EMBEDDINGS OF $n$-DIMENSIONAL SEPARABLE METRIC SPACES INTO THE PRODUCT OF SIERPIŃSKI CURVES

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Abstract. We give a short proof of the following fact: the set of embeddings of any $n$-dimensional separable metric space $X$ into a certain $n$-dimensional subset of the $(n+1)$-product of Sierpiński curves $\Sigma$ is residual in $C(X, \Sigma^{n+1})$.

Introduction and notation

In a Sierpiński curve we can specify the 0-dimensional subset of a “rational” points. In [5], Ivanšić and Milutinović proved that the $(n+1)$-product of Sierpiński curves, with points whose coordinates are all rational removed, is a universal space for $n$-dimensional metric separable spaces.

In this paper we present a new short proof of the last result. The proof is similar in spirit to proof of Theorem 1.1 in [6]. Sternfeld proved in his paper that any $n$-dimensional compact metric space may be embedded in the $(n+1)$-product of dendrites $D$. He also showed that the set of the basic embeddings is dense in $C(X, D^{n+1})$. However, it is worth pointing out that concerned in [6] is the fact that dendrites are ARs. Since a Sierpiński curve has not a good extension property, we use Lemma 3 to approximate it by ANRs. We also use in our proof the idea of the disjoint disk property that has been used in the topology of infinite- or finite-dimensional manifolds (cf. [7]).

All maps in this paper are continuous. Maps $f, g : X \to Y$ are said to be $\varepsilon$-near if $\sup_{x \in X} \text{dist}(f(x), g(x)) < \varepsilon$. A map $f : X \to Y$ is an $\varepsilon$-map if each point $y \in Y$ has an open neighbourhood $V_y$ such that $\text{diam}(f^{-1}(V_y)) < \varepsilon$. We denote by $B(x, r)$ the open ball with centre $x$ and radius $r$.

We shall use the following proposition (it is an easy application of Eilenberg’s Theorem; see [1])

Proposition 1. Let $X$ be a compact metric space and $Y$ be a metric ANR. Then for each map $f : X \to Y$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each surjective $\delta$-map $p : X \to X'$ there exists a map $q : X' \to Y$ such that $\text{dist}(f, q \circ p) < \varepsilon$.
Main theorem

Let us recall the construction of the triangular Sierpiński curve. Consider the homotheties \( \varphi_i : \mathbb{R}^3 \to \mathbb{R}^3 \) with scale \( \frac{1}{3} \) and centres \( e_i \), where \( e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1) \). With \( \Sigma = \text{conv}(e_1,e_2,e_3) \) we let

\[
\Sigma_m := \bigcup_{\lambda_1, \ldots, \lambda_m \in \Lambda} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m}(\Sigma), \quad \text{where } \Lambda = \{1,2,3\}, \quad \text{and } \Sigma(3) := \bigcap_{m \in \mathbb{N}} \Sigma_m.
\]

We also let \( \Sigma_n(3) := \Sigma \setminus \text{Int } \Sigma_n \). Clearly, \( \Sigma_n(3) \) is a graph and hence an ANR, for each \( n \in \mathbb{N} \).

We call the vertices of triangles obtained in the construction of \( \Sigma(3) \) rational points, and other points of \( \Sigma(3) \) irrational points. Let

\[
L_n(3) := \{ x \in \Sigma(3)^{n+1} : \text{at least one coordinate of } x \text{ is irrational} \}.
\]

**Theorem 2.** Suppose that \( X \) is a metric separable space. If \( \dim(X) \leq n \), then the set of embeddings of \( X \) in \( L_n(3) \) is dense in the space \( C(X,\Sigma(3)^{n+1}) \).

The original proof \[5\] of this result depended on complex arguments involving brick decompositions. Instead, here we use a simple property of compact metric spaces and the fact that a Sierpiński curve can be approximated by the graphs \( \Sigma_n(3) \):

**Lemma 3.** For each \( \varepsilon > 0 \) there exists a natural number \( k \) and a retraction \( r_k \) of \( \Sigma(3) \) onto \( \Sigma_k(3) \) such that \( d(r_k, \text{id}_{\Sigma(3)}) < \varepsilon \).

**Proof.** Let \( k \) be such that \( \frac{1}{k} < \varepsilon \). For each \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda^k \) the Sierpiński curve \( \Sigma(3) \) intersects the simplex \( S_\lambda = \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k}(\Sigma) \) along a set containing the boundary of \( \text{Bd}(S_\lambda) \) but different from \( S_\lambda \). Therefore \( \Sigma(3) \cap S_\lambda \) may be retracted onto \( \text{Bd}(S_\lambda) \). The union of these retractions, taken over all \( \lambda \in \Lambda^k \), does the job. \( \square \)

**Lemma 4.** Let \( X \) be a compact metric space and \( f : X \to \Sigma(3)^{n+1} \) be a map. Then for each \( \varepsilon > 0 \) and for each pair of disjoint compact sets \( A, B \subset X \) such that \( \dim A \leq n \), there exists a map \( f_\varepsilon : X \to \Sigma(3)^{n+1} \) such that \( \text{dist}(f,f_\varepsilon) < \varepsilon \), \( f_\varepsilon(A) \cap f_\varepsilon(B) = \emptyset \) and \( f_\varepsilon(A) \subset L_n(3) \).

**Proof.** The proof is by induction on \( n \). First let \( n = 0 \). We fix \( \varepsilon > 0 \) and take \( k \) so large that the retraction \( r_k : \Sigma(3) \to \Sigma_k(3) \) satisfies \( \text{dist}(\text{id}_{\Sigma(3)},r_k) < \varepsilon/3 \). We also cover \( A \) by finitely many disjoint \( \delta \)-small compact sets, where \( \delta < \text{dist}(A,B) \), and denote by \( X' \) the space obtained from \( X \) by squeezing each of them to a point. Let us observe that by \[3\] (Theorem 4.4.15) \( X' \) is a metrizable space. By Proposition\[1\] for \( \delta \) small enough the map \( r_k \circ f \) is \( \varepsilon/3 \)-close to the composition of the projection \( p : X \to X' \) and of a map \( f' : X' \to \Sigma_k(3) \subset \Sigma(3) \). Thus, by replacing \( f \) by \( f' \), and \( A \) and \( B \) by \( p(A) \) and \( p(B) \), respectively, we may assume that the set \( A \) is finite, and by treating each of its points individually - that \( A \) has only one point, which we denote \( a \). We assume these arrangements have been made; in particular, \( \text{Im}(f) \subset \Sigma_k(3) \).

Let \( B(a,\eta) \) be a ball in \( X \) centred at \( a \), \( W = X/\text{Bd}B(a,\eta) \) be the decomposition space, obtained by squeezing \( \text{Bd}B(a,\eta) \) to a point, and let \( p : X \to W \) be the projection. By taking \( \eta \) small enough we ensure that \( \text{diam}(f(B(a,\eta))) < \varepsilon/3 \) and there exists a map \( q : W \to \Sigma_k(3) \subset \Sigma(3) \) such that \( q \circ p \) is \( \varepsilon/3 \)-near to \( f \). (We use Proposition\[1\].)
Let us fix $b \in \text{Bd}(B(a, \eta))$. There exists an $\epsilon/3$-short arc $J$ in $\Sigma(3)$ joining $q(p(b))$ with an irrational point $c \in \Sigma(3) \setminus \Sigma_k(3)$. Let $h : W \to \Sigma(3)$ be a map such that $h(x) = q(x)$ for $x \in W \setminus p(B(a, \eta))$ and $h(p(a)) = c$. The arc $J$ is an AR so we can extend the map $h$ to the whole space $W$ so that $h(p(B(a, \eta)) \subset J$.

Let us define $f_x = h \circ \rho$. If $x \in X \setminus B(a, \eta)$, then $f_x(x) = q \circ \rho(x)$ and hence $d(f_x(x), f(x)) < \epsilon/3$, while for $x \in B(a, \eta)$ we have $f(x), f_x(x) \in J$ and $f(b), f(x) \in f(B(a, \eta))$, whence

$$d(f_x(x), f(x)) \leq d(f_x(x), f_x(b)) + d(f_x(b), f(b)) + d(f(b), f(x)) < 3 \cdot \epsilon/3$$

because $\text{diam}_J, \text{diam}_B(B(a, \eta)) < \epsilon/3$. Thus $d(f, f_x) < \epsilon$ and since $f_x(a) \in \Sigma(3) \setminus \Sigma_k(3)$ and $f_x(B) \subset \Sigma_k(3)$ the assertion of the lemma is true if $n = 0$.

Assume now that the lemma holds true if dim $A \leq n - 1$. From the Baire Theorem and from the fact that $L_0(3)$ is of $G_\delta$ type we conclude that

(*)

the lemma is true if $A$ is $\sigma$-compact and dim $A \leq n - 1$.

Now, given $f = (f_1, f_2) : X \to \Sigma(3)^n \times \Sigma(3)$ and a compact set $A \subset X$ with dim $A \leq n$, we can represent $A$ as $A = Y \cup Z$, where dim $Y \leq n - 1$, dim $Z \leq 0$ and $Z$ is $\sigma$-compact. (See [2], §1.5.) By (*) we can assume that $f_2(Z) \cap f_2(B) = \emptyset$ and $f_2(Z) \subset L_0(3)$. The set $Y' = (f_2 \mid_A)^{-1}(f_2(B) \cup (\Sigma(3) \setminus L_0(3)))$ is disjoint from $Z$, $n - 1$-dimensional, and $\sigma$-compact (for $L_0(3)$ is of type $G_\delta$). Thus by (*) there exists a map $f_1' : X \to \Sigma(3)^n$ $\varepsilon$-near to $f_1$ such that $f_1'(Y') \subset L_{n-1}(3)$ and $f_1'(Y') \cap f_2'(B) = \emptyset$. We take $f_x(x) = (f_1'(x), f_2(x))$ to complete the proof. \hfill \Box

**Proof of Theorem** [2]. Let $f \in C(X, \Sigma(3)^{n+1})$. By [4] there exists a dimension preserving compactification $X^*$ of the space $X$ such that the map $f : X \to \Sigma(3)^{n+1}$ can be extended to $f^* : X^* \to \Sigma(3)^{n+1}$. Hence, without loss of generality, we may assume that $X$ is compact.

Let $\mathcal{B}$ be a countable family of closed subsets of $X$ whose interiors are a base of the topology of $X$. Let us observe that

$$\{h \in C(X, \Sigma(3)^{n+1}) : h \text{ is an embedding}\}$$

$$= \bigcap_{A, B \in \mathcal{B}} \{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}.$$ 

Fix $A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$. By Lemma [3] the set $\{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}$ is dense in $C(X, \Sigma(3)^{n+1})$. Also by Lemma [4] (with $A = X$ and $B = \emptyset$) the set $\{f \in C(X, \Sigma(3)^{n+1}) : f(X) \subset L_n(3)\}$ is dense in $C(X, \Sigma(3)^{n+1})$. The sets $\{f \in C(X, \Sigma(3)^{n+1}) : f(X) \subset L_n(3)\}$ and $\bigcap_{A, B \subset X} \{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}$ are obviously of type $G_\delta$. Now the proof is completed by an application of Baire’s Theorem. \hfill \Box

**Remark 5.** The following property of a Sierpiński curve can be proved analogously to Lemma [3].

Let $F$ be a $\sigma$-closed 0-dimensional subset of a compact metric space $X$. Then $\{f \in C(X, \Sigma(3)) : f^{-1}f(x) = \{x\} \text{ for all } x \in F \text{ and } f(F) \text{ contains no rational points} \}$ is a dense $G_\delta$-set in $C(X, \Sigma(3))$.

This fact is similar to Theorem 1.1 in [5] that is a key result of Sternfeld’s paper. Using this property of a Sierpiński curve and the argument that the set of trivial fibers of map $f$ is $G_\delta$-set we can construct an embedding into $L_n(3)$. This way of construction is analogous to Sternfeld’s methods in [6].
Remark 6. In [6], Sternfeld noted that in the case of embeddings into \((n + 1)\)-product of dendrites the last space in that product can be replaced by an interval. The same can be done also for the product of Sierpiński curves.

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References


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