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IDEALS DEFINING GORENSTEIN RINGS ARE (ALMOST) NEVER PRODUCTS

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Abstract. This note proves that if S is an unramified regular local ring and I, J proper ideals of height at least two, then S/IJ is never Gorenstein.

This paper answers in the unramified case a question that was asked by Eisenbud and Herzog: can an ideal in a regular local ring that defines a Gorenstein quotient ring ever be the product of two proper ideals, except in the obvious case in which a principal ideal is a product?

A related (unpublished) result was proved by W. Heinzer, D. Lantz and K. Shah in the early 1990s. They proved that an ideal generated by a system of parameters in a Gorenstein ring can never be the product of two proper ideals. Explicitly, they prove the following:

Proposition 1. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension at least two. Then an ideal I generated by a system of parameters is never the product of two proper ideals.*

Proof (Heinzer, Lantz, Shah). Without loss of generality we may assume that the residue field of R is infinite. Assume that $I = JK$. We claim that we may assume that J and K are also generated by systems of parameters. For let J' and K' be minimal reductions of J and K. Since the residue field of R is infinite, both J' and K' are generated by systems of parameters. Then $I = JK$ is integral over $J'K'$ since they have the same extension to any valuation overring. Since I is not integral over any proper subideal, we must have $I = J'K'$. So we may assume that J and K are generated by systems of parameters. In particular, J has a unique minimal overideal. Hence by Matlis duality the annihilator L/I of J/I has a unique minimal underideal in R/I. Thus $L/I = a(R/I)$ for any $a \in L$ not in that underideal. Since $I = JK$ is contained in JL which is contained in I, we have $I = JL = J(aR + I) = aJ + JI$. But this means $I = aJ$, so I is not $\mathfrak{m}$-primary. $\square$

Our main result is the following theorem.

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Theorem 2. *Let $(S, \mathfrak{m})$ be an unramified regular local ring, and let I be an ideal of height at least two. If S/I is Gorenstein, then $I \neq JK$ for any two proper ideals J, K.*

Proof. By way of contradiction we assume that $I = JK$, where J and K are proper ideals. We reduce to the case in which I is $\mathfrak{m}$-primary using a simple reduction suggested by the referee, which is easier than what this author originally did. First we make the residue field infinite in case it is not already infinite by replacing S by $S(t) = S[t]_{\mathfrak{m}S[t]}$, where t is a variable. This does not change the assumption that S is an unramified regular local ring, and $IS(t) = JS(t) \cdot KS(t)$. We will prove that the height of $IS(t)$ is one, which will be a contradiction. Henceforth we assume that the residue field of S is infinite.

We next reduce to the case that I is $\mathfrak{m}$-primary. Choose a maximal regular sequence $x_1, ..., x_d$ on S/I consisting of elements whose images in $\mathfrak{m}/(\mathfrak{m}^2 + I)$ are independent. We can furthermore assume that if S is of mixed characteristic p, then $p \notin (x_1, ..., x_d) + \mathfrak{m}^2$ by choosing $x_1, ..., x_d$ generally enough, unless $\mathfrak{m} = (x_1, ..., x_d)$. In this latter case, $I = 0$, a contradiction. One can then replace S by $S' = S/(x_1, ..., x_d)$, which is still an unramified regular local ring, and replace I by $I' = IS'$. In this case, S'/I' is still Gorenstein, I' is still a product, and I' is primary for the maximal ideal. Moreover the height of I and the height of I' are the same.

We have reduced to the case in which I is $\mathfrak{m}$-primary. Among all representations of I as a product, $I = JK$ for two proper ideals J and K, choose J and K each maximal with respect to this property. Since $I = JK \subseteq (I : K)K \subseteq I$, the maximality shows that $I : K = J$ and similarly $I : J = K$. Set $R = S/I$. For an arbitrary R-module M, we let $\lambda(M)$ denote the length of M. From duality, using the fact that R is Gorenstein, we see that

$$\begin{aligned}\lambda(R/(J+K)R) &= \lambda(\mathrm{Hom}_R(R/(J+K)R, R)) = \lambda(0 : (J+K)R) \\ &= \lambda((0 : JR) \cap (0 : KR)) = \lambda((K \cap J)R).\end{aligned}$$

Lifting these ideals back to S, we obtain that $\lambda(S/(J+K)) = \lambda((J \cap K)/JK)$, i.e. that

$$\lambda(\mathrm{Tor}_0^S(S/J, S/K)) = \lambda(\mathrm{Tor}_1^S(S/J, S/K)).$$

Since both of these modules have finite length, $\chi(S/J, S/K) = 0$, where whenever M and N are finitely generated S-modules such that $M \otimes_S N$ has finite length we define

$$\chi(M, N) = \sum_{i=0}^{\infty} (-1)^i \lambda(\mathrm{Tor}_i^S(M, N)).$$

Hence

$$\sum_{i=2}^{\infty} (-1)^i \lambda(\mathrm{Tor}_i^S(S/J, S/K)) = 0.$$

By [3] (see also [2]), this forces $\mathrm{Tor}_i^S(S/J, S/K) = 0$ for all $i \geq 2$. We claim this forces the dimension of S to be at most one: let $\mathbf{F}$ be the minimal free resolution of S/J. After tensoring with S/K, the last map in the resulting complex can never be injective since every socle element of S/K must go to zero. Since the last free module in the minimal resolution of S/J occurs at the dimension of S, the vanishing of all higher Tors past 2 forces the dimension to be one. Hence the height of I is one, a contradiction. $\square$

Theorem 2 has several possible extensions. Of course, to begin with, one would like to know the result in the ramified case. The most reasonable related question seems to be the following:

Question: *Let S be a regular local ring and I an ideal of height c. Assume that $I = JK$ for two proper ideals J, K, and that S/I is Cohen-Macaulay. Then is the type of S/I at least c?*

Perhaps there is an elementary argument that answers this question in the affirmative, which would simplify and generalize the result above. For example, in the case I is $\mathfrak{m}$-primary and $J = \mathfrak{m}$, the result holds since the type of S/I is just the socle dimension, and if $I = \mathfrak{m}K$, then $K/\mathfrak{m}K$ is contained in the socle. Hence the type is at least the minimal number of generators of K, which is at least the height of I by Krull's height theorem. At the very least this special case shows that some homological algebra dealing with Krull's theorem might be needed.

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