( Q \) be the orthogonal projection on \( \ker(T^n) \). Then \( T^{d-1}Q \) is compact and so \( P(T^{d-1})P(Q) = 0 \). We claim that \( \ell^\infty(\ker(T^d)) + m(H) = \ell^\infty(\ker(T^{d-1})) + m(H) \). Let \( \{x_n\}_n \in \ell^\infty(\ker(T^d)) \); then we have
\[
P(T^{d-1})\{x_n\}_n(m(H)) = P(T^{d-1})P(Q)\{x_n\}_n + m(H) = 0,
\]
and hence \( \{x_n\}_n + m(H) \in \ker(P(T^{d-1})) = \ell^\infty(\ker(T^{d-1})) + m(H) \) because \( \dim \ker(T^d)/\ker(T^{d-1}) \) is finite, a contradiction. Therefore \( n \geq d \), and consequently, Theorem 5.3 of [11] implies that \( \text{ran}(T^n) \) and \( \text{ran}(T^{n+1}) \) are closed. Finally, \( n = d \) by Theorem 3.2.

\[\square\]

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**References**


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