ACTIONS OF POINTED HOPF ALGEBRAS
WITH REDUCED PI INVARIANTS

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Abstract. Let \( R \) be an \( H \)-module algebra, where \( H \) is a pointed Hopf algebra acting on \( R \) finitely of dimension \( N \). Suppose that \( L^H \neq 0 \) for every nonzero \( H \)-stable left ideal of \( R \). It is proved that if \( R^H \) satisfies a polynomial identity of degree \( d \), then \( R \) satisfies a polynomial identity of degree \( dN \) provided at least one of the following additional conditions is fulfilled:

1. \( R \) is semiprime and \( R^H \) is almost central in \( R \),
2. \( R \) is reduced.

If we also assume that \( R^H \) is central, then \( R \) satisfies the standard polynomial identity of degree \( 2\lfloor \sqrt{N} \rfloor \), where \( \lfloor \sqrt{N} \rfloor \) is the greatest integer in \( \sqrt{N} \).

1. Introduction

This paper is motivated by the following general question: if \( H \) is a finite-dimensional Hopf algebra over the field \( K \), and \( R \) is a left \( H \)-module algebra such that the algebra of invariants \( R^H \) satisfies a polynomial identity, must \( R \) also satisfy a polynomial identity? The answer to this question is positive in many concrete situations, e.g.,

1. when \( H = K[G] \), where \( G \) is a finite group, and either \( |G|^{-1} \in K \) or \( R \) is reduced (see [K1] and [K2]);
2. when \( H = K[G]^* \) (see [BC] and [BaZ]);
3. when \( H = u(L) \), where \( L \) is a finite-dimensional restricted Lie algebra of derivations of a prime ring \( R \) with \( \text{char} \, R = p > 0 \) such that \( R^H \) is semiprime and the elements inducing the \( X \)-inner part of \( L \) generate a quasi-Frobenius algebra (see [K3]);
4. when \( H \) is such that for every \( H \)-module algebra \( R \) such that \( R^H \) is nilpotent, also \( R \) is nilpotent (see [BaL]);
5. when \( H \) is pointed and \( R \) contains an element \( \gamma \) such that \( t \cdot \gamma = 1 \), for some \( 0 \neq t \in \mathbb{Q}^H \), the space of left integrals of \( H \) (see [BeT]).

However for actions of finite groups, where \( |G|R = 0 \), it is known that the answer can be negative. In an example of Bergman, there is an action of a group \( G \) of order \( p^2 \) (where \( p \) is the characteristic of \( K \)) on the algebra \( R = M_2(K[x,y]) \) of \( 2 \times 2 \) matrices over a free algebra \( K[x,y] \) such that \( R^G \) is commutative. Recall that in

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this example, \( G \) is generated by the inner automorphisms induced by

\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.
\]

Then \( R \) is a prime ring where every nonzero \( G \)-stable (right) ideal of \( R \) contains nontrivial invariants. This shows that the assumption that every nonzero \( H \)-stable left ideal of \( R \) contains nontrivial invariants is not sufficient to obtain a positive answer to the above question. Notice that in the above example \( R^H \) contains nilpotent elements. The main goal of this paper is to present a condition, which guarantees, for a semiprime algebra \( R \), that if \( R^H \) satisfies a PI, then \( R \) also satisfies a PI. We will show that if \( H \) is pointed and every nonzero \( H \)-stable left ideal contains a nontrivial central invariant, then \( R^H \) satisfying a PI implies that \( R \) satisfies a PI. This extends a situation considered in [BCF] and [BG]. In the second main result, we show that if \( R \) has no nonzero nilpotent elements, then the assumption that nonzero \( H \)-stable left ideals contain nontrivial invariants is sufficient for lifting the PI property from \( R^H \) to \( R \). Note that the most typical nontrivial examples of pointed Hopf algebras, which are neither group algebras nor universal enveloping rings of invariants under the action of \( p \)-nilpotent groups, nilpotent Lie algebras and Lie superalgebras was also considered in [BCF] and [BG]. In the second main result, we show that if \( R \) has no nonzero nilpotent elements, then the assumption that nonzero \( H \)-stable left ideals contain nontrivial invariants is sufficient for lifting the PI property from \( R^H \) to \( R \).

Throughout the paper \( K \) will be a field, \( H \) a pointed Hopf algebra over \( K \), and \( R \) an algebra over \( K \). We let \( \Delta : H \to H \otimes H \) be the comultiplication of \( H \), \( \epsilon : H \to K \) is the counit of \( H \), and \( S : H \to H \) the antipode of \( H \). We say that \( R \) is a left \( H \)-module algebra if \( R \) is a left \( H \)-module such that \( h \cdot ab = \sum (h_1 \cdot a)(h_2 \cdot b) \) and \( h \cdot 1_R = \epsilon(h)1_R \), where \( h \in H \), \( \Delta(h) = \sum h_1 \otimes h_2 \), \( a, b \in R \). If \( A \) is a subset of \( R \) such that \( h \cdot A \subseteq A \), for all \( h \in H \), then we say that \( A \) is \( H \)-stable. When \( R \) is a left \( H \)-module algebra one can consider the smash product \( R \# H \). As a vector space \( R \# H = R \otimes H \). The elements of \( R \# H \) can be written as finite sums \( \sum a_h h \), where \( h \in H \) and \( a_h \in R \). Then the multiplication in \( R \# H \) is determined by the formula \( (ah)(bl) = \sum a(h_1 \cdot b)h_2l \), for all \( a, b, h, l \in H \). The ring of invariants \( R^H \) is defined as \( \{ r \in R \mid h \cdot r = \epsilon(h)r, \text{ for all } h \in H \} \).

If \( R \) is a left \( H \)-module algebra, then \( R \) becomes a left \( R \# H \)-module using the left action \( (ah).r = a(h \cdot r) \), where \( a, r \in R \) and \( h \in H \). Then the commuting ring \( \text{End}_{R \# H}(R) \) is isomorphic to \( R^H \) and the submodules of \( R \) over \( R \# H \) are precisely \( H \)-stable ideals of \( R \).

If \( M \) is a left \( H \)-module, then there is a homomorphism \( \pi : H \to \text{End}_K(M) \) defined by \( \pi(h)(m) = hm \), for all \( h \in H \) and \( m \in M \). If \( \dim_K \pi(H) = N < \infty \), then we say that \( H \) acts finitely of dimension \( N \). Clearly \( \dim_K \pi(H) \leq \dim_K H \), so if \( H \) is finite dimensional, then \( H \) acts finitely on each \( H \)-module.

If \( R \) is semiprime, we let \( Q = Q(R) \) denote the symmetric Martindale quotient ring. Its center, known as the extended centroid of \( R \), we denote by \( C \). The following properties of \( Q \), when \( R \) is acted on by a Hopf algebra, are proved in Propositions 1, 2 and 5 of [GH].
Lemma 1. Let $R$ be a semiprime $H$-module algebra such that the $H$-action on $R$ extends to an $H$-action on $Q$ and any nonzero $H$-stable ideal of $R$ contains nontrivial invariants. Then

1. the ring $C^H = C \cap Q^H$ is von Neumann regular and selfinjective.
2. For any nonempty subset $X$ of $Q$ there exists an idempotent $\tilde{e}_X \in C^H$ such that $\text{ann}_{C^H}(X) = (1 - \tilde{e}_X)C^H$. If $X$ is an injective $C^H$-submodule of $Q$, then there exists $x \in X$ such that $\text{ann}_{C^H}(X) = \text{ann}_{C^H}(x) = (1 - \tilde{e}_x)C^H$.
3. If $L \subseteq Q$ is an $H$-stable subalgebra of $Q$ which is injective as a $C^H$-module, then $L^H$ is also injective as a $C^H$-module.
4. If a nonempty subset $S \subseteq C^H \setminus \{0\}$ is closed under a multiplication, then the localization $Q_S$ of $Q$ at $S$ is semiprime and $Z(Q_S) = C_S$.
5. If $H$ acts finitely on $Q$ and $S = C^H \setminus M$, where $M$ is a maximal ideal of $C^H$, then the $H$-action on $Q$ extends to an $H$-action on $Q_S$ and $(Q^H)_S = (Q_S)^H$, where $(C^H)_S = (C^H)_S = C_S \cap (Q^H)$ is a field contained in the center of $Q_S$.

It is also known (GH Proposition 2)) that under the assumptions of Lemma 1 the ring $Q$ is nonsingular and injective as a $C^H$-module. This immediately implies that if $\varphi : M \rightarrow N$ is an onto $C^H$-module map, where $0 \neq N \subseteq Q$ and $M$ is injective, then $N$ is also an injective $C^H$-module. In particular, any principal left ideal $Qg$ of $Q$ is nonsingular and injective as a $C^H$-module. Hence each finitely generated left ideal of $Q$ (finitely generated as a left $Q$-module) is also injective over $C^H$.

An important role will be played by the following result of Bergen, Cohen and Fischman on irreducible actions of Hopf algebras (see [BCF], Theorem 2.2).

Theorem 2. Let $A$ be a left $H$-module algebra such that $A \# H$ acts irreducibly on $A$. $A$ has a finite left Goldie rank, and $H$ acts finitely of dimension $N$ on $A$. Then $[A : A^H]_r \leq N$, where $[A : A^H]_r$ is the dimension of $A$ as a right vector space over the division ring $A^H$.

2. Main results

Throughout this section $H$ will be a pointed Hopf algebra over a field $K$. Recall that a ring $R$ is said to be reduced if it does not contain nonzero nilpotent elements. It is well known that if $r_1, r_2, \ldots, r_n$ are elements of a reduced ring $R$ such that $r_1r_2 \cdots r_n = 0$, then $r_{f(1)}r_{f(2)} \cdots r_{f(n)} = 0$ for any bijection $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$.

If $R$ is a left $H$-module algebra with center $Z(R)$, we say that the ring of invariants is almost central in $R$ if $L^H \cap Z(R) \neq 0$ for every nonzero $H$-stable left ideal $L$ of $R$. Notice that if $R$ is semiprime and $R^H$ is almost central in $R$, then $R^H$ is reduced. Indeed, suppose there exists $0 \neq a \in R^H$ such that $a^2 = 0$. The left ideal $Ra$ is $H$-stable, so one can find a nonzero element $ra \in (Ra)^H \cap Z(R)$. Then $ara = a(ra) = (ra)a = 0$ and thus $(ra)^2 = 0$, which is impossible since $Z(R)$ is reduced.

Our first main goal is to prove the following.

Theorem 3. Let $R$ be a semiprime $K$-algebra with center $Z$ and suppose $R$ is a left $H$-module algebra, where $H$ is a pointed Hopf algebra acting on $R$ finitely of dimension $N$. If the subalgebra of invariants $R^H$ is almost central in $R$, and $R^H$
satisfies a polynomial identity of degree \( d \), then \( R \) satisfies the standard polynomial identity of degree \( dN \). If in addition \( R^H \subseteq \mathbb{Z} \), then \( R \) satisfies the standard polynomial identity of degree \( 2[\sqrt{N}] \), where \( \lfloor \sqrt{N} \rfloor \) is the greatest integer in \( \sqrt{N} \).

Our next result concerns the situation when the algebra \( R \) is reduced.

**Theorem 4.** Let \( R \) be a reduced \( H \)-module \( K \)-algebra, where \( H \) is a pointed Hopf algebra acting on \( R \) finitely of dimension \( N \). Suppose that \( L^H \neq 0 \) for every nonzero \( H \)-stable left ideal \( L \) of \( R \). If \( R^H \) satisfies a polynomial identity of degree \( d \), then \( R \) satisfies the standard polynomial identity of degree \( dN \).

The proofs require some preparation. Recall that a module \( M \) is called uniform if the intersection of any two nonzero submodules is nonzero. We start with the following general observation.

**Lemma 5.** Let \( M \) be an irreducible (uniform) left \( R\#H \)-module and suppose that \( H \) acts finitely on \( M \). Then \( M \) has finite length (finite Goldie rank) as a left \( R \)-module.

**Proof.** Let \( M \) be an arbitrary (not necessarily irreducible) left \( R\#H \)-module. Let \( \pi : H \to \text{End}_R(M) \) be a homomorphism of algebras induced by the action of \( H \) on \( M \). By using the Taft-Wilson Theorem (see [M1, Theorem 5.4.1]) we can decompose \( H \) as a finite union \( \bigcup_{i=1}^{N} H_i \) of an increasing chain of subspaces \( \{ H_i \} \) such that

1. \( \pi(H_i) = \pi(H_{i-1}) + K \cdot \pi(h_i) \), where \( h_i = 1_H \) and \( h_i \in H \) for \( 2 \leq i \leq N \),
2. \( \Delta(h_i) \in \sigma \otimes h_i + h_i \otimes \tau + H_{i-1} \otimes H_{i-1} \), where \( \sigma, \tau \in G \) and \( 2 \leq i \leq N \).

Moreover, we can assume in (ii) that if \( h_i \neq \tau \) (that is, if \( h_i \) is not a group-like element), then \( \tau \in H_{i-1} \).

If \( A \) is an \( R \)-submodule of \( M \) and \( j \geq 1 \), let

\[
A_{(j)} = \{ m \in M \mid h_1m, \ldots, h_jm \in A \}.
\]

If \( h_i \in H \) satisfies (ii), then

\[
h_i(rm) = \sigma(r)h_i m + (h_i \cdot r)\tau m + \sum (h_{i1} \cdot r)h_{i2}m,
\]

where \( r \in R \), \( m \in M \) and \( h_{i1}, h_{i2} \in H_{i-1} \). Thus an easy induction argument shows that \( A_{(j)} \) is also an \( R \)-submodule of \( M \). Since \( \{ \pi(h_1), \ldots, \pi(h_N) \} \) is a \( K \)-basis of \( \pi(H) \), we obtain immediately that \( hA_{(N)} \subseteq A_{(N)} \), for all \( h \in H \); thus \( A_{(N)} \) is an \( R\#H \)-submodule. In fact \( A_{(N)} \) is the largest \( R\#H \)-submodule contained in \( A \).

Now if \( \{ A_{\alpha} \} \) is a chain of \( R \)-submodules of \( M \), each of which contains no nonzero \( R\#H \)-submodule, then \( \bigcup A_{\alpha} \) also contains no nonzero \( R\#H \)-submodule. Indeed, if \( B \subseteq \bigcup A_{\alpha} \) is a nonzero \( R\#H \)-submodule and \( 0 \neq b \in B \), then \( \{ h_1b, \ldots, h_Nb \} \subseteq A_{\alpha_0} \) for some \( \alpha_0 \). Therefore \( (R\#H)b \subseteq A_{\alpha_0} \), and so \( A_{\alpha_0} \) contains a nonzero \( R\#H \)-submodule, a contradiction. Consequently, by Zorn’s Lemma, there exists an \( R \)-submodule \( \widehat{A} \) of \( M \) which is maximal with respect to containing no nonzero \( R\#H \)-submodule. We can now consider the chain of \( R \)-submodules

\[
M \supseteq \widehat{A} = \widehat{A}_{(1)} \supseteq \widehat{A}_{(2)} \supseteq \cdots \supseteq \widehat{A}_{(N-1)} \supseteq \widehat{A}_{(N)} = 0.
\]

Now suppose that \( M \) is irreducible (resp. uniform) as a left \( R\#H \)-module. Since \( B_{(N)} \neq 0 \), for any \( R \)-submodule \( B \) properly containing \( \widehat{A} \), we see that the factor \( R \)-module \( M/\widehat{A} \) is irreducible (resp. uniform). If \( 1 \leq i \leq N-1 \), then we can consider the maps

\[
\varphi_i : \widehat{A}_{(i)} \to M/\widehat{A}
\]
Hence \( a \) is a nontrivial element of \( \Delta(i) \). By (ii) there exist \( \sigma, \tau \in G \) such that

\[
\Delta(h_{i+1}) - \sigma \otimes h_{i+1} - h_{i+1} \otimes \tau \in H_i \otimes H_i.
\]

Hence if \( r \in R \) and \( a \in \widehat{A}(i) \), then since \( H_i a \subseteq \widehat{A} \), we have

\[
\varphi_i(ra) = h_{i+1}(ra) + \widehat{A} = (h_{i+1}r)a + \widehat{A} = \sigma(r)h_{i+1}a + (h_{i+1} \cdot r)\tau a + \widehat{A} = \sigma(r)h_{i+1}a + \widehat{A} = \sigma(r)\varphi_i(a).
\]

It is easy to see that \( \ker \varphi_i = \widehat{A}(i) \). Therefore each \( \varphi_i \) induces an embedding of the lattice of \( R \)-submodules of \( \widehat{A}(i)/\widehat{A}(i+1) \) into the lattice of \( R \)-submodules of \( M/\widehat{A} \).

In our situation the \( R \)-module \( M/\widehat{A} \) is irreducible, so each \( \widehat{A}(i)/\widehat{A}(i+1) \) is either the zero module or irreducible (resp. uniform) as an \( R \)-module. Therefore \( M \) has a finite length (resp. finite Goldie rank), not exceeding \( N \), as an \( R \)-module. \( \square \)

Let \( Q = Q(R) \) be the symmetric Martindale quotient ring of \( R \). From the result of Montgomery (see [M2 Corollary 3.5]) it follows that when \( H \) is pointed, the \( H \)-action on \( R \) can be extended to an \( H \)-action on \( Q \). Moreover, it is easy to see that if \( H \) acts finitely on \( R \), then every essential ideal of \( R \) contains an \( H \)-stable ideal which is also essential in \( R \) (see [GH] Lemma 9). As a consequence of some basic properties of \( Q \), we obtain the following.

**Lemma 6.** Let \( H \) be a pointed Hopf algebra and let \( R \) be a semiprime left \( H \)-module algebra such that \( R^H \) is reduced and \( L^H \neq 0 \) for every nonzero \( H \)-stable left ideal \( L \) of \( R \). Suppose \( R^H \) satisfies a multilinear identity of degree \( d \) and \( H \) acts on \( R \) finitely of dimension \( N \). Then

1. \( L^H \neq 0 \), for every nonzero \( H \)-stable left ideal \( L \) of \( Q \).
2. \( Z(R^H) \subseteq Z(Q^H) \).
3. \( Q^H \) is reduced and satisfies the same multilinear identity as \( R^H \).
4. \( H \) acts finitely of dimension \( N \) on \( Q \).
5. \( R^H \) is almost central in \( R \), then \( Q^H \) is almost central in \( Q \).
6. \( R^H \subseteq Z \), then \( Q^H \subseteq C \).

**Proof.** For (1), if \( L \) is a nonzero \( H \)-stable left ideal of \( Q \), then \( L = L \cap R \) is a nonzero \( H \)-stable left ideal of \( R \). By assumption \( L^H \neq 0 \), so \( L^H \neq 0 \).

Before proving (2), notice that if \( I \) is an \( H \)-stable essential ideal of \( R \), then \( \text{1ann}_R(I^H \cap Z(R^H)) = 0 \). Indeed, it is clear that \( L = 1 \text{ann}_R(I^H \cap Z(R^H)) \) is an \( H \)-stable left ideal of \( R \). If \( L \neq 0 \), then \( 0 \neq I \cdot L \subseteq I \cap L \) and since \( R^H \) is reduced, we obtain that \( I \cap L^H \) is a two-sided ideal of \( R^H \). By assumption \( R^H \) satisfies a PI, so \( 0 \neq Z((I \cap L)^H) \subseteq Z(R^H) \). Thus one can choose a nonzero element \( c \in (I \cap L)^H \cap Z(R^H) \). But then \( c^2 \in L \cdot (I^H \cap Z(R)) = 0 \), which is impossible, since \( R^H \) is reduced. This also implies that \( 1 \text{ann}_Q(I^H \cap Z(R^H)) = 0 \). By using an easy induction argument, we obtain that for any \( d \geq 1 \),

\[
1 \text{ann}_Q((I^H \cap Z(R^H))^d) = 0.
\]

This immediately implies that \( Z(R^H) \subseteq Z(Q^H) \). To see this, take a nonzero \( q \in Q^H \) and an essential \( H \)-stable ideal \( J \) of \( R \) satisfying \( Jq \subseteq R \) and \( qJ \subseteq R \). Then, for any \( x \in J^H \) and \( c \in Z(R^H) \), we have \( qx \in R^H \) and

\[
0 = [q, x] = [q, c]x + q[x, c] = [q, c]x.
\]

Hence \( [q, c]J^H = 0 \) and by (1.1), \( [q, c] = 0 \). Consequently, \( Z(R^H) \subseteq Z(Q^H) \). This ends the proof of (2).
For the first part of (3), take \( q \in Q^H \) such that \( q^2 = 0 \), and \( I \) an essential \( H \)-stable ideal of \( R \) satisfying \( qI \cup Iq \subseteq R \). Then \( (I^H \cap Z(R^H))q \subseteq R \) and using (2) we obtain \( ((I^H \cap Z(R^H))q)^2 = q^2(I^H \cap Z(R^H))^2 = 0 \). Since \( R^H \) is reduced, \( (I^H \cap Z(R^H))q = 0 \) and (2.1) forces that \( q = 0 \). Therefore \( Q^H \) is reduced.

Now let \( f(x_1, x_2, \ldots, x_d) = \sum_{\sigma \in S_d} a_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(d)} \) be a multilinear polynomial such that the identity \( f(x_1, x_2, \ldots, x_d) = 0 \) is satisfied by \( R^H \). Take \( q_1, q_2, \ldots, q_d \in Q^H \) and an \( H \)-stable essential ideal \( I \) of \( R \) such that \( q_j I \subseteq R \) for \( j = 1, 2, \ldots, d \). Then for all \( c_1, c_2, \ldots, c_d \in I^H \cap Z(R^H) \) we have \( c_i q_j \in R^H \), so by using (2),

\[
0 = f(c_1 q_1, c_2 q_2, \ldots, c_d q_d) = f(q_1, q_2, \ldots, q_d) c_1 c_2 \cdots c_d.
\]

This means that \( f(q_1, q_2, \ldots, q_d) \in \text{l.ann}_Q((I^H \cap Z)^d) = 0 \). Thus the identity

\[
f(x_1, x_2, \ldots, x_d) = 0
\]

is satisfied also by \( Q^H \). This proves (3).

For (4), let \( \hat{\pi} : H \to \text{End}_K(Q) \) be the natural \( K \)-algebra homomorphism, corresponding to the action of \( H \) on \( Q \). We need to show that \( \ker \pi = \ker \hat{\pi} \). The inclusion \( \ker \pi \supseteq \ker \hat{\pi} \) is clear. Suppose \( h \in \ker \pi \). Take \( q \in Q \) and \( I \) an essential \( H \)-stable ideal of \( R \) such that \( qI \subseteq R \). Since \( \pi(h) \) is an \( R^H \)-bimodule map, we obtain that

\[
\hat{\pi}(h)(q)a = \hat{\pi}(h)(qa) = \pi(h)(qa) = 0
\]

for any \( a \in I^H \). Hence \( \hat{\pi}(h)(q) \in \text{l.ann}_Q(I^H) \subseteq \text{l.ann}_Q((I^H \cap Z(R^H))) = 0 \). Thus \( h \in \ker \hat{\pi} \) and consequently \( \ker \hat{\pi} = \ker \pi \). Thus \( \text{dim}_K \hat{\pi}(H) = \text{dim}_K \pi(H) \).

For (5), if \( L \) is a nonzero \( H \)-stable left ideal of \( Q \), then \( \hat{L} = L \cap R \) is a nonzero \( H \)-stable left ideal of \( R \). Since \( Z(R) \subseteq C \) and \( \hat{L}^H \cap Z(R) \neq 0 \), we obtain that \( L^H \cap C \neq 0 \). Thus \( Q^H \) is almost central in \( Q \).

For (6), take \( q \in Q^H \) and an \( H \)-stable essential ideal \( I \) of \( R \) such that \( qI \subseteq R \). If \( c \in I^H \), then \( qc \in R^H \subseteq Z \) and hence

\[
(qr - rq)c = (qr)c - r(qc) = (qc)r - (qc)r = 0,
\]

for any \( r \in R \). Thus \( rq - qr \in \text{l.ann}_Q(I^H \cap Z(R^H)) = 0 \). Therefore \( q \) centralizes \( R \), so \( q \in C \).

We are now ready to prove the first main result of the paper.

**Proof of Theorem 3.** By Lemma 3 all assumptions on \( R \) can be lifted to \( Q \). Let \( h_1, h_2, \ldots, h_N \in H \) be such that \( \{\pi(h_1), \pi(h_2), \ldots, \pi(h_N)\} \) is a basis for \( \pi(H) \subseteq \text{End}_K(Q) \). Notice that for any \( q \in Q \) the left ideal \( L = \sum_{i=1}^{N} \text{Q}(h_i \cdot q) \) is \( H \)-stable.

By applying the remarks after Lemma 1 we see that any finitely generated (as a left \( Q \)-module) left ideal of \( Q \) is contained in an \( H \)-stable finitely generated left ideal which is also injective as a \( C^H \)-module.

Let \( M \) be a maximal ideal of \( C^H \) and \( \eta_M : Q \to Q_M \) be a natural ring homomorphism, where \( Q_M \) is the localization of \( Q \) at \( S = C^H \setminus M \). By Lemma 1 it follows that \( Q_M \) is semiprime and Lemma 6 shows that \( (Q_M)^H = (Q^H)_M \) satisfies a multilinear identity of degree \( d \). We claim that \( (Q^H)_M \) is almost central in \( Q_M \). Take a nonzero \( H \)-stable left ideal \( T \) of \( Q_M \) and choose a finitely generated \( H \)-stable left ideal \( L \) of \( Q \) such that \( 0 \neq \eta_M(L) \subseteq T \). Then \( L \) is injective as a left \( C^H \)-module and by Lemma 1(3), \( L^H \) is also injective as a left \( C^H \)-module. Since \( C \) is injective over \( C^H \), the intersection \( L^H \cap C \) is injective as a \( C^H \)-module.

By Lemma 1(2), there exist \( x \in L^H \cap C \) and an idempotent \( \tilde{c}_x \in C^H \) such that
ann_{C^H}(L^H \cap C) = \text{ann}_{C^H}(x) = (1 - \widehat{e}_x)C^H. We claim that 
(1 - \widehat{e}_x)L = 0. If not, then 
(1 - \widehat{e}_x)L is a nonzero $H$-stable left ideal of $Q$. Since $Q^H$ is almost central in $Q$, we can choose a nonzero $c \in ((1 - \widehat{e}_x)L)^H \cap C$. Then $c \in L^H \cap C$ and $c = (1 - \widehat{e}_x)c \in (1 - \widehat{e}_x)C^H = \text{ann}_{C^H}(L^H \cap C)$. Therefore $c^2 = 0$, which is impossible because $C$ is a field. This proves the claim. Since $\eta_M(L) \neq 0$, $1 - \widehat{e}_x \in M$. Hence $\text{ann}_{C^H}(x) \subseteq M$ and thus $0 \neq Q^H(x) \in T^H \cap C_M$. Therefore $(Q_M)^H$ is almost central in $Q_M$.

On the other hand by Lemma 1(5), $(Q_M)^H \cap C_M$ is a field, so $Q_M$ does not contain proper $H$-stable left ideals. Thus $Q_M$ is an irreducible left $Q_M\#H$-module. By Lemma 5, $Q_M$ has finite length as a left $Q_M$-module, so in particular $Q_M$ has finite left Goldie rank. We are now in a position to apply Theorem 2. It asserts that $(Q_M)^H$ is a division ring and 
$\text{dim}_Q ((Q_M)^H) \leq n \leq N$. If we let $A_M$ denote the annihilator ideal $\{w \in Q_M\#H \mid wQ_M = 0\}$, then $Q_M\#H/A_M \cong M_n((Q_M)^H)$. The division algebra $(Q_M)^H$ satisfies a polynomial identity of degree $\eta$, so $M_n((Q_M)^H)$ satisfies the standard polynomial identity $s_{dN}$ of degree $dn \leq dN$. Since $Q_M$ is semiprime we have an embedding $Q_M \hookrightarrow Q_M\#H/A_M$. Thus for any maximal ideal $M$ of $C^H$, the localization $Q_M$ satisfies the standard polynomial identity $s_{dN}$. The fact that $C^H$ is von Neumann regular implies immediately the existence of an embedding $Q \hookrightarrow \prod M Q_M$, where the product is taken over all maximal ideals of $C^H$. Therefore $Q$ satisfies $s_{dN}$.

If we additionally assume that $R^H \subseteq Z$, then by Lemma 6, $Q^H \subseteq C$. Thus for a given maximal ideal $M$ of $C^H$, $Q_M$ is a semisimple finite-dimensional algebra containing a central subfield $(C_M)^H$ such that $\text{dim}_{C_M^H} Q_M \leq N$. Therefore, the Amitsur-Levitzki Theorem asserts that $Q_M$ satisfies the standard polynomial identity of degree $2\sqrt{N}$. As a result, if the invariants $R^H$ are central in $R$, then $R$ satisfies $s_{2\sqrt{N}}$, thereby concluding the proof. \hfill \Box

Proof of Theorem 4 Let us first consider the special case where $R^H$ is a domain. Then, by Posner’s Theorem, $R^H$ is a Goldie ring. Furthermore, if we put $T = Z(R^H) \setminus \{0\}$, then the localization $T^{-1}R^H$ is a division algebra with center $Z = T^{-1}Z(R^H)$ and $\text{dim}_Z T^{-1}R^H \leq (\frac{4}{3})^2$. It is easy to see that every nonzero element $z \in Z(R^H)$ is regular in $R$. In fact, since $R$ is reduced, $J = \text{Ann}_R(z) = r. \text{Ann}_R(z)$ is a two-sided $H$-stable ideal of $R$. If $J$ is nonzero, then $Z(J^H) \neq 0$ (because $R^H$ satisfies a PI), and clearly $Z(J^H) \subseteq Z(R^H)$. But $Z(J^H)z = 0$, and this contradicts our assumption that $R^H$ is a domain. We claim that the subset $T$ satisfies the left Ore condition in $R$. To see this, note that by Lemma 5, $R$ has a finite left Goldie rank. Furthermore, $R$, as a reduced ring, certainly has zero singular ideal. Thus $R$ is left Goldie. Now it is enough to show that any essential left ideal of $R$ intersects $T$ nontrivially. Since the group $G = G(H)$ of group-like elements acts finitely, we need only consider essential left ideals which are $G$-stable. Let $L$ be a $G$-stable essential left ideal of $R$ and, using the notation in Lemma 3, let $L_{(j)} = \{x \in L \mid h_1 \cdot x, \ldots, h_j \cdot x \in L\}$. We will show by induction that $L_{(j)}$ is essential for all $j \geq 1$. To see this, let $I$ be any nonzero left ideal of $R$. Given $0 \neq a \in L_{(j-1)} \cap I$, the left ideal $E = \bigcap_{1 \leq j} \{r \in R \mid r(h_i \cdot a) \in L\}$ is essential. Since $R$ is nonsingular, we can choose $r \in \bigcap_{\sigma \in G} E^\sigma$ with $ra \neq 0$. Then

$$h_j \cdot (ra) = \sigma(r)(h_j \cdot a) + (h_j \cdot r)\tau(a) + \sum (h_{j_1} \cdot r)(h_{j_2} \cdot a) \in L.$$
Thus 0 ≠ ra ∈ L(j), so L(j) is essential. In particular ˆL = L(N) is nonzero and H-stable. By assumption ˆL is a nonzero left ideal of R^H. But R^H is a PI domain, so any nonzero left ideal intersects T nontrivially. This proves the claim.

Notice that T^{-1}R is in a natural way a left H-module algebra and (T^{-1}R)^H = T^{-1}R^H is a division algebra satisfying a polynomial identity of degree d. It is also clear that T^{-1}R has no proper left H-stable ideals, so T^{-1}R becomes an irreducible left (T^{-1}R)#H-module. Applying the same argument as in the proof of our previous theorem, we obtain that T^{-1}R satisfies s_{dN}. Therefore R also satisfies the standard identity s_{dN}.

For the general case, since R is reduced, the symmetric Martindale quotient ring Q is also reduced. Similarly, as in Theorem 3 let us consider a maximal ideal M of C^H and the canonical map η_M : Q → Q_M. We claim that (Q^H)_M is a domain. To this end, let a, b ∈ Q^H be such that ab = 0, and let e_a, e_b ∈ C^H be idempotents such that ann_Q(QaQ) = (1 − e_a)Q and ann_Q(QbQ) = (1 − e_b)Q. Since Q is reduced, ann_Q(a) = r.ann_Q(QaQ) = ann_Q(QaQ) and e_a.e_b = 0. On the other hand ann_Q(b) = (1 − e_b)Q, so there exists x ∈ Q satisfying e_a = (1 − e_b)x. Now it is clear that e_a.e_b = 0, and thus either e_a ∈ M or e_b ∈ M. This immediately implies that either η_M(a) = 0 or η_M(b) = 0. Therefore (Q^H)_M is a domain, as claimed. Notice that the ring Q_M is reduced. Indeed, if q ∈ Q and c ∈ C^H \ M are such that cq^2 = 0, then (cq)^2 = 0 and cq = 0, since Q is reduced. Consequently η_M(q) = 0. As a result Q_M is a reduced left H-module algebra and its subalgebra of invariants (Q_M)^H = (Q^H)_M is a domain satisfying a polynomial identity of degree d. By the previous paragraph Q_M satisfies s_{dN}. Since this holds for any maximal ideal M of C^H, the ring Q satisfies s_{dN}.

We close the paper with a remark concerning actions on reduced algebras. We see that the result of Kharchenko for group actions (mentioned in the introduction) is a direct consequence of his fundamental result on the existence of fixed elements and Theorem 3. Moreover, Beidar and Grzeszczuk proved in [BeG] an analogous result on the existence of nontrivial constants for actions of Lie algebras. Finally, Theorem 3 now provides us with a common proof of the following.

Corollary 7. Let R be a reduced algebra. Then

1. (cf. [K2]) if R is acted on by a finite group G and R^G satisfies a PI of degree d, then R satisfies a PI of degree d|G|.

2. If R is acted finitely on by a finite-dimensional Lie algebra L and R^L satisfies a PI of degree d, then R satisfies a PI of degree dN, where N is the dimension of the action.

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