THE GEOGRAPHY OF SYMPLECTIC 4-MANIFOLDS
WITH AN ARBITRARY FUNDAMENTAL GROUP

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Abstract. In this article, for each finitely presented group $G$, we construct a
family of minimal symplectic 4-manifolds with $\pi_1 = G$ which cover most lattice
points $(x, c)$ with $x$ large in the region $0 \leq c < 9x$. Furthermore, we show that
all these 4-manifolds admit infinitely many distinct smooth structures.

1. Introduction

Applications of gauge theory to symplectic 4-manifolds have led to many re-
markable results. For example, C. Taubes proved that every minimal symplectic
4-manifold with $b_2^+ > 1$ satisfies $c_1^2 \geq 0$ \cite{T}. Topologists were able to prove that
most lattice points $(x, c)$ satisfying $0 \leq c < 9x$ are realized as chern numbers
$(\chi, c_1^2)$ of simply connected minimal symplectic 4-manifolds, and most known sim-
ply connected minimal symplectic 4-manifolds with $b_2^+$ large enough admit infinitely
many distinct symplectic structures \cite{FS, GS, P1-P3}. Note that $\chi = \frac{2e+\sigma}{2}$ and
$c_1^2 = 3\sigma + 2e$, where $\sigma$ and $e$ denote the signature and the Euler characteristic of a
given 4-manifold. But little is known in the non-simply connected case.

The aim of this paper is to extend the results above partially to the non-simply
connected case. Explicitly we have

Theorem 1.1. For each finitely generated group $G$, there are constants $r_G$ and $t_G$
such that every lattice point $(x, c)$ satisfying $0 \leq c \leq r_G x$ and $x \geq t_G$ is realized
as $(\chi, c_1^2)$ of a minimal symplectic 4-manifold with $\pi_1 = G$ which admits infinitely
many distinct smooth structures. Furthermore, the constant $r_G$ can be chosen so
that it is close to $9$.

Remarks. 1. R. Gompf first constructed a new family of symplectic 4-manifolds
with any prescribed fundamental group \cite{G}, and recently S. Baldridge and P. Kirk
constructed a family of symplectic 4-manifolds with an arbitrary fundamental group
near the Bogomolov-Miyaoka-Yau line \cite{BK2}. Theorem 1.1 above addresses the
uniqueness of smooth structures on such 4-manifolds as well as the existence of
such 4-manifolds.
Remarks 2. The key ingredient in the proof of Theorem 1.1 above is to use S. Boyer’s result on simply connected 4-manifolds with a given boundary [3].

2. The main construction

The main tactic in proving that many simply connected symplectic 4-manifolds admit infinitely many distinct smooth structures was to show that a family of simply connected homeomorphic 4-manifolds obtained by some topological surgeries have mutually different Seiberg-Witten invariants. We use the same idea in the non-simply connected case. We first briefly review such topological surgeries — called a fiber sum and a knot surgery — and state related theorems.

Definition. For $i = 1, 2$, let $X_i$ be a symplectic 4-manifold containing a symplectic (or Lagrangian) genus $g$ surface $\Sigma_g$ of square zero. Suppose that $X^0_i = X_i - \nu_i(\Sigma_g)$ is a complement of a tubular neighborhood $\nu_i(\Sigma_g)$ of $\Sigma_g$ in $X_i$. Then, by choosing an orientation-reversing, fiber-preserving diffeomorphism $\varphi : \nu_i(\Sigma_g) \to \nu_2(\Sigma_g)$ and by gluing $X^0_1$ to $X^0_2$ along their boundaries via the diffeomorphism $\varphi| : \partial \nu_1(\Sigma_g) \to \partial \nu_2(\Sigma_g)$, one can get a new symplectic 4-manifold $X^*_{1\Sigma_g}X_2$, called a (symplectic) fiber sum of $X_1$ and $X_2$ along $\Sigma_g$.

Remarks 1. The Euler characteristic and the signature of a fiber sum 4-manifold $X^*_{1\Sigma_g}X_2$ are easily computed, so that it has
\[
\chi(X^*_{1\Sigma_g}X_2) = \chi(X_1) + \chi(X_2) + (g - 1) \quad \text{and} \quad \varepsilon(X^*_{1\Sigma_g}X_2) = \varepsilon(X_1) + \varepsilon(X_2) + 8(g - 1).
\]

Remarks 2. The minimality is preserved under a symplectic fiber sum operation. That is, if both $X_1$ and $X_2$ are minimal symplectic 4-manifolds, so is $X^*_{1\Sigma_g}X_2$ (Theorem 2.5 in [2]).

Remarks 3. The Seiberg-Witten invariants of a fiber sum 4-manifold can be completely computed under certain cases. For example, the following product formula is widely known to the experts ([4], [5]).

Theorem 2.2 ([4], Corollary 15 and Corollary 20). Suppose $X_1$ and $X_2$ are closed symplectic 4-manifolds which contain a symplectic torus $T$ in a cusp neighborhood. Then the Seiberg-Witten invariant of $X^*_{1\Sigma_g}X_2$ is given by
\[
\operatorname{SW}_{X^*_{1\Sigma_g}X_2} = \operatorname{SW}_{X_1} \cdot \operatorname{SW}_{X_2} \cdot (\exp([T]) - \exp(-[T]))^2.
\]

Definition. Suppose $K$ is a fibered knot in $S^3$. Let $M_K$ be a 3-manifold obtained by performing 0-framed surgery on $K$. Then the 3-manifold $M_K$ can be considered as a fiber bundle over circle, so that $M_K \times S^1$ fibers over torus $T$. If $X$ is a symplectic 4-manifold with a symplectically embedded torus $T$ of square 0, then one can get a new symplectic 4-manifold $X_K := X^*_{T}(M_K \times S^1)$, obtained by taking a (symplectic) fiber sum along $T$; we call this a knot surgery with a knot $K$. R. Fintushel and R. Stern proved that $X_K$ is homotopy equivalent to $X$ under a mild condition on $X$ and they also computed the Seiberg-Witten invariant of $X_K$.

Theorem 2.2 ([4]). Suppose $X$ is a simply connected symplectic 4-manifold which contains a symplectic torus $T$ of square 0 in a cusp neighborhood with $\pi_1(X \setminus T) = 1$. If $K$ is a fibered knot in $S^3$, then $X_K$ is a symplectic 4-manifold which is homeomorphic to $X$ and whose Seiberg-Witten invariant is
\[
\operatorname{SW}_{X_K} = \operatorname{SW}_{X} \cdot \Delta_K(t),
\]
where $\Delta_K(t)$ is the Alexander polynomial of a knot $K$ and $t = \exp(2[T])$. 

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Next, we introduce some basic symplectic 4-manifolds which will serve as building blocks of our construction ([G], [GS] for details).

**Building Block 1.** Let \( Q := Z_1 \sharp_c Z_2 \) be a symplectic 4-manifold constructed as follows: First, consider a Thurston’s manifold \( Z := \mathbb{R}^4/G \), where \( G \) is a discrete subgroup of symplectomorphisms generated by unit translations parallel to the \( x^1-, x^2-, \) and \( x^3-\) axes, together with the map \( (x^1, \ldots, x^4) \mapsto (x^1 + x^2, x^2, x^3, x^4 + 1) \). Note that projection onto the last two coordinates induces a bundle structure \( \pi : Z \to \mathbb{T}^2 \) with symplectic torus fibers. Next, using two copies, \( \pi_i : Z_i \to \mathbb{T}^2 \) \( (i = 1, 2) \) of Thurston’s manifold and using an orientation-reversing bundle map \( \psi \) induced from 90° rotation \( \psi_n : \pi_1^{-1}(0) \to \pi_2^{-1}(0) \) defined by \( \psi_n(x^1, x^2) = (-x^2, x^1) \), one obtains a symplectic fiber sum \( Q := Z_1 \sharp \psi Z_2 \). This is a torus bundle over a genus 2 surface \( \mathbb{T}^2 \sharp \mathbb{T}^2 \) and has a symplectic section \( \Sigma_2 \) of square 0 given by sections in \( Z_1 \) and \( Z_2 \) such that \( \pi_1(Q - \Sigma_2)/\pi_1(\Sigma_2') = 1 \), where \( \Sigma_2' \) is a parallel copy of \( \Sigma_2 \). Similarly, there is also a Lagrangian torus \( T \subset Q \) of square 0, disjoint from \( \Sigma_2 \), in \( Q \). For example, one obtains such a torus \( T \) by setting \( x^1 = x^4 = 1/2 \) in \( Z_1 \).

**Building Block 2.** Let \( E(n) \) be a simply connected elliptic surface with no multiple fibers and holomorphic Euler characteristic \( n \). Then \( E(n) \) can be obtained as an algebraic surface \( B(2, 3, 6n-1) \cup_{\Sigma_2} V(2, 3, 6n-1) \) \( C(n) \), where \( B(2, 3, 6n-1) \) is a Brieskorn manifold and \( C(n) \), usually called a Gompf nucleus, is the neighborhood of a cusp fiber and a section which is an embedded 2-sphere of square \( -n \). By performing a knot surgery with a fibered knot \( K \subset S^3 \) in \( C(n) \), one obtains a simply connected symplectic 4-manifold \( E(n)_K \) which is homeomorphic, but not diffeomorphic, to \( E(n) \). Since a Brieskorn manifold \( B(2, 3, 6n-1) \) with \( n \geq 2 \) contains a Lagrangian torus \( T \) of square 0, which intersects 2-sphere transversely at a single point, lying in another Gompf nucleus, we conclude that \( E(n)_K \) also contains a Lagrangian torus \( T \) in a Gompf nucleus.

Using the building blocks above together with a Lefschetz fibration lying on the BMY-line [S], we constructed various families of simply connected, minimal, symplectic 4-manifolds which solve many geography problems in the simply connected case. Among them, we quote the following basic constructions.

**Lemma 2.1** ([P2], Lemma 2.1). For each integer \( k, \) \( 10 \leq k \leq 18 \), there exists a simply connected, minimal, symplectic 4-manifold \( X_{3,k} \) with \( \chi = 2 \) and \( c_1^2 = 19 - k \) which contains a symplectic genus 2 surface \( \Sigma_2 \) of square 0 and a symplectic torus \( T \) of square 0, disjoint from \( \Sigma_2 \), in a fishtail neighborhood, and \( \pi_1(X_{3,k} - \Sigma_2) = \pi_1(X_{3,k} - T) = 1 \).

**Proposition 2.1** ([P3], Proposition 2.1). For each odd integer \( m \geq 1 \) and \( 10 \leq k \leq 18 \), there exists a simply connected, minimal, symplectic 4-manifold \( Z_{m,k} \) which contains a symplectic genus 2 surface \( \Sigma_2 \) of square 0 and a torus \( T \) of square 0, disjoint from \( \Sigma_2 \), in a fishtail neighborhood which satisfies

\[
\pi_1(Z_{m,k} - \Sigma_2) = \pi_1(Z_{m,k} - T) = 1.
\]

Furthermore, it has \( \chi(Z_{m,k}) = 25m^2 + 31m + 5 \) and \( c_1^2(Z_{m,k}) = 225m^2 + 248m + 35 - k \).

**Building Block 3.** Let \( G \) be a finitely presented group with \( g \) generators \( \{x_1, \ldots, x_g\} \) and \( r \) relations \( \{w_1, \ldots, w_r\} \). Let \( F \) be an oriented genus \( g \) Riemann surface with an oriented circles \( \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\} \) representing a standard symplectic basis.
for $H_1(F)$. For $i = 1, \ldots, r$, let $\gamma_i$ be a smoothly immersed, oriented circle in $F$ representing the word $w_i$ in this group, and, for $i = 1, \ldots, g$, let $\gamma_{r+i} = \beta_i$ and identifying $x_i = \alpha_i$, we have $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_{g+r} \rangle \cong G$. By forming connected sums of $F$ with a copy of torus $T$ and perturbing and rearranging $\gamma_i$, R. Gompf constructed a collection of symplectic tori $T_i$ corresponding to $\gamma_i \times \alpha$ in $F \times T$, where $\alpha$ is one of the oriented circles representing a standard basis of $H_1(T)$. Let $M = (F \times T) \cup_1 E(1) \cup_1 \mathbb{T}$ be a symplectic 4-manifold obtained by symplectically fiber summing $F \times \mathbb{T}$ with $E(1)$'s along each $T_i$ and $\{pt\} \times \mathbb{T}$. Then, since $E(1) - T$ is simply connected, $\pi_1(M) \cong G$. Furthermore, $M$ contains a symplectic torus $T = \{pt\} \times T$ such that $\pi_1(M)/\pi_1(T) \cong \pi_1(M) = G$. Recently, by modifying the construction above, S. Baldridge and P. Kirk constructed a similar symplectic 4-manifold with $\pi_1 = G$.

**Theorem 2.3 ([BK1], Theorem 6).** Let $G$ have a presentation with $g$ generators $\{x_1, \ldots, x_g\}$ and $r$ relations $\{w_1, \ldots, w_r\}$. Then there exists a closed symplectic 4-manifold $M$ with $\pi_1(M) = G$ which has $\chi(M) = g + r + 1$ and $c_1^2(M) = 0$.

**Remark 1.** The symplectic 4-manifold $M$ constructed in Theorem 2.3 above also contains a symplectic torus $T$ of square 0 such that the inclusion $i : T \to M$ induces a zero map $i_* \equiv 0 : \pi_1(T) \to \pi_1(M) = G$. In fact, since $E(1) - T$ is simply connected, the inclusion $i : T'' \to M - T$ also induces a zero map $i_* \equiv 0 : \pi_1(T'') \to \pi_1(M - T)$, where $T''$ is a parallel copy of $T$.

Suppose $X_{3,k}, Z_{m,k}$ and $M$ are symplectic 4-manifolds constructed in Lemma 2.1 Proposition 2.1 and Theorem 2.3 respectively. Since $X_{3,k}$ contains a symplectic surface $\Sigma_2$ of square 0, taking a fiber sum $l$ times along $\Sigma_2$ in $X_{3,k}$ with $l$ copies of $Q$, we get a symplectic 4-manifold $X_{3,k}(\Sigma_2)^Q := (\cdots((X_{3,k}(\Sigma_2)^Q)\Sigma_2)\cdots)\Sigma_2)^Q$. Again, since $Q$ contains a Lagrangian torus $T'$ of square 0 disjoint from $\Sigma_2$, we can also construct a symplectic 4-manifold $(X_{3,k}(\Sigma_2)^Q)\Sigma_2(E(2))$ by taking a fiber sum along $T'$ in $Q$ and $C(2) \subset E(2)$. Furthermore, there is another symplectic torus $T$ lying in $M$ and $X_{3,k} - \Sigma_2$. Hence, by taking a fiber sum along $T$, we obtain a new symplectic 4-manifold

$$X_{3,k,l} := M^l_{\Sigma_2}((X_{3,k}(\Sigma_2)^Q)\Sigma_2(E(2))).$$

Similarly, using $Z_{m,k}$ instead of $X_{3,k}$, we obtain a symplectic 4-manifold

$$Y_{m,k,l} := M^l_{\Sigma_2}((Z_{m,k}(\Sigma_2)^Q)\Sigma_2(E(2))).$$

Then we have

**Corollary 2.1.** For each integer $k$ and $l$, $10 \leq k \leq 18$ and $l \geq 0$, $X_{3,k,l}$ is a minimal symplectic 4-manifold with $\pi_1(X_{3,k,l}) \cong \pi_1(M) = G$ which satisfies $\chi(X_{3,k,l}) = l + g + r + 5$ and $c_1^2(X_{3,k,l}) = 8l + 19 - k$.

**Proof.** It suffices to prove that $\pi_1(X_{3,k,l}) \cong \pi_1(M)$. For this, we first show that $\pi_1((X_{3,k} - T)(\Sigma_2)^Q) = 1$. Note that the Van-Kampen theorem on $\pi_1(X_{3,k} - T) = 1$ implies that $\pi_1((\Sigma_2)^Q) \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is a zero map and $\pi_1(\partial(\Sigma_2)) \to \pi_1((X_{3,k} - T) - \Sigma_2)$ is surjective, where $\Sigma_2$ is a parallel copy of $\Sigma_2$. Since $\pi_1(\Sigma_2)/\pi_1(\Sigma_2) = 1$, we have

$$\pi_1((X_{3,k} - T)(\Sigma_2)^Q) = \pi_1((X_{3,k} - T) - \nu(\Sigma_2)) * \pi_1(Q - \nu(\Sigma_2))/\langle \pi_1(\partial(\nu(\Sigma_2))) \rangle \cong \pi_1((X_{3,k} - T) - \nu(\Sigma_2))/\langle \pi_1(\partial(\nu(\Sigma_2))) \rangle \cong 1.$$
Now, by induction on \(l\), we conclude that \(\pi_1((X_{3,k} - T)\sharp_2IQ) = 1\).

Next, since both \((X_{3,k} - T)\sharp_2IQ\) and \(E(2) - T'\) are simply connected, the fiber sum 4-manifold \(((X_{3,k} - T)\sharp_2IQ)\sharp_2T; E(2) \simeq ((X_{3,k} - \nu(T))\sharp_2IQ)\sharp_2T; E(2)\) is also simply connected.

Finally, since the inclusion \(i : T'' \to M - T\) induces a zero map \(i_* \equiv 0 : \pi_1(T'') \to \pi_1(M - T) \cong \pi_1(M - \nu(T))\) (see Remark \([\text{I}]\) above), the Van-Kampen theorem implies that

\[
\pi_1(X_{3,k,l}) = \pi_1(M - \nu(T)) \ast \pi_1(((X_{3,k}\sharp_2IQ)\sharp_2T; E(2) - \nu(T))/\langle \pi_1(\partial\nu(T)) \rangle
\]

\[
\cong \pi_1(M - \nu(T)) \ast \pi_1(((X_{3,k} - \nu(T))\sharp_2IQ)\sharp_2T; E(2))/\langle \pi_1(\partial\nu(T)) \rangle
\]

\[
\cong \pi_1(M).
\]

\[\square\]

**Corollary 2.2.** For each odd integer \(m \geq 1\), \(10 \leq k \leq 18\) and \(l \geq 0\), \(Y_{m,k,l}\) is a minimal symplectic 4-manifold with \(\pi_1(Y_{m,k,l}) \cong \pi_1(M) = G\) which satisfies \(\chi(Y_{m,k,l}) = 25m^2 + 31m + 3 + 8r + 8\) and \(c_1^2(Y_{m,k,l}) = 225m^2 + 248m + 8l + 35 - k\).

**Proof.** Similar to the proof of Corollary \([\text{II}]\) above, \(\pi_1(Z_{m,k} - T) = 1\) implies that \(\pi_1(((Z_{m,k} - \nu(T))\sharp_2IQ)\sharp_2T; E(2)) = 1\). Hence Van-Kampen theorem again induces

\[
\pi_1(Y_{m,k,l}) = \pi_1(M - \nu(T)) \ast \pi_1(((Z_{m,k}\sharp_2IQ)\sharp_2T; E(2) - \nu(T))/\langle \pi_1(\partial\nu(T)) \rangle
\]

\[
\cong \pi_1(M - \nu(T)) \ast \pi_1(((Z_{m,k} - \nu(T))\sharp_2IQ)\sharp_2T; E(2))/\langle \pi_1(\partial\nu(T)) \rangle
\]

\[
\cong \pi_1(M).
\]

\[\square\]

Next, we introduce S. Boyer’s result on simply connected 4-manifolds with a given boundary \([\text{III}]\), which will be a key ingredient in proving our result.

Let \(V_1\) and \(V_2\) be simply connected, compact, oriented 4-manifolds with connected boundary \(\partial V_1 = \partial V_2\) and suppose that \(f : \partial V_1 \to \partial V_2\) is an orientation preserving homeomorphism. Then there are two obstructions to extending \(f\) to a homeomorphism \(F : V_1 \to V_2\): The first is to find an isometry \(\Lambda : (H_2(V_1; \mathbb{Z}), Q_{V_1}) \to (H_2(V_2; \mathbb{Z}), Q_{V_2})\) for which the following diagram commutes:

\[
\begin{array}{ccc}
0 & \longrightarrow & H_2(\partial V_1) \\
\downarrow f_* & & \downarrow \Lambda \\
0 & \longrightarrow & H_2(\partial V_2)
\end{array}
\]

Here an isometry means an isomorphism \(\Lambda : H_2(V_1; \mathbb{Z}) \to H_2(V_2; \mathbb{Z})\) which preserves an intersection form \(Q_{V_1}\), and \(\Lambda^*\) is the adjoint of \(\Lambda\) with respect to the identification of \(H_2(V_i, \partial V_i; \mathbb{Z})\) with \(\text{Hom}(H_2(V_i; \mathbb{Z})\) arising from Lefschetz duality. We call such a pair \((f, \Lambda)\) a *morphism* and denote it symbolically as \((f, \Lambda) : V_1 \to V_2\).

The second obstruction encountered is to realize a given morphism \((f, \Lambda)\) geometrically, i.e. to find a homeomorphism \(F : V_1 \to V_2\) such that \((f, \Lambda) = (F|_{\partial V_1}, F_*).\)

S. Boyer showed when a given morphism could be realized geometrically. Explicitly, he proved

**Theorem 2.4 ([\text{III}], Theorem 0.7 and Proposition 0.8).** If \((f, \Lambda) : V_1 \to V_2\) is a morphism between two simply connected smooth 4-manifolds \(V_1\) and \(V_2\) with boundary \(\partial V_1 = \partial V_2\), there is an obstruction \(\theta(f, \Lambda) \in I^1(\partial V_1)\) such that \((f, \Lambda)\) is realized
geometrically if and only if $\theta(f, \Lambda) = 0$. Furthermore, if $H_1(\partial V_1; \mathbb{Q}) = 0$, then $\theta(f, \Lambda) = 0$.

**Corollary 2.3.** Suppose that $\partial V_1 = \partial V_2$ is a homology 3-sphere. Then, for any morphism $(f, \Lambda) : V_1 \to V_2$ between two simply connected smooth 4-manifolds $V_1$ and $V_2$, there is a homeomorphism $F : V_1 \to V_2$ such that $(f, \Lambda) = (F|_{\partial V_1}, F_\ast)$.

Finally, we state and prove a key proposition.

**Proposition 2.2.** Suppose $X$ is a symplectic 4-manifold which contains a symplectic torus $T$ lying in a cusp neighborhood. Then, for each integer $n \geq 2$, a family of symplectic 4-manifolds $\{X_{\pm T}E(n)_K \mid K \text{ is a fibered knot in } S^3\}$ are mutually non-diffeomorphic, but all homeomorphic. Furthermore, $\pi_1(X_{\pm T}E(n)_K) \cong \pi_1(X)$.

**Proof.** Since $E(n)_K$ contains a Gompf nucleus, say $C_n$, embedded in $B(2,3,6n-1)$ (Building Block 2), we can decompose $E(n)_K$ into $C_n \cup_\Sigma E(n)_K'$, where $\Sigma$ is a homology 3-sphere and $E(n)_K'$ denotes a complement of $C_n$ in $E(n)_K$. Note that the Gompf nucleus $C_n$ is independent of $K$, i.e. $C_n$ is a common submanifold of codimension zero in all $E(n)_K$'s. Hence, after taking a symplectic fiber sum of $X$ with $E(n)_K$ along a Lagrangian torus $T \subset C_n$, we get a decomposition

$$X_{\pm T}E(n)_K = (X_{\pm T}C_n) \cup_\Sigma E(n)_K'.$$

In order to show that $X_{\pm T}E(n)_K$ is homeomorphic to $X_{\pm T}E(n)_K'$, for any fibered knots $K$ and $K'$ in $S^3$, we first choose an identity map $f = \text{id} : X_{\pm T}C_n \to X_{\pm T}C_n$ and an isomorphism $\Lambda : H_2(E(n)_K'; \mathbb{Z}) \to H_2(E(n)_K'; \mathbb{Z})$. In particular, since both intersection forms $Q_{E(n)_K}$ and $Q_{E(n)_K'}$ are unimodular and indefinite, they are uniquely determined by rank, signature and type, so that they are isomorphic. (The reason is following: Since $E(n)_K$ and $E(n)_K'$ are homeomorphic, their intersection forms have the same rank, signature and type. Hence $Q_{E(n)_K}$ and $Q_{E(n)_K'}$ also have the same rank, signature and type.) Hence we may choose $\Lambda$ to be an isometry, i.e. $\Lambda : (H_2(E(n)_K'; Q_{E(n)_K}) \to (H_2(E(n)_K'; Q_{E(n)_K'}))$ is an isomorphism which preserves the intersection forms. Then, applying Corollary 2.3 with the fact that $\Sigma$ is a homology 3-sphere, we conclude that there exists a homeomorphism $F : E(n)_K \to E(n)_K'$ such that $(f, \Lambda) = (F|_{\Sigma}, F_\ast)$. Thus there is a homeomorphism between two symplectic 4-manifolds $X_{\pm T}E(n)_K$ and $X_{\pm T}E(n)_K'$.

Next, applying Theorem 2.1 and Theorem 2.2 above, we also conclude that $X_{\pm T}E(n)_K$ and $X_{\pm T}E(n)_K'$ have different Seiberg-Witten invariants as long as $\Delta_K(t) \neq \Delta_{K'}(t)$, i.e. $X_{\pm T}E(n)_K$ is not diffeomorphic to $X_{\pm T}E(n)_K'$.

The last statement follows from the Van-Kampen theorem using the fact that $T$ is a regular fiber in a cusp neighborhood and $\pi_1(E(n)_K - T) = 1$:

$$\pi_1(X_{\pm T}E(n)_K) = \pi_1(X - \nu(T)) * \pi_1(E(n)_K - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle$$

$$\cong \pi_1(X - \nu(T))/\langle \pi_1(\partial \nu(T)) \rangle$$

$$\cong \pi_1(X - \nu(T)) * \pi_1(\nu(T))/\langle \pi_1(\partial \nu(T)) \rangle$$

$$\cong \pi_1(X).$$

**Proof of Theorem 1.1.** Let $\{X_{3,k,l}\}$ and $\{Y_{m,k,l}\}$ be minimal symplectic 4-manifolds appeared in Corollary 2.1 and Corollary 2.2 above. For each odd integer $m \geq 1$ with a finitely presented group $G$ fixed (so that $g$ and $r$ are also fixed), define two
numbers \( r_{G,m} \) and \( t_{G,m} \) by
\[
\begin{align*}
\begin{cases}
  r_{G,m} := & \frac{c_1^2(Y_{m,10,112.5m+173.5})}{\chi(Y_{m,10,112.5m+173.5})} = \frac{225m^2 + 1148m + 1413}{25m^2 + 141.5m + 181.5 + g + r}, \\
  t_{G,m} := & 25m^2 + 143.5m + 181.5 + g + r.
\end{cases}
\end{align*}
\]
Then \( \{r_{G,m}\} \) is an increasing sequence converging to 9 and any lattice point \((x,c)\) satisfying \(0 \leq c \leq r_{G,m}x\) and \(x \geq t_{G,m}\) is realized as chern numbers \((\chi, c_1^2)\) of a minimal symplectic 4-manifold

\[X_{3,k,l} \sharp T E(n)_K \text{ or } Y_{m,k',l'} \sharp T E(n)_K\]

for some integers \(m, k, k', l, l'\) and \(n\). Finally, since \(X_{3,k,l}\) and \(Y_{m,k,l}\) contain a symplectic torus \(T\) in a Gompf nucleus \(\subset E(2)\), we conclude from Proposition 2.2 above that both \(X_{3,k,l} \sharp T E(n)_K\) and \(Y_{m,k,l} \sharp T E(n)_K\) admit infinitely many distinct smooth structures and they have \(\pi_1 = G\).

\[\square\]

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