THE GEOGRAPHY OF SYMPLECTIC 4-MANIFOLDS WITH AN ARBITRARY FUNDAMENTAL GROUP

JONGIL PARK

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Abstract. In this article, for each finitely presented group $G$, we construct a family of minimal symplectic 4-manifolds with $\pi_1 = G$ which cover most lattice points $(x, c)$ with $x$ large in the region $0 \leq c < 9x$. Furthermore, we show that all these 4-manifolds admit infinitely many distinct smooth structures.

1. Introduction

Applications of gauge theory to symplectic 4-manifolds have led to many remarkable results. For example, C. Taubes proved that every minimal symplectic 4-manifold with $b_2^+ > 1$ satisfies $c_1^2 \geq 0$ [1]. Topologists were able to prove that most lattice points $(x, c)$ satisfying $0 \leq c < 9x$ are realized as Chern numbers $(\chi, c_1^2)$ of simply connected minimal symplectic 4-manifolds, and most known simply connected minimal symplectic 4-manifolds with $b_2^+$ large enough admit infinitely many distinct symplectic structures ([PS], [GS], [P1]-[P3]). Note that $\chi = \frac{2 + e}{2}$ and $c_1^2 = 3\sigma + 2e$, where $\sigma$ and $e$ denote the signature and the Euler characteristic of a given 4-manifold. But little is known in the non-simply connected case.

The aim of this paper is to extend the results above partially to the non-simply connected case. Explicitly we have

Theorem 1.1. For each finitely generated group $G$, there are constants $r_G$ and $t_G$ such that every lattice point $(x, c)$ satisfying $0 \leq c \leq r_G x$ and $x \geq t_G$ is realized as $(\chi, c_1^2)$ of a minimal symplectic 4-manifold with $\pi_1 = G$ which admits infinitely many distinct smooth structures. Furthermore, the constant $r_G$ can be chosen so that it is close to 9.

Remarks 1. R. Gompf first constructed a new family of symplectic 4-manifolds with any prescribed fundamental group [3], and recently S. Baldridge and P. Kirk constructed a family of symplectic 4-manifolds with an arbitrary fundamental group near the Bogomolov-Miyaoka-Yau line [BK2]. Theorem 1.1 above addresses the uniqueness of smooth structures on such 4-manifolds as well as the existence of such 4-manifolds.
Remarks 2. The key ingredient in the proof of Theorem [11] above is to use S. Boyer’s result on simply connected 4-manifolds with a given boundary [3].

2. THE MAIN CONSTRUCTION

The main tactic in proving that many simply connected symplectic 4-manifolds admit infinitely many distinct smooth structures was to show that a family of simply connected homeomorphic 4-manifolds obtained by some topological surgeries have mutually different Seiberg-Witten invariants. We use the same idea in the non-simply connected case. We first briefly review such topological surgeries — called a fiber sum and a knot surgery — and state related theorems.

Definition. For $i = 1, 2$, let $X_i$ be a symplectic 4-manifold containing a symplectic (or Lagrangian) genus $g$ surface $\Sigma_g$ of square zero. Suppose that $X_i^0 = X_i - \nu_i(\Sigma_g)$ is a complement of a tubular neighborhood $\nu_i(\Sigma_g)$ of $\Sigma_g$ in $X_i$. Then, by choosing an orientation-reversing, fiber-preserving diffeomorphism $\varphi : \nu_1(\Sigma_g) \to \nu_2(\Sigma_g)$ and by gluing $X_1^0$ to $X_2^0$ along their boundaries via the diffeomorphism $\varphi| : \partial \nu_1(\Sigma_g) \to \partial \nu_2(\Sigma_g)$, one can get a new symplectic 4-manifold $X_1 \sharp_\Sigma_g X_2$, called a (symplectic) fiber sum of $X_1$ and $X_2$ along $\Sigma_g$.

Remarks 1. The Euler characteristic and the signature of a fiber sum 4-manifold $X_1 \sharp_\Sigma_g X_2$ are easily computed, so that it has

$$\chi(X_1 \sharp_\Sigma_g X_2) = \chi(X_1) + \chi(X_2) + (g-1)$$

and

$$c_1^2(X_1 \sharp_\Sigma_g X_2) = c_1^2(X_1) + c_1^2(X_2) + 8(g-1).$$

Remarks 2. The minimality is preserved under a symplectic fiber sum operation. That is, if both $X_1$ and $X_2$ are minimal symplectic 4-manifolds, so is $X_1 \sharp_\Sigma_g X_2$ (Theorem 2.5 in [11]).

Remarks 3. The Seiberg-Witten invariants of a fiber sum 4-manifold can be completely computed under certain cases. For example, the following product formula is widely known to the experts ([FS], [Pd]).

Theorem 2.1 ([Pd], Corollary 15 and Corollary 20). Suppose $X_1$ and $X_2$ are closed symplectic 4-manifolds which contain a symplectic torus $T$ in a cusp neighborhood. Then the Seiberg-Witten invariant of $X_1 \sharp_T X_2$ is given by

$$SW_{X_1 \sharp T X_2} = SW_{X_1} \cdot SW_{X_2} \cdot (\exp(|T|) - \exp(-|T|))^2.$$ 

Definition. Suppose $K$ is a fibered knot in $S^3$. Let $M_K$ be a 3-manifold obtained by performing 0-framed surgery on $K$. Then the 3-manifold $M_K$ can be considered as a fiber bundle over circle, so that $M_K \times S^1$ fibers over torus $T$. If $X$ is a symplectic 4-manifold with a symplectically embedded torus $T$ of square 0, then one can get a new symplectic 4-manifold $X_K := X \sharp T(M_K \times S^1)$, obtained by taking a (symplectic) fiber sum along $T$; we call this a knot surgery with a knot $K$. R. Fintushel and R. Stern proved that $X_K$ is homotopy equivalent to $X$ under a mild condition on $X$ and they also computed the Seiberg-Witten invariant of $X_K$.

Theorem 2.2 ([FS]). Suppose $X$ is a simply connected symplectic 4-manifold which contains a symplectic torus $T$ of square 0 in a cusp neighborhood with $\pi_1(X \setminus T) = 1$. If $K$ is a fibered knot in $S^3$, then $X_K$ is a symplectic 4-manifold which is homeomorphic to $X$ and whose Seiberg-Witten invariant is

$$SW_{X_K} = SW_X \cdot \Delta_K(t),$$

where $\Delta_K(t)$ is the Alexander polynomial of a knot $K$ and $t = \exp(2|T|).$
Next, we introduce some basic symplectic 4-manifolds which will serve as building blocks of our construction ([G], [GS] for details).

**Building Block 1.** Let $Q := Z_1 \sharp Z_2$ be a symplectic 4-manifold constructed as follows: First, consider a Thurston’s manifold $Z := \mathbb{R}^4 / G$, where $G$ is a discrete subgroup of symplectomorphisms generated by unit translations parallel to the $x^1$- and $x^2$-axes, together with the map $(x^3, \ldots, x^4) \mapsto (x^1 + x^2, x^2, x^3, x^4 + 1)$. Note that projection onto the last two coordinates induces a bundle structure $\pi : Z \rightarrow \mathbb{T}^2$ with symplectic torus fibers. Next, using two copies, $\pi_1 : Z_i \rightarrow Z^2 (i = 1, 2)$ of Thurston’s manifold and using an orientation-reversing bundle map $\psi$ induced from $90^\circ$ rotation $\psi : \tau_1^{-1}(0) \rightarrow \tau_2^{-1}(0)$ defined by $\psi(x^1, x^2) = (-x^2, x^1)$, one obtains a symplectic fiber sum $Q := Z_1 \sharp Z_2$. This is a torus bundle over a genus 2 surface $T^2 \times \mathbb{T}^2$ and has a symplectic section $\Sigma_2$ of square 0 given by sections in $Z_1$ and $Z_2$ such that $\pi_1(Q - \Sigma_2)/\pi_1(\Sigma_2') = 1$, where $\Sigma_2'$ is a parallel copy of $\Sigma_2$. Similarly, there is also a Lagrangian torus $T \subset Q$ of square 0, disjoint from $\Sigma_2$, in $Q$. For example, one obtains such a torus $T$ by setting $x^1 = x^4 = 1/2$ in $Z_1$.

**Building Block 2.** Let $E(n)$ be a simply connected elliptic surface with no multiple fibers and holomorphic Euler characteristic $n$. Then $E(n)$ can be obtained as an algebraic surface $B(2, 3, 6n - 1) \cup_{C(2,3,6n-1)} C(n)$, where $B(2, 3, 6n - 1)$ is a Brieskorn manifold and $C(n)$, usually called a Gompf nucleus, is the neighborhood of a cusp fiber and a section which is an embedded 2-sphere of square $-n$. By performing a knot surgery with a fibered knot $K \subset S^3$ in $C(n)$, one obtains a simply connected symplectic 4-manifold $E(n)_K$ which is homeomorphic, but not diffeomorphic, to $E(n)$. Since a Brieskorn manifold $B(2, 3, 6n - 1)$ with $n \geq 2$ contains a Lagrangian torus $T$ of square 0, which intersects 2-sphere transversely at a single point, lying in another Gompf nucleus, we conclude that $E(n)_K$ also contains a Lagrangian torus $T$ in a Gompf nucleus.

Using the building blocks above together with a Lefschetz fibration lying on the BMY-line [S], we constructed various families of simply connected, minimal, symplectic 4-manifolds which solve many geography problems in the simply connected case. Among them, we quote the following basic constructions.

**Lemma 2.1** ([P2], Lemma 2.1). For each integer $k$, $10 \leq k \leq 18$, there exists a simply connected, minimal, symplectic 4-manifold $X_{3,k}$ with $\chi = 2$ and $c_1^2 = 19 - k$ which contains a symplectic genus 2 surface $\Sigma_2$ of square 0 and a symplectic torus $T$ of square 0, disjoint from $\Sigma_2$, in a fishtail neighborhood, and $\pi_1(X_{3,k} - \Sigma_2) = \pi_1(X_{3,k} - T) = 1$.

**Proposition 2.1** ([P3], Proposition 2.1). For each odd integer $m \geq 1$ and $10 \leq k \leq 18$, there exists a simply connected, minimal, symplectic 4-manifold $Z_{m,k}$ which contains a symplectic genus 2 surface $\Sigma_2$ of square 0 and a torus $T$ of square 0, disjoint from $\Sigma_2$, in a fishtail neighborhood which satisfies

$$\pi_1(Z_{m,k} - \Sigma_2) = \pi_1(Z_{m,k} - T) = 1.$$  

Furthermore, it has $\chi(Z_{m,k}) = 25m^2 + 31m + 5$ and $c_1^2(Z_{m,k}) = 225m^2 + 248m + 35 - k$.

**Building Block 3.** Let $G$ be a finitely presented group with $g$ generators $\{x_1, \ldots, x_g\}$ and $r$ relations $\{w_1, \ldots, w_r\}$. Let $F$ be an oriented genus $g$ Riemann surface with an oriented circles $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ representing a standard symplectic basis
for $H_1(F)$. For $i = 1, \ldots, r$, let $\gamma_i$ be a smoothly immersed, oriented circle in $F$ representing the word $w_i$ in this group, and, for $i = 1, \ldots, g$, let $\gamma_{T+i} = \beta_i$ and identifying $x_i = \alpha_i$, we have $\pi_1(F)/(\gamma_1, \ldots, \gamma_{g+r}) \cong G$. By forming connected sums of $F$ with a copy of torus $T$ and perturbing and rearranging $\gamma_i$, R. Gompf constructed a collection of symplectic tori $T_i$ corresponding to $\gamma_i \times \alpha$ in $F \times T$, where $\alpha$ is one of the oriented circles representing a standard basis of $H_1(T)$. Let $M = (F \times T) \# T \cup_r E(1) \# (pt) \times \mathbb{T}$ be a symplectic 4-manifold obtained by symplectically fiber summing $F \times T$ with $E(1)$’s along each $T_i$ and $\{pt\} \times T$. Then, since $E(1) - T$ is simply connected, $\pi_1(M) \cong G$. Furthermore, $M$ contains a symplectic torus $T = \{pt\} \times \mathbb{T}$ such that $\pi_1(M)/\pi_1(T) \cong \pi_1(M) = G$. Recently, by modifying the construction above, S. Baldridge and P. Kirk constructed a similar symplectic 4-manifold with $\pi_1 = G$.

**Theorem 2.3 ([BK1], Theorem 6).** Let $G$ have a presentation with $g$ generators $\{x_1, \ldots, x_g\}$ and $r$ relations $\{w_1, \ldots, w_r\}$. Then there exists a closed symplectic 4-manifold $M$ with $\pi_1(M) = G$ which has $\chi(M) = g + r + 1$ and $c_1^2(M) = 0$.

**Remark 1.** The symplectic 4-manifold $M$ constructed in Theorem 2.3 above also contains a symplectic torus $T$ of square 0 such that the inclusion $i : T \to M$ induces a zero map $i_* \equiv 0 : \pi_1(T) \to \pi_1(M) = G$. In fact, since $E(1) - T$ is simply connected, the inclusion $i : T'' \to M - T$ also induces a zero map $i_* \equiv 0 : \pi_1(T'') \to \pi_1(M - T)$, where $T''$ is a parallel copy of $T$.

Suppose $X_{3,k}, Z_{m,k}$ and $M$ are symplectic 4-manifolds constructed in Lemma 2.1 Proposition 2.1 and Theorem 2.3 respectively. Since $X_{3,k}$ contains a symplectic surface $\Sigma_2$ of square 0, taking a fiber sum $l$ times along $\Sigma_2$ in $X_{3,k}$ with $l$ copies of $Q$, we get a symplectic 4-manifold $X_{3,k} \#_{\Sigma_2} Q := (\cdots((X_{3,k} \#_{\Sigma_2} Q) \#_{\Sigma_2} Q) \#_{\Sigma_2} \cdots \#_{\Sigma_2} Q)$. Again, since $Q$ contains a Lagrangian torus $T'$ of square 0 disjoint from $\Sigma_2$, we can also construct a symplectic 4-manifold $(X_{3,k} \#_{\Sigma_2} Q) \#_{T'} E(2)$ by taking a fiber sum along $T'$ in $Q$ and $C(2) \subset E(2)$. Furthermore, there is another symplectic torus $T$ lying in $M$ and $X_{3,k} - \Sigma_2$. Hence, by taking a fiber sum along $T$, we obtain a new symplectic 4-manifold

$$X_{3,k,l} := M \#_{T'} ((X_{3,k} \#_{\Sigma_2} Q) \#_{T'} E(2)).$$

Similarly, using $Z_{m,k}$ instead of $X_{3,k}$, we obtain a symplectic 4-manifold

$$Y_{m,k,l} := M \#_{T'} ((Z_{m,k} \#_{\Sigma_2} Q) \#_{T'} E(2)).$$

Then we have

**Corollary 2.1.** For each integer $k$ and $l$, $10 \leq k \leq 18$ and $l \geq 0$, $X_{3,k,l}$ is a minimal symplectic 4-manifold with $\pi_1(X_{3,k,l}) \cong \pi_1(M) = G$ which satisfies $\chi(X_{3,k,l}) = l + g + r + 5$ and $c_1^2(X_{3,k,l}) = 8l + 19 - k$.

**Proof.** It suffices to prove that $\pi_1(X_{3,k,l}) \cong \pi_1(M)$. For this, we first show that $\pi_1((X_{3,k} - T) \#_{\Sigma_2} Q) = 1$: Note that the Van-Kampen theorem on $\pi_1(X_{3,k} - T) = 1$ implies that $\pi_1(\Sigma_2^g) \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is a zero map and $\pi_1(\partial \nu(\Sigma_2)) \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is surjective, where $\Sigma_2^g$ is a parallel copy of $\Sigma_2$. Since $\pi_1(Q - \Sigma_2)/\pi_1(\Sigma_2^g) = 1$, we have

$$\pi_1((X_{3,k} - T) \#_{\Sigma_2} Q) = \pi_1((X_{3,k} - T) - \nu(\Sigma_2)) * \pi_1(Q - \nu(\Sigma_2))/\pi_1(\partial \nu(\Sigma_2))$$

$$\cong \pi_1((X_{3,k} - T) - \nu(\Sigma_2))/\pi_1(\partial \nu(\Sigma_2))$$

$$\cong 1.$$
Now, by induction on \(l\), we conclude that \(\pi_1((X_{3,k} - T)\sharp \Sigma_l Q) = 1\).

Next, since both \((X_{3,k} - T)\sharp \Sigma_l Q\) and \(E(2) - T'\) are simply connected, the fiber sum 4-manifold \(((X_{3,k} - T)\sharp \Sigma_l Q)\sharp T' E(2) \simeq ((X_{3,k} - \nu(T))\sharp \Sigma_l Q)\sharp T' E(2)\) is also simply connected.

Finally, since the inclusion \(i : T'' \rightarrow M \rightarrow T\) induces a zero map \(i_* \equiv 0 : \pi_1(T'') \rightarrow \pi_1(M - T) \simeq \pi_1(M - \nu(T))\) (see Remark 1 above), the Van-Kampen theorem implies that

\[
\pi_1(X_{3,k,l}) = \pi_1(M - \nu(T)) * \pi_1(((X_{3,k}, \nu(T))\sharp \Sigma_l Q)\sharp T' E(2))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
= \pi_1(M - \nu(T)) * \pi_1(((X_{2,k} - \nu(T))\sharp \Sigma_l Q)\sharp T' E(2))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
\cong \pi_1(M - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
\cong \pi_1(M). \quad \Box
\]

**Corollary 2.2.** For each odd integer \(m \geq 1\), \(10 \leq k \leq 18\) and \(l \geq 0\), \(Y_{m,k,l}\) is a minimal symplectic 4-manifold with \(\pi_1(Y_{m,k,l}) \cong \pi_1(M) = G\) which satisfies \(\chi(Y_{m,k,l}) = 25m^2 + 131m + 1 + g + r + 8\) and \(c_2(Y_{m,k,l}) = 225m^2 + 248m + 8l + 35 - k\).

**Proof.** Similar to the proof of Corollary 2.1 above, \(\pi_1(Z_{m,k} - T = 1\) implies that \(\pi_1((Z_{m,k} - \nu(T))\sharp \Sigma_l Q)\sharp T' E(2)) = 1\). Hence Van-Kampen theorem again induces

\[
\pi_1(Y_{m,k,l}) = \pi_1(M - \nu(T)) * \pi_1(((Z_{m,k} - \nu(T))\sharp \Sigma_l Q)\sharp T' E(2))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
= \pi_1(M - \nu(T)) * \pi_1(((Z_{m,k} - \nu(T))\sharp \Sigma_l Q)\sharp T' E(2))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
\cong \pi_1(M - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle
\]

\[
\cong \pi_1(M). \quad \Box
\]

Next, we introduce S. Boyer’s result on simply connected 4-manifolds with a given boundary [3], which will be a key ingredient in proving our result.

Let \(V_1\) and \(V_2\) be simply connected, compact, oriented 4-manifolds with connected boundary \(\partial V_1 = \partial V_2\) and suppose that \(f : \partial V_1 \rightarrow \partial V_2\) is an orientation preserving homeomorphism. Then there are two obstructions to extending \(f\) to a homeomorphism \(F : V_1 \rightarrow V_2\): The first is to find an isometry \(\Lambda : (H_2(V_1; \mathbb{Z}), Q_{V_1}) \rightarrow (H_2(V_2; \mathbb{Z}), Q_{V_2})\) for which the following diagram commutes:

\[
\begin{array}{ccc}
0 & \rightarrow & H_2(\partial V_1) \\
\downarrow f_* & & \downarrow \Lambda \\
0 & \rightarrow & H_2(\partial V_2)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & H_2(\partial V_1) \\
\downarrow \Lambda^* & & \downarrow f_* \\
0 & \rightarrow & H_2(\partial V_2)
\end{array}
\]

Here an isometry means an isomorphism \(\Lambda : H_2(V_1; \mathbb{Z}) \rightarrow H_2(V_2; \mathbb{Z})\) which preserves an intersection form \(Q_{V_1}\), and \(\Lambda^*\) is the adjoint of \(\Lambda\) with respect to the identification of \(H_2(V_i, \partial V_i) \rightarrow Hom(H_2(V_i); \mathbb{Z})\) arising from Lefschetz duality. We call such a pair \((f, \Lambda)\) a morphism and denote it symbolically as \((f, \Lambda) : V_1 \rightarrow V_2\). The second obstruction encountered is to realize a given morphism \((f, \Lambda)\) geometrically, i.e. to find a homeomorphism \(F : V_1 \rightarrow V_2\) such that \((f, \Lambda) = (F|_{\partial V_1}, F_*).\)

S. Boyer showed when a given morphism could be realized geometrically. Explicitly, he proved

**Theorem 2.4 ([3], Theorem 0.7 and Proposition 0.8).** If \((f, \Lambda) : V_1 \rightarrow V_2\) is a morphism between two simply connected smooth 4-manifolds \(V_1\) and \(V_2\) with boundary \(\partial V_1 = \partial V_2\), there is an obstruction \(\theta(f, \Lambda) \in \Pi^1(\partial V_1)\) such that \((f, \Lambda)\) is realized
geometrically if and only if \( \theta(f, \Lambda) = 0 \). Furthermore, if \( H_1(\partial V_1; \mathbb{Q}) = 0 \), then \( \theta(f, \Lambda) = 0 \).

**Corollary 2.3.** Suppose that \( \partial V_1 = \partial V_2 \) is a homology 3-sphere. Then, for any morphism \((f, \Lambda) : V_1 \to V_2\) between two simply connected smooth 4-manifolds \(V_1\) and \(V_2\), there is a homeomorphism \( F : V_1 \to V_2 \) such that \((f, \Lambda) = (F|_{\partial V_1}, F_*).\)

Finally, we state and prove a key proposition.

**Proposition 2.2.** Suppose \(X\) is a symplectic 4-manifold which contains a symplectic torus \(T\) lying in a cusp neighborhood. Then, for each integer \(n \geq 2\), a family of symplectic 4-manifolds \(\{X_T^*(E(n)_K) \mid K\}\) is a fibered knot in \(S^3\) are mutually non-diffeomorphic, but all homeomorphic. Furthermore, \(\pi_1(X_T^*(E(n)_K)) \cong \pi_1(X)\).

**Proof.** Since \(E(n)_K\) contains a Gompf nucleus, say \(C_n\), embedded in \(B(2, 3, 6n - 1)\) (Building Block 2), we can decompose \(E(n)_K\) into \(C_n \cup E(n)_{K'}^n\), where \(\Sigma\) is a homology 3-sphere and \(E(n)_{K'}^n\) denotes a complement of \(C_n\) in \(E(n)_K\). Note that the Gompf nucleus \(C_n\) is independent of \(K\), i.e. \(C_n\) is a common submanifold of codimension zero in all \(E(n)_K\)'s. Hence, after taking a symplectic fiber sum of \(X\) with \(E(n)_K\) along a Lagrangian torus \(T \subset C_n\), we get a decomposition

\[X_T^*(E(n)_K) = (X_T^*(C_n) \cup (E(n)_{K'}^n).\]

In order to show that \(X_T^*(E(n)_K)\) is homeomorphic to \(X_T^*(E(n)_{K'}^n)\) for any fibered knots \(K\) and \(K'\) in \(S^3\), we first choose an identity map \(f = \text{id} : X_T^*(C_n) \to X_T^*(C_n)\) and an isomorphism \(\Lambda : H_2(E(n)_K; \mathbb{Z}) \to H_2(E(n)_{K'}^n; \mathbb{Z})\). In particular, since both intersection forms \(Q_{E(n)_K}Q_{E(n)_{K'}^n}\) are unimodular and indefinite, they are uniquely determined by rank, signature and type, so that they are isomorphic. (The reason is following: Since \(E(n)_K\) and \(E(n)_{K'}^n\) are homeomorphic, their intersection forms have the same rank, signature and type. Hence \(Q_{E(n)_K}\) and \(Q_{E(n)_{K'}^n}\) also have the same rank, signature and type.) Hence we may choose \(\Lambda\) to be an isometry, i.e. \(\Lambda : (H_2(E(n)_{K'}^n; Q_{E(n)_{K'}^n}) \to (H_2(E(n)_K; Q_{E(n)_K})).\) Thus there is a homomorphism between two symplectic 4-manifolds \(X_T^*(E(n)_K)\) and \(X_T^*(E(n)_{K'}^n)\).

Next, applying Theorem 2.1 and Theorem 2.2 above, we also conclude that \(X_T^*(E(n)_K)\) and \(X_T^*(E(n)_{K'}^n)\) have different Seiberg-Witten invariants as long as \(\Delta_K(t) \neq \Delta_{K'}(t)\), i.e. \(X_T^*(E(n)_K)\) is not diffeomorphic to \(X_T^*(E(n)_{K'}^n)\).

The last statement follows from the Van-Kampen theorem using the fact that \(T\) is a regular fiber in a cusp neighborhood and \(\pi_1(E(n)_K - T) = 1:\)

\[\pi_1(X_T^*(E(n)_K)) = \pi_1(X - \nu(T)) \ast \pi_1(E(n)_K - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle\]
\[\cong \pi_1(X - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle\]
\[\cong \pi_1(X - \nu(T)) \ast \pi_1(\nu(T))/\langle \pi_1(\partial \nu(T)) \rangle\]
\[\cong \pi_1(X).\]

\[\square\]

**Proof of Theorem 2.1.** Let \(x_{3,k,l}\) and \(y_{m,k,l}\) be minimal symplectic 4-manifolds appeared in Corollary 2.1 and Corollary 2.2 above. For each odd integer \(m \geq 1\) with a finitely presented group \(G\) fixed (so that \(g\) and \(r\) are also fixed), define two
numbers \( r_{G,m} \) and \( t_{G,m} \) by
\[
\begin{align*}
\begin{cases}
  r_{G,m} := \frac{c_1^2(Y_{m,10.112.5m+173.5})}{\chi(Y_{m,10.112.5m+173.5})} = \frac{225m^2 + 1148m + 1413}{25m^2 + 143.5m + 181.5 + g + r} \\
  t_{G,m} := 25m^2 + 143.5m + 181.5 + g + r.
\end{cases}
\end{align*}
\]
Then \( \{r_{G,m}\} \) is an increasing sequence converging to 9 and any lattice point \((x,c)\) satisfying \(0 \leq c \leq r_{G,m}x\) and \(x \geq t_{G,m}\) is realized as Chern numbers \((\chi, c_1^2)\) of a minimal symplectic 4-manifold
\[
X_{3,k,l} \sharp T E(n)_K \text{ or } Y_{m,k',l'} \sharp T E(n)_K
\]
for some integers \(m, k, k', l, l'\) and \(n\). Finally, since \(X_{3,k,l}\) and \(Y_{m,k,l}\) contain a symplectic torus \(T\) in a Gompf nucleus \(\subset E(2)\), we conclude from Proposition 2.2 above that both \(X_{3,k,l} \sharp T E(n)_K\) and \(Y_{m,k,l} \sharp T E(n)_K\) admit infinitely many distinct smooth structures and they have \(\pi_1 = G\). \(\square\)

References


