INNER SEQUENCE BASED INVARIANT SUBSPACES IN $H^2(D^2)$

MICHIRO SETO AND RONGWEI YANG

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Abstract. A closed subspace $H^2(D^2)$ is said to be invariant if it is invariant under the Toeplitz operators $T_z$ and $T_w$. Invariant subspaces of $H^2(D^2)$ are well-known to be very complicated. So discovering some good examples of invariant subspaces will be beneficial to the general study. This paper studies a type of invariant subspace constructed through a sequence of inner functions. It will be shown that this type of invariant subspace has direct connections with the Jordan operator. Related calculations also give rise to a simple upper bound for $\sum_j 1 - |\lambda_j|$, where $\{\lambda_j\}$ are zeros of a Blaschke product.

1. Introduction

In $H^2(D^2)$ with coordinates $z$ and $w$, multiplications by $z$ and $w$ (denoted by $T_z$ and $T_w$, respectively) are shift operators with infinite multiplicity. A subspace $M$ is said to be $z$-(or $w$-) invariant if $M$ is invariant under $T_z$ (or $T_w$, respectively), and $M$ is said to be invariant if it is invariant under both $T_z$ and $T_w$. It is well-known that in general invariant subspaces of $H^2(D^2)$ can be very complex (cf. [Ru]), and their study demands new ideas and techniques (cf. [DP], [DM], [Ya1]), as well as good understanding of some examples. This paper studies a type of invariant subspace that is constructed through an inner sequence. A sequence of inner functions $\{q_j(z) : 0 \leq j \leq m\}$, where $m$ may be infinite, is called an inner sequence if $q_{j+1} | q_j$ for each $j$. We will see that this type of invariant subspace has a simple structure, and it has direct connections with the Jordan operator.

The classical Hardy space $H^2(D)$ in the variable $z$ and that in the variable $w$ are different subspaces in $H^2(D^2)$, and we denote them by $H^2(z)$ and $H^2(w)$, respectively. For every $g \in H^2(w)$, we define an operator $\pi_g : H^2(D^2) \rightarrow H^2(z)$ by

$$\pi_g(f)(z) = \int_T f(z,w)\overline{g}(w)dm(w), \quad f \in H^2(D^2),$$

where $T$ is the unit circle and $dm(w)$ is the normalized Lebesgue measure on $T$. It is easy to check that $\pi_g$ is bounded. In fact, one verifies that

$$\pi_g^* h = g(w)h(z), \quad h \in H^2(z),$$
and hence \( \| \pi_g \| = \| g \| \). We now look at a few facts regarding \( \pi_g \). If \( M \) is \( z \)-invariant, then for every \( g \in H^2(w) \)

\[
z\pi_g(M) = \pi_g(zM) \subset \pi_g(M).
\]

If \( M \) is \( w \)-invariant, then

\[
\pi_w(M) = \pi_{w^{i+1}}(wM) \subset \pi_{w^{i+1}}(M)
\]

for every integer \( i \geq 0 \). So in the case when \( M \) is invariant, the closures \( \pi_{w^i}(M) \), \( i = 0, 1, 2, \ldots \), is an increasing sequence of invariant subspaces of \( H^2(z) \). By Beurling’s theorem, this gives rise to a sequence of inner functions \( \{q_i(z) : i \geq 0\} \) with \( q_{j+1} \parallel q_j \), i.e., an inner sequence, such that \( \pi_{w^i}(M) = q_i H^2(z) \). It then follows easily that

\[
M \subset \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j.
\]

One observes that \( \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j \) is clearly \( z \)-invariant. Moreover, since \( q_{j+1} \parallel q_j \),

\[
wq_j H^2(z)w^j = q_j H^2(z)w^{j+1} \subset q_{j+1} H^2(z)w^{j+1},
\]

and this shows that \( \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j \) is also \( w \)-invariant.

Invariant subspace of the form \( \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j \), where \( \{q_i(z) : i \geq 0\} \) is an inner sequence, shall be said to be inner-sequence-based in this paper, and it is the primary subject of this paper. The above observations indicate that every invariant subspace has a smallest inner-sequence-based invariant subspaces containing it (i.e., an inner-sequence-based envelope). Of course, this envelope is non-trivial only when \( q_0 \) is not a constant, or equivalently, \( \pi_1(M) \) is not dense in \( H^2(z) \), since if \( q_0 \) is a constant, the inner sequence is a sequence of non-zero scalars and hence

\[
\bigoplus_{j=0}^{\infty} q_j H^2(z)w^j = H^2(D^2).
\]

Let \( M \) be \( z \)-invariant and \( N = H^2(D^2) \ominus M \), and we denote the compression of \( T_z \) to \( N \) by \( S_z \). It is well-known that \( S_z \) serves as a model for the so-called \( C_0 \) class contractions, namely the class of contractions \( A \) with \( (A^*)^n \) converging strongly to \( 0 \). Clearly, the \( C_0 \) class is very large, for instance, every strict contraction is in it. If \( M \) is invariant (i.e., also invariant for \( T_w \)), then \( S_z \) on \( N \) is much less general, but it still represents a good class of interesting operators. For example, the Bergman shift and unilateral shifts with any multiplicity are all unitarily equivalent to \( S_z \) for \( M \) an invariant subspace. For convenience, we let \( S \) denote the collection of operators that are unitarily equivalent to \( S_z \) or \( S_w \) on \( N = H^2(D^2) \ominus M \) for some invariant subspace \( M \); here \( S_w \) is the compression of \( T_w \) to \( N \). In Section 2, we will show that every Jordan operator is in \( S \). In Section 3, we show that Rudin’s invariant subspace (cf. [R]), formerly believed to be pathological, is in fact inner-sequence-based. The core operator is an important associate of invariant subspaces of \( H^2(D^2) \) (cf. [CY]). In Section 4 we show how one can compute the core operator explicitly for the case of an inner-sequence-base invariant subspace. Calculations in Section 4 lead to an interesting upper bound for \( \sum_j 1 - |\lambda_j| \), where \( \{\lambda_j\} \) are zeros of a Blaschke product. We will address this point in Section 5.
2. A note on Jordan operators

Let $H^2(D)$ be the Hardy space over the unit disk. Multiplication by coordinate function $z$ on $H^2(D)$ is the unilateral shift, and its invariant subspace is of the form $\theta H^2(D)$, where $\theta$ is an inner function. The compression $S(\theta)$ of the unilateral shift to the quotient space $N_\theta := H^2(D) \ominus \theta H^2(D)$ is called a Jordan block. To be precise, $S(\theta)f = P_{N_\theta}zf$, $f \in N_\theta$, where $P_{N_\theta}$ is the projection from $H^2(D)$ onto $N_\theta$. For an inner sequence $\{q_j(z) : 0 \leq j \leq m\}$, the direct sum $\bigoplus_{j=0}^m S(q_j)$ is often called a Jordan operator. A celebrated result in the 70’s is that every $C_0$ class operator on a separable Hilbert space is quasisimilar to a Jordan operator (cf. [Be]).

Jordan operators are directly connected with inner-sequence-based invariant subspaces. For an inner sequence $\{q_j(z) : j \geq 0\}$, and $M = \bigoplus_{j=0}^\infty q_j H^2(z) w^j$, we observe that:

$$N = \bigoplus_{j=0}^\infty H^2(z) w^j \ominus \bigoplus_{j=0}^\infty q_j H^2(z) w^j = \bigoplus_{j=0}^\infty (H^2(z) \ominus q_j H^2(z)) w^j.$$

Let $P_j$ denote the orthogonal projection from $H^2(D^2)$ onto $(H^2(z) \ominus q_j H^2(z)) w^j$. Clearly, $P_N = \bigoplus_{j=0}^\infty P_j$. For every $f = \sum_{j=0}^\infty f_j(z) w^j \in N$, where $f_j \in H^2(z) \ominus q_j H^2(z)$,

$$S_z f = \left( \bigoplus_{j=0}^\infty P_j \right) \left( \sum_{i=0}^\infty z f_i(z) w^j \right)$$

$$= \sum_{i=0}^\infty \left( \bigoplus_{j=0}^\infty P_j z f_i(z) w^j \right)$$

$$= \sum_{i=0}^\infty P_j z f_i(z) w^j$$

$$= \sum_{i=0}^\infty (S(q_i)f_i) w^j.$$

So with respect to the decomposition $N = \bigoplus_{j=0}^\infty (H^2(z) \ominus q_j H^2(z)) w^j$, (2-1)

$$S_z = \bigoplus_{j=0}^\infty S(q_j),$$

and we obtain the following fact.

**Corollary 2.1.** Every Jordan operator is in $S$.

It is not hard to see that if $q_j$ is a scalar starting from $j = k$, then the direct sum above is a finite sum of $k$ terms.

It is also worth noting that (2-1) gives a necessary condition for an invariant subspace to be of inner-sequence-based type. For instance, (2-1) implies that $\sigma(S_z) \cap D$, being the zeros of $S(q_0)$, is discrete. One can easily come up with invariant subspaces for which $\sigma(S_z) \cap D$ is not discrete. For example, if we let $M = [z - w]$ be the closure of the ideal $(z - w)$ in the polynomial ring $C[z, w]$, then $S_z$ on $H^2(D^2) \ominus M$ is equivalent to the Bergman shift, and hence $\sigma(S_z) = D$. This indicates that $[z - w]$ is not inner sequence based. It is not difficult to check that the envelope of $[z - w]$ is $z H^2(z) \oplus \bigoplus_{j=1}^\infty H^2(z) w^j$. 

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3. Rudin’s invariant subspace

In [Ru], Rudin constructed an invariant subspace of infinite rank as follows. Let $M$ be the Hardy invariant subspace consisting of all functions in $H^2(D^2)$ which have a zero of order greater than or equal to $n$ at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for any positive integer $n$. For quite a long time this invariant subspace was viewed as somewhat pathological. Here, we will show that it is in fact an inner-sequence-based invariant subspace. It is interesting because this means that Rudin’s invariant subspace, despite the fact that it has infinite rank, has a very simple structure.

For simplicity, we denote $H^2(D^2)$ by $H^2$ and set $b_n(z) = (z - \alpha_n)/(1 - \alpha_n z)$. Then we have $M = \bigcap_{n \geq 1} M_n$, where

$$M_n = b_n^n(z) H^2 \vee b_n^{n-1}(z) w H^2 \vee \cdots \vee w^n H^2,$$

that is, $M_n$ is the invariant subspace consisting of all functions in $H^2$ which have a zero of order greater than or equal to $n$ at $(\alpha_n, 0)$. Further, we have

$$M_n = b_n^n(z) H^2(z) \oplus b_n^{n-1}(z) w H^2(z) \oplus \cdots \oplus w^n H^2$$

$$= \sum_{k=0}^{n-1} \oplus b_n^{n-k}(z) w^k H^2(z) \oplus w^n H^2.$$

We define a family of inner functions inductively as follows:

$$q_0(z) = \prod_{n=1}^\infty b_n^n(z),$$

$$q_j(z) = q_{j-1}(z)/\prod_{n=j}^\infty b_n(z) \quad (j \geq 1).$$

Then $q_j(z)$ is divisible by $q_{j+1}(z)$ for every $j \geq 0$, and we have

$$M = \bigcap_{n \geq 1} M_n = \bigoplus_{j=0}^\infty q_j(z) H^2(z) w^j.$$

The fact that Rudin’s example has infinite rank prompts the following conjecture.

**Conjecture.** If $\{q_j(z)\}$ is an inner sequence such that $q_{j+1}$ is a proper factor of $q_j$ for each $j \geq 0$, then $\bigoplus_{j=0}^\infty q_j H^2(z) w^j$ has infinite rank.

4. The Core Operator

For an invariant subspace $M$, we let $(R_z, R_w)$ denote the restriction of $(T_z, T_w)$ to $M$. So it is clear that $(R_z, R_w)$ is a pair of commuting isometries. The core operator $C$ for $M$ is defined as

$$C = I - R_z R_z^* - R_w R_w^* + R_z R_w R_z^* R_w^*.$$

A parallel associate for $(S_z, S_w)$ is

$$\Delta_S := I - S_z^* S_z - S_w^* S_w + S_z^* S_w S_z S_w.$$

Both $C$ and $\Delta_S$ are useful tools in the study of invariant subspaces. We refer the readers to [GY] and [Ya2] for details. In this section, we will see that both $C$ and $\Delta_S$ can be written as a direct sum with respect to the decomposition $M = \bigoplus_{j=0}^\infty q_j(z) H^2(z) w^j$. For simplicity we let $q_j' = \frac{q_j}{q_{j-1}}$, $j = 1, 2, 3, \ldots$. 

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Let \( g = q_j f_j w^j \) be any function in \( q_j H^2(z)w^j \), where \( j \geq 1 \). Then

\[
R_w^*(q_j f_j w^j) = [(I - P_{j-1})(q_j f_j)]w^{j-1} = \left( \sum_{i=0}^{\infty} \langle q_j f_j, q_{j-1}z^i \rangle q_{j-1}z^i \right) w^{j-1}
\]

and

\[
(I - R_z R_w^*)g = \sum_{i=0}^{\infty} \langle g, q_i w^i \rangle q_i w^i = \langle q_j f_j w^j, q_j w^j \rangle q_j w^j = f_j(0)q_j w^j.
\]

It follows that

\[
Cg = [I - R_z R_w^* - R_w(I - R_z R_w^*)R_w^*]g = f_j(0)q_j w^j - \langle q_j f_j, q_{j-1} w^{j-1} \rangle = (\langle f_j, 1 \rangle 1 - \langle f_j, q_j' q_j \rangle) q_j w^j.
\]

This shows that \( q_j H^2(z)w^j \) is invariant for \( C \) and on \( q_j H^2(z)w^j \)

\[ C \cong 1 \otimes 1 - q_j' \otimes q_j. \]

Moreover, since \( R_w^* \) is 0 on \( q_0 H^2(z) \), it is easy to verify that \( C \cong 1 \otimes 1 \) on \( q_0 H^2(z) \).

Summarizing these observations we have

**Corollary 4.1.** With respect to the decomposition \( M = \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j \),

\[ C \cong 1 \otimes 1 \bigoplus_{j=1}^{\infty} (1 \otimes 1 - q_j' \otimes q_j'). \]

Now we take a look at \( \Delta_S \). Let \( f_j \) be any function in \( H^2(z) \otimes q_j H^2(z) \), where \( j \geq 1 \). Then as indicated before, \( S_z(f_j w^j) = (S(q_j)f_j)w^j \), and \( S_w(f_j w^j) = (P_{j+1}f_j)w^j \).

Writing

\[
\Delta_S = (I - S_z^* S_z) - S_w^*(I - S_z^* S_z) S_w
\]

and \( I_j - S^*(q_j)S(q_j) \) as \( D_j \), where \( I_j \) stands for the identity map on \( H^2(z) \otimes q_j H^2(z) \),

we compute that

\[
\Delta_S(f_j w^j) = w^j(I_j - S^*(q_j)S(q_j))f_j - S_w^*(w^{j+1}(I_{j+1} - S^*(q_{j+1})S(q_{j+1}))(P_{j+1}f_j))
\]

\[ = w^j D_j f_j - w^j D_{j+1}P_{j+1}f_j \]

\[ = w^j D_j f_j - w^j D_{j+1}(P_j - P_j - P_{j+1})f_j, \]

One verifies that \( D_{j+1}(P_j - P_{j+1}) = 0 \), and hence

\[
\Delta_S(f_j w^j) = (D_j - D_{j+1})f_j w^j, \quad j \geq 0.
\]

We summarize these observations in the following corollary.

**Corollary 4.2.** With respect to the decomposition \( N = \bigoplus_{j=0}^{\infty} (H^2(z) \otimes q_j H^2(z))w^j \),

\[
\Delta_S \cong \bigoplus_{j=0}^{\infty} (D_j - D_{j+1}).
\]
It is well-known that $D_j$ is of rank 1 when $q_j$ is non-trivial, and it can be explicitly expressed in terms of $q_j$. We will return to this point in Section 5.

**Lemma 4.3.** Let $f$ and $g$ be non-zero functions in $H^2(z)$ and $A = f \otimes f - g \otimes g$. Then

$$trA^2 = \|f\|^4 + \|g\|^4 - 2|\langle f, g \rangle|^2.$$  

**Proof.** It is a simple calculation. Clearly, $A$ is selfadjoint. Since $Af = \|f\|^2f - \langle f, g \rangle g, \quad Ag = \langle g, f \rangle f - \|g\|^2g,$

with respect to the basis $\{f, g\}$, $A$ has the matrix form

$$
\begin{pmatrix}
\|f\|^2 & \langle g, f \rangle \\
-\langle f, g \rangle & -\|g\|^2
\end{pmatrix}.
$$

If we denote the eigenvalues of $A$ by $\lambda_1$ and $\lambda_2$, then

$$\lambda_1 + \lambda_2 = \|f\|^2 - \|g\|^2, \quad \lambda_1\lambda_2 = -\|f\|^2\|g\|^2 + |\langle f, g \rangle|^2.$$  

So

$$trA^2 = \lambda_1^2 + \lambda_2^2 = \|f\|^4 + \|g\|^4 - 2|\langle f, g \rangle|^2.$$  

□

The following fact follows directly from Corollary 4.1 and the above lemma.

**Corollary 4.4.** $trC^2 = 1 + 2\sum_{j=1}^{\infty} 1 - |q_j'(0)|^2.$

Recall that for an invariant subspace $M$, its fringe operator $F$ is defined on $M \ominus zM$ by

$$Ff = P_{M \ominus zM}wf, \quad f \in M \ominus zM.$$  

It is indicated in [Ya1] that the fringe operator is also a very useful tool for the study in this area. It is interesting to see how the fringe operator acts on $M = \bigoplus_{j=0}^{\infty} q_j H^2(z)w^j$. It is not difficult to check that in this case

$$M \ominus zM = \bigoplus_{j=0}^{\infty} C q_j(z)w^j,$$  

and clearly $\{q_j(z)w^j : j \geq 0\}$ is an orthonormal basis for $M \ominus zM$. For every $j$,

$$F(q_jw^j) = P_{M \ominus zM}q_jw^{j+1}$$

$$= \langle q_jw^{j+1}, q_{j+1}w^{j+1} \rangle q_{j+1}(z)w^{j+1}$$

$$= \langle q_{j+1}'q_{j+1}, q_{j+1}q_{j+1}(z)w^{j+1} \rangle q_{j+1}'(0)q_{j+1}(z)w^{j+1}.$$  

This shows that the fringe operator in this case is a weighted shift with weighs

$$q_1'(0), \quad q_2'(0), \quad q_3'(0), \quad \cdots.$$
5. An inequality about Blaschke products

For any bounded linear operator $A$ on a Hilbert space $H$, the so-called minimum modulus

$$\gamma(A) := \inf \{\|Ax\| : x \in (\ker A)_{\bot}, \|x\| = 1\}$$

measures the norm of $A$’s “partial inverse”. Clearly, $A$ has closed range if and only if $\gamma(A) > 0$. When $A$ is invertible, $\gamma^{-1}(A) = \|A^{-1}\|$.

It is shown in [Ya2] that for every invariant subspace $M$

$$trC^2 \leq 2\gamma^{-2}(S_z) + 2 \dim \ker(S_z) - 1.$$  

It is interesting to see what this inequality means for an inner-sequence-based invariant subspace. Without loss of generality we assume $q_0(0) \neq 0$. Then by (2-1), $\ker(S_z)$ is trivial. To calculate $\gamma(S_z)$, we consider a general Jordan operator $S(\theta)$ on the space $N_\theta := H^2(D) \ominus \theta H^2(D)$. One verifies that

$$I - S^*(\theta)S(\theta) = P_{N_\theta}(\bar{\theta}) \otimes P_{N_\theta}(\bar{\theta}).$$

Therefore, for $g \in N_\theta$,

$$\|S(\theta)g\|^2 = \|g\|^2 - |\langle g, P_{N_\theta}(\bar{\theta}) \rangle|^2 \geq (1 - \|P_{N_\theta}(\bar{\theta})\|^2)\|g\|^2 = |\theta(0)|^2\|g\|^2,$$

and the equality is obtained when $g = P_{N_\theta}(\bar{\theta})$. This shows that $\gamma(S(\theta)) = |\theta(0)|$.

(2-1) then implies that

$$\gamma(S_z) = \inf \{|q_j(0)| : j \geq 0\}.$$  

But since every $q_j$ is a factor of $q_0$,

$$\gamma(S_z) = |q_0(0)|.$$  

Combining Corollary 4.4, (5-1) and (5-2), we have

$$\sum_{j=1}^{\infty} 1 - |q_j(0)|^2 \leq |q_0(0)|^{-2} - 1.$$  

Or equivalently, we have

**Corollary 5.1.** For $M = \bigoplus_{j=0}^{\infty} q_j H^2(z) w^j$, $trC^2 \leq 2|q_0(0)|^{-2} - 1$.

If we let $B(z)$ be a Blaschke product with zeros $\{\lambda_j : 1 \leq j \leq m\}$ counting multiplicity, where $m$ can be infinity, and set

$$q_j = \prod_{i=j+1}^{m} \frac{\lambda_i - z}{1 - \bar{\lambda}_i z}, \quad j \geq 0,$$

then (5-3) leads to an interesting inequality for Blaschke products, namely

$$\sum_{j=1}^{\infty} 1 - |\lambda_j|^2 \leq |B(0)|^{-2} - 1.$$  

Let us pursue this point a little further. Clearly, (5-4) is equivalent to the general inequality

$$\sum_{j=1}^{\infty} 1 - |\lambda_j|^2 \leq (\prod_{j=1}^{\infty} |\lambda_j|)^{-2} - 1.$$
for any sequence \( \{\lambda_j : j \geq 1\} \subseteq \bar{D} \) which can be proved easily by induction, and the equality holds only in the case when both sides are zero. So substituting \( \lambda_j \) by its square root, (5-4) leads to the following inequality.

\[
\sum_j 1 - |\lambda_j| \leq |B(0)|^{-1} - 1,
\]

where \( B(z) \) is a Blaschke product with zeros \( \{\lambda_j\} \) counting multiplicity. (5-5) is a simple inequality, but surprisingly it appears to have been unknown before. If \( f \in H^p(D) \), where \( 1 \leq p \leq \infty \), and \( \|f\|_p \leq 1 \), and \( \{\lambda_j : j \geq 1\} \) are its zeros counting multiplicity, then \( B \) is a factor of \( f \), and \( |B(0)| \geq |f(0)| \). So (5-5) can be generalized to the following corollary.

**Corollary 5.2.** If \( f \in H^p(D) \) with \( \|f\|_p \leq 1 \), where \( 1 \leq p \leq \infty \), and \( \{\lambda_j : j \geq 1\} \) are its zeros counting multiplicity, then

\[
\sum_{j=1}^\infty 1 - |\lambda_j| \leq |f(0)|^{-1} - 1.
\]

If \( \|f\|_\infty \leq 1 \), and \( \eta \in D \), then \( \frac{f(z)-\eta}{f(\eta)} \) is also an analytic self map of \( D \), and hence Corollary 5.2 implies

\[
\sum_{f(z)=\eta} 1 - |\eta| \leq \frac{1 - \bar{\eta}f(0)}{f(0)-\eta} \leq 1.
\]

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**References**


