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THE GEOGRAPHY OF SYMPLECTIC 4-MANIFOLDS WITH AN ARBITRARY FUNDAMENTAL GROUP

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Abstract. In this article, for each finitely presented group G, we construct a family of minimal symplectic 4-manifolds with $\pi_1 = G$ which cover most lattice points $(x, \mathbf{c})$ with x large in the region $0 \leq \mathbf{c} < 9x$. Furthermore, we show that all these 4-manifolds admit infinitely many distinct smooth structures.

1. Introduction

Applications of gauge theory to symplectic 4-manifolds have led to many remarkable results. For example, C. Taubes proved that every minimal symplectic 4-manifold with $b_2^+ > 1$ satisfies $c_1^2 \geq 0$ [T]. Topologists were able to prove that most lattice points $(x, \mathbf{c})$ satisfying $0 \leq \mathbf{c} < 9x$ are realized as chern numbers (χ, c_1^2) of simply connected minimal symplectic 4-manifolds, and most known simply connected minimal symplectic 4-manifolds with b_2^+ large enough admit infinitely many distinct symplectic structures ([FS], [GS], [P1]-[P3]). Note that $\chi = \frac{\sigma+e}{4}$ and $c_1^2 = 3\sigma + 2e$, where σ and e denote the signature and the Euler characteristic of a given 4-manifold. But little is known in the non-simply connected case.

The aim of this paper is to extend the results above partially to the non-simply connected case. Explicitly we have

Theorem 1.1. *For each finitely generated group G, there are constants r_G and t_G such that every lattice point $(x, \mathbf{c})$ satisfying $0 \leq \mathbf{c} \leq r_G x$ and $x \geq t_G$ is realized as (χ, c_1^2) of a minimal symplectic 4-manifold with $\pi_1 = G$ which admits infinitely many distinct smooth structures. Furthermore, the constant r_G can be chosen so that it is close to 9.*

Remarks 1. R. Gompf first constructed a new family of symplectic 4-manifolds with any prescribed fundamental group [G], and recently S. Baldridge and P. Kirk constructed a family of symplectic 4-manifolds with an arbitrary fundamental group near the Bogomolov-Miyaoka-Yau line [BK2]. Theorem 1.1 above addresses the uniqueness of smooth structures on such 4-manifolds as well as the existence of such 4-manifolds.

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Remarks 2. The key ingredient in the proof of Theorem 1.1 above is to use S. Boyer's result on simply connected 4-manifolds with a given boundary [B].

2. The main construction

The main tactic in proving that many simply connected symplectic 4-manifolds admit infinitely many distinct smooth structures was to show that a family of simply connected homeomorphic 4-manifolds obtained by some topological surgeries have mutually different Seiberg-Witten invariants. We use the same idea in the non-simply connected case. We first briefly review such topological surgeries — called a fiber sum and a knot surgery — and state related theorems.

Definition. For $i=1,2$, let X_i be a symplectic 4-manifold containing a symplectic (or Lagrangian) genus g surface Σ_g of square zero. Suppose that $X_i^0 = X_i - \nu_i(\Sigma_g)$ is a complement of a tubular neighborhood $\nu_i(\Sigma_g)$ of Σ_g in X_i. Then, by choosing an orientation-reversing, fiber-preserving diffeomorphism $\varphi : \nu_1(\Sigma_g) \to \nu_2(\Sigma_g)$ and by gluing X_1^0 to X_2^0 along their boundaries via the diffeomorphism $\varphi| : \partial\nu_1(\Sigma_g) \to \partial\nu_2(\Sigma_g)$, one can get a new symplectic 4-manifold $X_1\sharp_{\Sigma_g}X_2$, called a (*symplectic*) *fiber sum* of X_1 and X_2 along Σ_g.

Remarks 1. The Euler characteristic and the signature of a fiber sum 4-manifold $X_1\sharp_{\Sigma_g}X_2$ are easily computed, so that it has

$$\chi(X_1\sharp_{\Sigma_g}X_2) = \chi(X_1)+\chi(X_2)+(g-1) \text{ and } c_1^2(X_1\sharp_{\Sigma_g}X_2)=c_1^2(X_1)+c_1^2(X_2)+8(g-1)\,.$$

Remarks 2. The minimality is preserved under a symplectic fiber sum operation. That is, if both X_1 and X_2 are minimal symplectic 4-manifolds, so is $X_1\sharp_{\Sigma_g}X_2$ (Theorem 2.5 in [P1]).

Remarks 3. The Seiberg-Witten invariants of a fiber sum 4-manifold can be completely computed under certain cases. For example, the following product formula is widely known to the experts ([FS], [Pd]).

Theorem 2.1 ([Pd], Corollary 15 and Corollary 20)**.** *Suppose X_1 and X_2 are closed symplectic 4-manifolds which contain a symplectic torus T in a cusp neighborhood. Then the Seiberg-Witten invariant of $X_1\sharp_T X_2$ is given by*

$$SW_{X_1\sharp_T X_2} = SW_{X_1}\cdot SW_{X_2}\cdot(\exp([T])-\exp(-[T]))^2.$$

Definition. Suppose K is a fibered knot in S^3. Let M_K be a 3-manifold obtained by performing 0-framed surgery on K. Then the 3-manifold M_K can be considered as a fiber bundle over circle, so that $M_K \times S^1$ fibers over torus T. If X is a symplectic 4-manifold with a symplectically embedded torus T of square 0, then one can get a new symplectic 4-manifold $X_K := X\sharp_T(M_K \times S^1)$, obtained by taking a (symplectic) fiber sum along T; we call this a *knot surgery* with a knot K. R. Fintushel and R. Stern proved that X_K is homotopy equivalent to X under a mild condition on X and they also computed the Seiberg-Witten invariant of X_K.

Theorem 2.2 ([FS])**.** *Suppose X is a simply connected symplectic 4-manifold which contains a symplectic torus T of square 0 in a cusp neighborhood with $\pi_1(X\setminus T) = 1$. If K is a fibered knot in S^3, then X_K is a symplectic 4-manifold which is homeomorphic to X and whose Seiberg-Witten invariant is*

$$SW_{X_K} = SW_X\cdot\Delta_K(t),$$

where $\Delta_K(t)$ is the Alexander polynomial of a knot K and $t=\exp(2[T])$.

Next, we introduce some basic symplectic 4-manifolds which will serve as building blocks of our construction ([G], [GS] for details).

Building Block 1. Let $Q := Z_1 \sharp_\psi Z_2$ be a symplectic 4-manifold constructed as follows: First, consider a Thurston's manifold $Z := \mathbb{R}^4/G$, where G is a discrete subgroup of symplectomorphisms generated by unit translations parallel to the x^1-, x^2-, and x^3-axes, together with the map $(x^1, \dots, x^4) \mapsto (x^1+x^2, x^2, x^3, x^4+1)$. Note that projection onto the last two coordinates induces a bundle structure $\pi : Z \to \mathbb{T}^2$ with symplectic torus fibers. Next, using two copies, $\pi_i : Z_i \to \mathbb{T}^2$ $(i=1,2)$ of Thurston's manifold and using an orientation-reversing bundle map ψ induced from 90° rotation $\psi_o : \pi_1^{-1}(0) \to \pi_2^{-1}(0)$ defined by $\psi_o(x^1, x^2) = (-x^2, x^1)$, one obtains a symplectic fiber sum $Q := Z_1 \sharp_\psi Z_2$. This is a torus bundle over a genus 2 surface $\mathbb{T}^2 \sharp \mathbb{T}^2$ and has a symplectic section Σ_2 of square 0 given by sections in Z_1 and Z_2 such that $\pi_1(Q - \Sigma_2)/\pi_1(\Sigma_2'') = 1$, where Σ_2'' is a parallel copy of Σ_2. Similarly, there is also a Lagrangian torus $T \subset Q$ of square 0, disjoint from Σ_2, in Q. For example, one obtains such a torus T by setting $x^1 = x^4 = 1/2$ in Z_1.

Building Block 2. Let $E(n)$ be a simply connected elliptic surface with no multiple fibers and holomorphic Euler characteristic n. Then $E(n)$ can be obtained as an algebraic surface $B(2,3,6n-1) \cup_{\Sigma(2,3,6n-1)} C(n)$, where $B(2,3,6n-1)$ is a Brieskorn manifold and $C(n)$, usually called *a Gompf nucleus*, is the neighborhood of a cusp fiber and a section which is an embedded 2-sphere of square $-n$. By performing a knot surgery with a fibered knot $K \subset S^3$ in $C(n)$, one obtains a simply connected symplectic 4-manifold $E(n)_K$ which is homeomorphic, but not diffeomorphic, to $E(n)$. Since a Brieskorn manifold $B(2,3,6n-1)$ with $n \geq 2$ contains a Lagrangian torus T of square 0, which intersects 2-sphere transversely at a single point, lying in another Gompf nucleus, we conclude that $E(n)_K$ also contains a Lagrangian torus T in a Gompf nucleus.

Using the building blocks above together with a Lefschetz fibration lying on the BMY-line [S], we constructed various families of simply connected, minimal, symplectic 4-manifolds which solve many geography problems in the simply connected case. Among them, we quote the following basic constructions.

Lemma 2.1 ([P2], Lemma 2.1)**.** *For each integer k, $10 \leq k \leq 18$, there exists a simply connected, minimal, symplectic 4-manifold $X_{3,k}$ with $\chi = 2$ and $c_1^2 = 19 - k$ which contains a symplectic genus 2 surface Σ_2 of square 0 and a symplectic torus T of square 0, disjoint from Σ_2, in a fishtail neighborhood, and $\pi_1(X_{3,k} - \Sigma_2) = \pi_1(X_{3,k} - T) = 1$.*

Proposition 2.1 ([P3], Proposition 2.1)**.** *For each odd integer $m \geq 1$ and $10 \leq k \leq 18$, there exists a simply connected, minimal, symplectic 4-manifold $Z_{m,k}$ which contains a symplectic genus 2 surface Σ_2 of square 0 and a torus T of square 0, disjoint from Σ_2, in a fishtail neighborhood which satisfies*

$$\pi_1(Z_{m,k} - \Sigma_2) = \pi_1(Z_{m,k} - T) = 1.$$

Furthermore, it has $\chi(Z_{m,k}) = 25m^2 + 31m + 5$ and $c_1^2(Z_{m,k}) = 225m^2 + 248m + 35 - k$.

Building Block 3. Let G be a finitely presented group with g generators $\{x_1, \dots, x_g\}$ and r relations $\{w_1, \dots, w_r\}$. Let F be an oriented genus g Riemann surface with an oriented circles $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ representing a standard symplectic basis

for $H_1(F)$. For $i = 1, \ldots, r$, let γ_i be a smoothly immersed, oriented circle in F representing the word w_i in this group, and, for $i = 1, \ldots, g$, let $\gamma_{r+i} = \beta_i$ and identifying $x_i = \alpha_i$, we have $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_{g+r} \rangle \cong G$. By forming connected sums of F with a copy of torus T and perturbing and rearranging γ_i, R. Gompf constructed a collection of symplectic tori T_i corresponding to $\gamma_i \times \alpha$ in $F \times \mathbb{T}$, where α is one of the oriented circles representing a standard basis of $H_1(\mathbb{T})$. Let $M = (F \times \mathbb{T}) \sharp_{T_i} \cup_i E(1) \sharp_{\{\mathrm{pt}\} \times \mathbb{T}} E(1)$ be a symplectic 4-manifold obtained by symplectically fiber summing $F \times \mathbb{T}$ with $E(1)$'s along each T_i and $\{\mathrm{pt}\} \times \mathbb{T}$. Then, since $E(1) - T$ is simply connected, $\pi_1(M) \cong G$. Furthermore, M contains a symplectic torus $T = \{\mathrm{pt}'\} \times \mathbb{T}$ such that $\pi_1(M)/\pi_1(T) \cong \pi_1(M) = G$. Recently, by modifying the construction above, S. Baldridge and P. Kirk constructed a similar symplectic 4-manifold with $\pi_1 = G$.

Theorem 2.3 ([BK1], Theorem 6)**.** *Let G have a presentation with g generators $\{x_1, \ldots, x_g\}$ and r relations $\{w_1, \ldots, w_r\}$. Then there exists a closed symplectic 4-manifold M with $\pi_1(M) = G$ which has $\chi(M) = g + r + 1$ and $c_1^2(M) = 0$.*

Remark 1. The symplectic 4-manifold M constructed in Theorem 2.3 above also contains a symplectic torus T of square 0 such that the inclusion $i : T \to M$ induces a zero map $i_* \equiv 0 : \pi_1(T) \to \pi_1(M) = G$. In fact, since $E(1) - T$ is simply connected, the inclusion $i : T'' \to M - T$ also induces a zero map $i_* \equiv 0 : \pi_1(T'') \to \pi_1(M - T)$, where T'' is a parallel copy of T.

Suppose $X_{3,k}$, $Z_{m,k}$ and M are symplectic 4-manifolds constructed in Lemma 2.1, Proposition 2.1 and Theorem 2.3 respectively. Since $X_{3,k}$ contains a symplectic surface Σ_2 of square 0, taking a fiber sum l times along Σ_2 in $X_{3,k}$ with l copies of Q, we get a symplectic 4-manifold $X_{3,k} \sharp_{\Sigma_2} lQ := (\cdots((X_{3,k} \sharp_{\Sigma_2} Q) \sharp_{\Sigma_2} Q) \sharp_{\Sigma_2} \cdots \sharp_{\Sigma_2} Q)$. Again, since Q contains a Lagrangian torus T' of square 0 disjoint from Σ_2, we can also construct a symplectic 4-manifold $(X_{3,k} \sharp_{\Sigma_2} lQ) \sharp_{T'} E(2)$ by taking a fiber sum along T' in Q and $C(2) \subset E(2)$. Furthermore, there is another symplectic torus T lying in M and $X_{3,k} - \Sigma_2$. Hence, by taking a fiber sum along T, we obtain a new symplectic 4-manifold

$$X_{3,k,l} := M \sharp_T ((X_{3,k} \sharp_{\Sigma_2} lQ) \sharp_{T'} E(2)).$$

Similarly, using $Z_{m,k}$ instead of $X_{3,k}$, we obtain a symplectic 4-manifold

$$Y_{m,k,l} := M \sharp_T ((Z_{m,k} \sharp_{\Sigma_2} lQ) \sharp_{T'} E(2)).$$

Then we have

Corollary 2.1. *For each integer k and l, $10 \leq k \leq 18$ and $l \geq 0$, $X_{3,k,l}$ is a minimal symplectic 4-manifold with $\pi_1(X_{3,k,l}) \cong \pi_1(M) = G$ which satisfies $\chi(X_{3,k,l}) = l + g + r + 5$ and $c_1^2(X_{3,k,l}) = 8l + 19 - k$.*

Proof. It suffices to prove that $\pi_1(X_{3,k,l}) \cong \pi_1(M)$. For this, we first show that $\pi_1((X_{3,k} - T) \sharp_{\Sigma_2} lQ) = 1$: Note that the Van-Kampen theorem on $\pi_1(X_{3,k} - T) = 1$ implies that $\pi_1(\Sigma_2'') \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is a zero map and $\pi_1(\partial \nu(\Sigma_2)) \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is surjective, where Σ_2'' is a parallel copy of Σ_2. Since $\pi_1(Q - \Sigma_2)/\pi_1(\Sigma_2'') = 1$, we have

$$\begin{aligned} \pi_1((X_{3,k} - T) \sharp_{\Sigma_2} Q) &= \pi_1((X_{3,k} - T) - \nu(\Sigma_2)) * \pi_1(Q - \nu(\Sigma_2))/\langle \pi_1(\partial \nu(\Sigma_2)) \rangle \\ &\cong \pi_1((X_{3,k} - T) - \nu(\Sigma_2))/\langle \pi_1(\partial \nu(\Sigma_2)) \rangle \\ &\cong 1\,. \end{aligned}$$

Now, by induction on l, we conclude that $\pi_1((X_{3,k}-T)\sharp_{\Sigma_2} lQ)=1$.

Next, since both $(X_{3,k}-T)\sharp_{\Sigma_2} lQ$ and $E(2)-T'$ are simply connected, the fiber sum 4-manifold $((X_{3,k}-T)\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2) \simeq ((X_{3,k}-\nu(T))\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2)$ is also simply connected.

Finally, since the inclusion $i: T'' \to M-T$ induces a zero map $i_* \equiv 0 : \pi_1(T'') \to \pi_1(M-T) \cong \pi_1(M-\nu(T))$ (see Remark 1 above), the Van-Kampen theorem implies that

$$\begin{aligned}
\pi_1(X_{3,k,l}) &= \pi_1(M-\nu(T)) * \pi_1(((X_{3,k}\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2))-\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&= \pi_1(M-\nu(T)) * \pi_1(((X_{3,k}-\nu(T))\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(M-\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(M-\nu(T)) * \pi_1(\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(M). \qquad \square
\end{aligned}$$

Corollary 2.2. *For each odd integer $m \geq 1$, $10 \leq k \leq 18$ and $l \geq 0$, $Y_{m,k,l}$ is a minimal symplectic 4-manifold with $\pi_1(Y_{m,k,l}) \cong \pi_1(M) = G$ which satisfies $\chi(Y_{m,k,l}) = 25m^2+31m+l+g+r+8$ and $c_1^2(Y_{m,k,l}) = 225m^2+248m+8l+35-k$.*

Proof. Similar to the proof of Corollary 2.1 above, $\pi_1(Z_{m,k}-T)=1$ implies that $\pi_1(((Z_{m,k}-\nu(T))\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2))=1$. Hence Van-Kampen theorem again induces

$$\begin{aligned}
\pi_1(Y_{m,k,l}) &= \pi_1(M-\nu(T)) * \pi_1(((Z_{m,k}\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2))-\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&= \pi_1(M-\nu(T)) * \pi_1(((Z_{m,k}-\nu(T))\sharp_{\Sigma_2} lQ)\sharp_{T'} E(2))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(M-\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(M). \qquad \square
\end{aligned}$$

Next, we introduce S. Boyer's result on simply connected 4-manifolds with a given boundary [B], which will be a key ingredient in proving our result.

Let V_1 and V_2 be simply connected, compact, oriented 4-manifolds with connected boundary $\partial V_1 = \partial V_2$ and suppose that $f: \partial V_1 \to \partial V_2$ is an orientation preserving homeomorphism. Then there are two obstructions to extending f to a homeomorphism $F: V_1 \to V_2$: The first is to find an isometry $\Lambda : (H_2(V_1;\mathbf{Z}), Q_{V_1}) \to (H_2(V_2;\mathbf{Z}), Q_{V_2})$ for which the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(\partial V_1) & \longrightarrow & H_2(V_1) & \longrightarrow & H_2(V_1,\partial V_1) & \longrightarrow & H_1(\partial V_1) \longrightarrow 0 \\
 & & \downarrow f_* & & \downarrow \Lambda & & \downarrow \Lambda^* & & \downarrow f_* \\
0 & \longrightarrow & H_2(\partial V_2) & \longrightarrow & H_2(V_2) & \longrightarrow & H_2(V_2,\partial V_2) & \longrightarrow & H_1(\partial V_2) \longrightarrow 0
\end{array}$$

Here an isometry means an isomorphism $\Lambda : H_2(V_1;\mathbf{Z}) \to H_2(V_2;\mathbf{Z})$ which preserves an intersection form Q_{V_i}, and Λ^* is the adjoint of Λ with respect to the identification of $H_2(V_i,\partial V_i)$ with $Hom(H_2(V_i);\mathbf{Z})$ arising from Lefschetz duality. We call such a pair (f,Λ) a *morphism* and denote it symbolically as $(f,\Lambda): V_1 \to V_2$. The second obstruction encountered is to realize a given morphism (f,Λ) *geometrically*, i.e. to find a homeomorphism $F: V_1 \to V_2$ such that $(f,\Lambda) = (F|_{\partial V_1}, F_*)$. S. Boyer showed when a given morphism could be realized geometrically. Explicitly, he proved

Theorem 2.4 ([B], Theorem 0.7 and Proposition 0.8)**.** *If $(f,\Lambda): V_1 \to V_2$ is a morphism between two simply connected smooth 4-manifolds V_1 and V_2 with boundary $\partial V_1 = \partial V_2$, there is an obstruction $\theta(f,\Lambda) \in I^1(\partial V_1)$ such that (f,Λ) is realized*

geometrically if and only if $\theta(f,\Lambda) = 0$. *Furthermore, if* $H_1(\partial V_1;\mathbf{Q}) = 0$, *then* $\theta(f,\Lambda) = 0$.

Corollary 2.3. *Suppose that* $\partial V_1 = \partial V_2$ *is a homology* 3*-sphere. Then, for any morphism* $(f,\Lambda) : V_1 \to V_2$ *between two simply connected smooth* 4*-manifolds* V_1 *and* V_2, *there is a homeomorphism* $F : V_1 \to V_2$ *such that* $(f,\Lambda) = (F|_{\partial V_1}, F_*)$.

Finally, we state and prove a key proposition.

Proposition 2.2. *Suppose* X *is a symplectic* 4*-manifold which contains a symplectic torus* T *lying in a cusp neighborhood. Then, for each integer* $n \geq 2$, *a family of symplectic* 4*-manifolds* $\{X\sharp_T E(n)_K \mid K \text{ is a fibered knot in } S^3\}$ *are mutually non-diffeomorphic, but all homeomorphic. Furthermore,* $\pi_1(X\sharp_T E(n)_K) \cong \pi_1(X)$.

Proof. Since $E(n)_K$ contains a Gompf nucleus, say C_n, embedded in $B(2,3,6n-1)$ (*Building Block* 2), we can decompose $E(n)_K$ into $C_n \cup_\Sigma E(n)^\circ_K$, where Σ is a homology 3-sphere and $E(n)^\circ_K$ denotes a complement of C_n in $E(n)_K$. Note that the Gompf nucleus C_n is independent of K, i.e. C_n is a common submanifold of codimension zero in all $E(n)_K$'s. Hence, after taking a symplectic fiber sum of X with $E(n)_K$ along a Lagrangian torus $T \subset C_n$, we get a decomposition

$$X\sharp_T E(n)_K = (X\sharp_T C_n) \cup_\Sigma E(n)^\circ_K .$$

In order to show that $X\sharp_T E(n)_K$ is homeomorphic to $X\sharp_T E(n)_{K'}$ for any fibered knots K and K' in S^3, we first choose an identity map $f = id : X\sharp_T C_n \to X\sharp_T C_n$ and an isomorphism $\Lambda : H_2(E(n)^\circ_K;\mathbf{Z}) \to H_2(E(n)^\circ_{K'};\mathbf{Z})$. In particular, since both intersection forms $Q_{E(n)^\circ_K}$ and $Q_{E(n)^\circ_{K'}}$ are unimodular and indefinite, they are uniquely determined by rank, signature and type, so that they are isomorphic. (The reason is following: Since $E(n)_K$ and $E(n)_{K'}$ are homeomorphic, their intersection forms have the same rank, signature and type. Hence $Q_{E(n)^\circ_K}$ and $Q_{E(n)^\circ_{K'}}$ also have the same rank, signature and type.) Hence we may choose Λ to be an isometry, i.e. $\Lambda : (H_2(E(n)^\circ_K; Q_{E(n)^\circ_K}) \to (H_2(E(n)^\circ_{K'}; Q_{E(n)^\circ_{K'}})$ is an isomorphism which preserves the intersection forms. Then, applying Corollary 2.3 with the fact that Σ is a homology 3-sphere, we conclude that there exists a homeomorphism $F : E(n)^\circ_K \to E(n)^\circ_{K'}$ such that $(f,\Lambda) = (F|_\Sigma, F_*)$. Thus there is a homeomorphism between two symplectic 4-manifolds $X\sharp_T E(n)_K$ and $X\sharp_T E(n)_{K'}$.

Next, applying Theorem 2.1 and Theorem 2.2 above, we also conclude that $X\sharp_T E(n)_K$ and $X\sharp_T E(n)_{K'}$ have different Seiberg-Witten invariants as long as $\Delta_K(t) \neq \Delta_{K'}(t)$, i.e. $X\sharp_T E(n)_K$ is not diffeomorphic to $X\sharp_T E(n)_{K'}$.

The last statement follows from the Van-Kampen theorem using the fact that T is a regular fiber in a cusp neighborhood and $\pi_1(E(n)_K - T) = 1$:

$$\begin{aligned}
\pi_1(X\sharp_T E(n)_K) &= \pi_1(X - \nu(T)) * \pi_1(E(n)_K - \nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(X - \nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(X - \nu(T)) * \pi_1(\nu(T))/\langle \pi_1(\partial\nu(T))\rangle \\
&\cong \pi_1(X). \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. Let $\{X_{3,k,l}\}$ and $\{Y_{m,k,l}\}$ be minimal symplectic 4-manifolds appeared in Corollary 2.1 and Corollary 2.2 above. For each odd integer $m \geq 1$ with a finitely presented group G fixed (so that g and r are also fixed), define two

numbers $r_{G,m}$ and $t_{G,m}$ by

$$\begin{cases} r_{G,m} := \frac{c_1^2(Y_{m,10,112.5m+173.5})}{\chi(Y_{m,10,112.5m+173.5})} = \frac{225m^2+1148m+1413}{25m^2+143.5m+181.5+g+r}, \\ \\ t_{G,m} := 25m^2 + 143.5m + 181.5 + g + r. \end{cases}$$

Then $\{r_{G,m}\}$ is an increasing sequence converging to 9 and any lattice point $(x, \mathbf{c})$ satisfying $0 \leq \mathbf{c} \leq r_{G,m}x$ and $x \geq t_{G,m}$ is realized as chern numbers (χ, c_1^2) of a minimal symplectic 4-manifold

$$X_{3,k,l}\sharp_T E(n)_K \quad \text{or} \quad Y_{m,k',l'}\sharp_T E(n)_K$$

for some integers m, k, k', l, l' and n. Finally, since $X_{3,k,l}$ and $Y_{m,k,l}$ contain a symplectic torus T in a Gompf nucleus $\subset E(2)$, we conclude from Proposition 2.2 above that both $X_{3,k,l}\sharp_T E(n)_K$ and $Y_{m,k,l}\sharp_T E(n)_K$ admit infinitely many distinct smooth structures and they have $\pi_1 = G$. □

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