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ON FUNDAMENTAL GROUPS OF COMPACT HAUSDORFF SPACES

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Abstract. We discuss which groups can be realized as the fundamental groups of compact Hausdorff spaces. In particular, we prove that the claim "every group can be realized as the fundamental group of a compact Hausdorff space" is consistent with the Zermelo–Fraenkel–Choice set theory.

In this note we prove that every group π can be realized as the fundamental group of a compact Hausdorff space, if we work in the pattern of the set theory without inaccessible cardinals. (See e.g. [2] concerning inaccessible cardinals.) Note that, because of a theorem of Kuratowski [2, Proposition 1.2], the absence of inaccessible cardinals is consistent with the Zermelo–Fraenkel–Choice set theory.

The idea is the following. Take a CW-space X with $\pi_1(X) = \pi$ and let βX be the Stone–Čech compactification of X, [4]. Then X is a path-connected component of βX, and therefore $\pi_1(\beta X, *) = \pi$ for all $* \in X$.

This result should be contrasted with the result of Saharon Shelah [5] that for path connected, locally path connected compact metric spaces X, $\pi_1(X)$ is either a finitely generated group or has cardinality $2^{\aleph_0}$.

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All spaces are assumed to be Hausdorff, all maps and functions are assumed to be continuous. We denote by I the unit segment $[0, 1]$. Let $\mathbb{N}^*$ denote the one-point compactification of the natural numbers $\mathbb{N}$ with $*$ being the point at infinity.

A dated, but useful, compendium of information on the Stone–Čech compactification can be found in [6]. See also [1]. The Stone–Čech compactification βX of the completely regular space X can be characterized as a topological embedding $X \subset \beta X$ with βX compact and such that X is dense in βX and such that every function $f : X \to I$ can be extended to a function $\widehat{f} : \beta X \to I$. There are several standard constructions of the Stone–Čech compactification. One construction uses the maximal ideals in the ring of bounded functions on X, $C^*(X)$, with the hull-kernel topology [1, Chapter 7].

Let υX denote the *Hewitt realcompactification* of X, [1, Chapter 8]. Recall that $X \subset \upsilon X \subset \beta X$ and is characterized by the property that every continuous function

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$X \to \mathbb{R}$ (not necessarily bounded) can be extended to υX. A space X is called *realcompact* if $\upsilon X = X$. Note that a realcompact space is not necessarily compact.

1. **Theorem.** *If X is a paracompact space of non-measurable cardinality, then $\upsilon X = X$.*

Proof. Katetov [3] proved that a paracompact space X is realcompact iff each of its closed discrete subspaces is realcompact. On the other hand, a discrete subspace is realcompact iff it has non-measurable cardinality, [1, Chapter 12]. □

2. **Theorem.** *Each non-discrete, closed subset in $\beta X \setminus \upsilon X$ contains a copy of $\beta\mathbb{N}$, and so its cardinality is at least $2^{\mathfrak{c}}$.*

Proof. See [1, Theorem 9.11]. □

3. **Theorem.** *If X is a path connected paracompact space of non-measurable cardinality, then X is a path component of βX.*

Proof. Suppose that there exists a path $\alpha : I \to \beta X$ with $\alpha(0) \in X$ and $\alpha(1) \in \beta X \setminus X$. Then, by Theorems 1 and 2, $\alpha(I) \cap (\beta X \setminus X)$ is a discrete set. So, we may assume without loss of generality that $\alpha([0,1)) \subset X$ and $\alpha(1) \in \beta(X) \setminus X$. Clearly, then $\alpha([0,1))$ is an infinite set; otherwise $\alpha(1) \in X$.

Now let $\{t_n\}_{n=1}^{\infty}$ be a sequence of distinct points in $\alpha([0,1))$ converging to $\alpha(1)$. Define $f(t_n) = n \mod 2 : \{t_n\}_{n=1}^{\infty} \to [0,1]$. Since X is paracompact and therefore normal, let $F : X \to [0,1]$ be any extension of f to all of X using Tietze's Extension Theorem. Then let $\widehat{F} : \beta X \to [0,1]$ be the Stone-Čech extension of F to βX. Then $\widehat{F}|\{t_n\}_{n=1}^{\infty} = f$ has an extension to $\{t_n\}_{n=1}^{\infty} \cup \alpha(1)$, which is clearly a contradiction. □

4. **Corollary.** *Every group of non-measurable cardinality is the fundamental group of a compact space.*

Proof. Let π be a given group. Let X be a connected CW-space having $\pi_1(X) = \pi$. We can assume that X has non-measurable cardinality. Since X is paracompact, we conclude that $\pi_1(\beta X, x_0) = \pi$ for any $x_0 \in X$ in view of Theorem 3. □

5. *Remark.* Similar arguments show that, given a sequence $\{A_n\}_{n=1}^{\infty}$ of groups of non-measurable cardinality, with A_n abelian for $n \geq 2$, there exists a compact space X with $\pi_n(X) = A_n$ for all $n \geq 1$.

6. *Remark.* Let us recall that every measurable cardinal is inaccessible, [2].

7. *Remark.* Of course, there exist groups of measurable cardinality that can be realized as the fundamental groups of compact spaces. For example, the product of any family of circles is compact, and so $\mathbb{Z}^m$ with any cardinal m can be realized.

8. *Remark.* In the first version of the paper (that we put in an electronic archive) we asked whether there exists a *path connected* compact space with prescribed fundamental group of non-measurable cardinality. Recently Adam Przeździecki notified us that the answer is affirmative. Also, he announced an example of a group of measurable cardinality that is not the fundamental group of any compact space.

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