SEQUENTIALLY COHEN-MACaulay Edge Ideals

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Abstract. Let $G$ be a simple undirected graph on $n$ vertices, and let $I(G) \subseteq R = k[x_1, \ldots, x_n]$ denote its associated edge ideal. We show that all chordal graphs $G$ are sequentially Cohen-Macaulay; our proof depends upon showing that the Alexander dual of $I(G)$ is componentwise linear. Our result complements Faridi’s theorem that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay and implies Herzog, Hibi, and Zheng’s theorem that a chordal graph is Cohen-Macaulay if and only if its edge ideal is unmixed. We also characterize the sequentially Cohen-Macaulay cycles and produce some examples of nonchordal sequentially Cohen-Macaulay graphs.

1. Introduction

Let $G$ be a simple graph on $n$ vertices (so $G$ has no loops or multiple edges between two vertices). Denote the vertex and edge sets of $G$ by $V_G$ and $E_G$ respectively. We associate to $G$ the quadratic squarefree monomial ideal $I(G) \subseteq R = k[x_1, \ldots, x_n]$, with $k$ a field, where $I(G) = \langle \{x_i x_j \mid \{x_i, x_j\} \in E_G\} \rangle$. The ideal $I(G)$ is called the edge ideal of $G$.

The primary focus of this paper is edge ideals of chordal graphs. A graph $G$ is chordal if every cycle of length $n > 3$ has a chord. Here, if $\{x_1, x_2\}, \ldots, \{x_n, x_1\}$ are the $n$ edges of a cycle of length $n$, we say the cycle has a chord in $G$ if there exist two vertices $x_i, x_j$ in the cycle such that $\{x_i, x_j\}$ is also an edge of $G$, but $\{x_i, x_j\}$ is not an edge of the cycle.

We say that a graph $G$ is Cohen-Macaulay if $R/I(G)$ is Cohen-Macaulay. As Herzog, Hibi, and Zheng point out, classifying all the Cohen-Macaulay graphs is probably not tractable right now; this problem is as difficult as classifying all Cohen-Macaulay simplicial complexes [11]. However, Herzog, Hibi, and Zheng proved in [11] that when $G$ is a chordal graph, then $G$ is Cohen-Macaulay (over any field) if and only if $I(G)$ is unmixed.

The property of being sequentially Cohen-Macaulay, a condition weaker than being Cohen-Macaulay, was introduced by Stanley [15] in connection with the theory of nonpure shellability.
Definition 1.1. Let $R = k[x_1, \ldots, x_n]$. A graded $R$-module $M$ is called sequentially Cohen-Macaulay (over $k$) if there exists a finite filtration of graded $R$-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

We say that a graph $G$ is sequentially Cohen-Macaulay (over $k$) if $R/I(G)$ is sequentially Cohen-Macaulay. We can expand upon Herzog, Hibi, and Zheng’s result by using this weakening of the Cohen-Macaulay condition. Our main result is the following theorem (which is independent of $\text{char}(k)$).

Theorem 1.2 (Theorem 3.2). All chordal graphs are sequentially Cohen-Macaulay.

Thus even chordal graphs whose edge ideals are not unmixed still satisfy a good algebraic property. Theorem 3.2 also generalizes the one-dimensional case of the work of Faridi on simplicial forests [4].

Our paper is organized as follows. In the next section, we gather some results from the literature on Alexander duality and on chordal graphs. In Section 3, we prove Theorem 3.2. We consider some nonchordal graphs in Section 4, classifying the sequentially Cohen-Macaulay cycles and investigating some properties of graphs containing $n$-cycles for $n > 3$. We also give a sufficient condition for a graph to fail to be sequentially Cohen-Macaulay.

2. Required ingredients

Throughout this paper $G$ will denote a simple graph on $n$ vertices with vertex set $V_G$ and edge set $E_G$. Associated to $G$ is the edge ideal $I(G) \subseteq R = k[x_1, \ldots, x_n]$, where $I(G) = (\{x_ix_j \mid \{x_i, x_j\} \in E_G\})$.

The complete graph on $n$ vertices, denoted $K_n$, is the graph with edge set $E_G = \{\{x_i, x_j\} \mid 1 \leq i < j \leq n\}$, i.e., the graph with the property that there is an edge between every pair of vertices. If $x$ is a vertex of $G$, we shall write $N(x)$ to denote the neighbors of $x$, that is, those vertices that share an edge with $x$. We shall be primarily interested in the case that $G$ is a chordal graph. Chordal graphs have the following property:

Lemma 2.1 ([6, Lemma 6.7.12]). Let $G$ be a chordal graph, and let $K$ be a complete subgraph of $G$. If $K \neq G$, then there is a vertex $x \not\in V_K$ such that the subgraph induced by the neighbor set $N(x)$ of $x$ is a complete subgraph. This also forces the subgraph induced on $N(x) \cup \{x\}$ to be a complete subgraph.

A vertex cover of a graph $G$ is a subset $A$ of $V_G$ such that every edge of $G$ is incident to at least one vertex of $A$. Note that we never need to include an isolated vertex in a vertex cover. For example, if we have a graph on three vertices $x_1$, $x_2$, and $x_3$, and $\{x_1, x_2\}$ is the only edge, then $\{x_1\}$ and $\{x_2\}$ are both vertex covers. The vertex covers of a graph $G$ are related to the Alexander dual of $I(G)$.

Definition 2.2. Let $I$ be a squarefree monomial ideal. The squarefree Alexander dual of $I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t})$ is the ideal

$$I^\vee = (x_{1,1}, \ldots, x_{1,s_1}) \cap \cdots \cap (x_{t,1}, \ldots, x_{t,s_t}).$$
A simple exercise will then verify:

**Lemma 2.3.** Let $G$ be a simple graph with edge ideal $\mathcal{I}(G)$. Then
$$\mathcal{I}(G)^\vee = \{x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \text{ is a vertex cover of } G\},$$
and the minimal generators of $\mathcal{I}(G)^\vee$ correspond to minimal vertex covers.

Associated to any homogeneous ideal $I$ of $R$ is a **minimal free graded resolution**
$$0 \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{j,j}(I)} \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$
where $R(-j)$ denotes the $R$-module obtained by shifting the degrees of $R$ by $j$. The number $\beta_{i,j}(I)$ is the **$ij$-th graded Betti number** of $I$ and equals the number of minimal generators of degree $j$ in the $i$-th syzygy module.

**Definition 2.4.** Suppose $I$ is a homogeneous ideal of $R$ whose generators all have degree $d$. Then $I$ has a **linear resolution** if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

For a homogeneous ideal $I$, we write $(I_d)$ to denote the ideal generated by all degree $d$ elements of $I$. Note that $(I_d)$ is different from $I_d$, the vector space of all degree $d$ elements of $I$. Herzog and Hibi introduced the following definition in [9].

**Definition 2.5.** A homogeneous ideal $I$ is **componentwise linear** if $(I_d)$ has a linear resolution for all $d$.

If $I$ is generated by squarefree monomials, let $I_{[d]}$ denote the ideal generated by the squarefree monomials of degree $d$ of $I$. Herzog and Hibi [9, Proposition 1.5] showed:

**Theorem 2.6.** Suppose $I$ is a monomial ideal generated by squarefree monomials. Then $I$ is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all $d$.

One can use linear quotients to determine if an ideal has a linear resolution.

**Definition 2.7.** Let $I$ be a monomial ideal of $R$. We say that $I$ has **linear quotients** if for some ordering $u_1, \ldots, u_m$ of the minimal generators of $I$ with\[ \deg u_1 \leq \deg u_2 \leq \cdots \leq \deg u_m \]and all $i > 1$, $(u_1, \ldots, u_{i-1}) : (u_i)$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

We then require [3, Lemma 5.2]:

**Lemma 2.8.** If $I = (u_1, \ldots, u_m)$ is a monomial ideal of $R = k[x_1, \ldots, x_n]$ that has linear quotients, and all the $u_i$ have the same degree, then $I$ has a linear resolution.

We end this section by applying these ideas to edge ideals.

**Lemma 2.9.** If $\mathcal{I}(G)$ is the edge ideal of a graph $G$, then
$$\mathcal{I}(G)_{[d]}^\vee = \{x_{i_1} \cdots x_{i_d} \mid \{x_{i_1}, \ldots, x_{i_d}\} \text{ is a vertex cover of } G \text{ of size } d\}.$$

**Proof.** Since $\mathcal{I}(G)^\vee$ is generated by the minimal vertex covers, any squarefree monomial of degree $d$ in $\mathcal{I}(G)^\vee$ corresponds to a set of $d$ vertices which contains a minimal vertex cover, and thus, the $d$ vertices also form a vertex cover of $G$. \[ \square \]

**Lemma 2.10.** Let $G = K_n$ be the complete graph on $n$ vertices. For each $d$, $\mathcal{I}(G)^\vee_{[d]}$ has linear quotients. Thus $\mathcal{I}(G)^\vee$ is componentwise linear.
I suffices to show that though they live in different rings. Thus

\[ I \]

is componentwise linear. To show \( I \) is componentwise linear, we have based our proof on Faridi’s proof of \([4]\) Theorem 5.4] that the squarefree part of the facet ideal of a simplicial forest has linear quotients in each degree. By Theorem 2.6

Proof. We show that for each \( d \), \( I(G)^{n_d} \) has linear quotients and hence a linear resolution, which means that \( I(G)^{n} \) is componentwise linear by Theorem 2.6.

The minimal vertex covers of \( K_n \) are all subsets of \( V(K_n) \) of size \( n - 1 \). Hence, by Lemma 2.10 \( I(K_n)^{n_d} = (0) \) if \( d < n - 1 \) or \( d > n \). When \( d = n \), \( I(K_n)^{n_d} = (x_1, x_2 \cdots x_n) \) is a principal ideal. These cases trivially have linear quotients. It thus suffices to show that \( I(K_n)^{n_{n-1}} \) has linear quotients.

Note that \( I(K_n)^{n} \) is minimally generated by all squarefree monomials of degree \( n - 1 \), and hence \( I(K_n)^{n} = I(K_n)^{n_{n-1}} \). Now \( I(K_n)^{n} \) is a squarefree Veronese ideal and thus has a linear resolution \([10]\). Hence \( I(K_n)^{n_{n-1}} \) has linear quotients if one orders the monomials in descending lexicographic order.

\[ \square \]

Remark 2.11. A statement more general than Lemma 2.10 is true. Let \( j \leq n \), and let \( R = k[x_1, \ldots, x_n] \). We can consider ideals whose components are all possible ideals generated by \( j \) of the \( n \) variables:

\[ I = (x_1, \ldots, x_j) \cap (x_1, \ldots, x_{j-1}, x_{j+1}) \cap \cdots \cap (x_{n-j+1}, \ldots, x_n). \]

We can view these ideals as the Alexander duals of either the Stanley-Reisner ideal of a simplicial complex with all possible \((j-2)\)-faces but no \((j-1)\)-faces or as the facet ideal of a simplicial complex with all possible \((j-1)\)-faces as its facets. \( I \) is minimally generated by all squarefree monomials of degree \( n - j + 1 \), and hence it is a squarefree Veronese ideal. Thus \( I \) has a linear resolution and is therefore componentwise linear.

For our last lemma we show that to determine if \( I(G)^{n} \) is componentwise linear, we may reduce to the case in which the graph \( G \) has no isolated vertices.

Lemma 2.12. Let \( G \) be a simple graph on \( n \) vertices with edge ideal \( I(G) \subseteq R = k[x_1, \ldots, x_n] \). Let \( H \) be the graph \( G \) with isolated vertices \( x_{n+1}, \ldots, x_m \) added. Assume that \( I(G)^{n} \) is componentwise linear. Then \( I(H)^{n} \subseteq k[x_1, \ldots, x_m] \) is componentwise linear.

Proof. Note that the edge ideals of \( G \) and \( H \) have the same minimal generators, though they live in different rings. Thus \( I(G)^{n} \) and \( I(H)^{n} \) have the same minimal generators. By \([7]\) Lemma 2.9], since \( I(G)^{n} \) is componentwise linear, \( I(H)^{n} \) is also componentwise linear.

\[ \square \]

3. Main theorem

In this section we prove the main result of this paper. Our proof hinges on the following result of Herzog and Hibi \([9]\) that links the notions of componentwise linearity and sequential Cohen-Macaulayness.

Theorem 3.1. Let \( I \) be a squarefree monomial ideal of \( R \). Then \( R/I \) is sequentially Cohen-Macaulay if and only if \( I^{n} \) is componentwise linear.

We have arrived at our main result.

Theorem 3.2. All chordal graphs are sequentially Cohen-Macaulay.

Proof. Let \( G \) be a chordal graph. By Theorem 3.1 it suffices to show that \( I(G)^{n} \) is componentwise linear. To show \( I(G)^{n} \) is componentwise linear, we have based our proof on Faridi’s proof of \([4]\) Theorem 5.4] that the squarefree part of the facet ideal of a simplicial forest has linear quotients in each degree. By Theorem 2.6
we need to show that $\mathcal{I}(G)_{[d]}$ has a linear resolution for each $d$. By Lemma 2.8, it suffices to show that $\mathcal{I}(G)_{[d]}$ has linear quotients for each $d$.

We induct on the number of vertices in the chordal graph. By Lemma 2.12, we may assume that $G$ has no isolated vertices. Thus the first case to consider is when we have a graph $G$ on two vertices connected by an edge. In this case $G = K_2$, so $\mathcal{I}(G)_{[d]}$ has linear quotients for each $d$ by Lemma 2.10.

Suppose now that $G$ is a chordal graph on $n \geq 3$ vertices that has no isolated vertices (so $G$ has at least two edges). If $G = K_n$, then we are done by Lemma 2.10. So, we may assume that $G$ is not complete. By Lemma 2.14 there is a vertex $x \in V_G$ such that the induced subgraph on $\{x\} \cup N(x)$ is a complete graph. (For example, take $K$ to be any edge of $G$, and then $x$ will be some vertex not incident to that edge.) Write $N(x) = \{y_1, \ldots, y_t\}$. Observe that $G \setminus \{x\}$ and $G \setminus (N(x) \cup \{x\})$ must be chordal. Note that it is possible that $G \setminus (N(x) \cup \{x\})$ is an isolated vertex (or vertices); in this case, its edge ideal is the zero ideal.

Now by Lemma 2.14 $\mathcal{I}(G)_{[d]}$ is generated by the squarefree monomials that correspond to the vertex covers of $G$ of size $d$. Note that any vertex cover $\{x_i, \ldots, x_i\}$ of $G$ must cover the complete subgraph $K_{t+1}$ formed by $\{x, y_1, \ldots, y_t\}$. So each vertex cover must contain at least $t$ vertices of $\{x, y_1, \ldots, y_t\}$.

If $\{x_i, \ldots, x_i\}$ is a vertex cover of $G$, then $\{x_i, \ldots, x_i\} \setminus \{x\}$ must be a vertex cover of $G \setminus \{x\}$. If a vertex cover $\{x_i, \ldots, x_i\}$ does not contain $\{x\}$, it must therefore contain $\{y_1, \ldots, y_t\}$. But then $\{x_i, \ldots, x_i\} \setminus \{y_1, \ldots, y_t\}$ must be a vertex cover of $G \setminus (N(x) \cup \{x\})$. (In the case when this subgraph is an isolated vertex, since there are no edges, the empty set is a vertex cover, as is any subset of vertices.)

Let $H_1 = G \setminus \{x\}$ and $H_2 = G \setminus (N(x) \cup \{x\})$, and let $\mathcal{I}(H_1) \subseteq R_1 = k[x_i \mid x_i \in V_G \setminus \{x\}]$ and $\mathcal{I}(H_2) \subseteq R_2 = k[x_i \mid x_i \in V_G \setminus \{x, y_1, \ldots, y_t\}]$ be their respective edge ideals. From the above discussion, it follows that

$\mathcal{I}(G)_{[d]} = y_1 \cdots y_t \mathcal{I}(H_2)_{[d-t]} + x \mathcal{I}(H_1)_{[d-1]}$.

Here, we are viewing $\mathcal{I}(H_2)_{[d-t]}$ and $\mathcal{I}(H_1)_{[d-1]}$ as ideals of $R$ with the same generators as $\mathcal{I}(H_2)_{[d-t]} \subseteq R_2$ and $\mathcal{I}(H_1)_{[d-1]} \subseteq R_1$.

Since $H_1$ and $H_2$ are both chordal with fewer vertices than $G$, by induction,

$\mathcal{I}(H_2)_{[d-t]} = (A_1, \ldots, A_a)$ and $\mathcal{I}(H_1)_{[d-1]} = (B_1, \ldots, B_b)$

have linear quotients. We assume that the $A_i$s and $B_j$s have been written in the correct order for linear quotients. We now show that

$\mathcal{I}(G)_{[d]} = (yA_1, \ldots, yA_a, xB_1, \ldots, xB_b)$, with $y = y_1 \cdots y_t$.

has linear quotients with respect to this order of the generators.

Since it is clear that $(yA_1, \ldots, yA_a) : (yA_i)$ has linear quotients for $i = 2, \ldots, a$, we need to check that the following ideal has linear quotients:

$(yA_1, \ldots, yA_a) : (xB_1)$.

First note that because $B_1$ corresponds to a vertex cover of $G \setminus \{x\}$, $B_1$ is divisible by at least $t-1$ of $\{y_1, \ldots, y_t\}$. (To see this, note that $B_1$ covers the complete graph $K_t$ formed by the $y_i$s.) So there exists at most one $y_k$ such that $y_k \mid B_1$.

Now suppose there exist monomials $m$ and $p$ and a $j$ such that

$mx B_1 = py A_j$. 

We can assume that $mxB_1$ and $pyA_j$ are squarefree. There are two cases to consider.

**Case 1.** If $y|B_1$, then $B_1 = yB'_1 = y_1 \cdot \ldots \cdot y_{t-1}B'_1$. Since $B_1$ corresponds to a vertex cover of $G\setminus \{x\}$, $B'_1$ corresponds to a vertex cover of size $d-t+1$ of $G\setminus (N(x) \cup \{x\})$. So $B'_1 \in \mathcal{I}(H_2)_{[d-t-1]}$. Note that if a variable $z|m$, then $z$ must be a variable of the ring $R_2$; otherwise $mxB_1$ would not be squarefree. So, for any variable $z$ such that $z|m$, $zB'_1 \in \mathcal{I}(H_2)_{[d-t+1]}$, and hence $zB'_1 = zB_1 = (yA_1, \ldots, yA_{a})$. Thus $zxB_1 \in (yA_1, \ldots, yA_{a})$ for any $z$ that divides $m$.

**Case 2.** Suppose $y \not{|} B_1$. By the observation above, there exists a $y_t \in \{y_1, \ldots, y_t\}$ such that $B_1 = y_1 \cdot \ldots \cdot y_{t-1}B'_1$. Since $mxB_1 = pyA_j$, and since $y_t$ divides the right-hand side, we must have $y_t|m$. Note that $y_tB_1 = yB'_1$ is a cover of $G$ of size $d$ with $B'_1$ a cover of $H_2$ of size $d-t$. Hence $B'_1 \in \mathcal{I}(H_2)_{[d-t]}$. Thus $y_txB_1 \in (yA_1, \ldots, yA_a)$.

The above two cases imply that $(yA_1, \ldots, yA_a) : (xB_1)$ has linear quotients. To finish the proof, we need to check whether

$$(yA_1, \ldots, yA_a, xB_1, \ldots, xB_{l-1}) : (xB_1)$$

is generated by a subset of the variables. If $mxB_i \in (xB_1, \ldots, xB_{l-1})$ for some monomial $m$, then since $(xB_1, \ldots, xB_b)$ has linear quotients, there exists a variable $x_i$ that divides $m$ such that $x_ixB_i \in (xB_1, \ldots, xB_{l-1})$. If there is a monomial $m$ such that $mxB_i \in (yA_1, \ldots, yA_a)$, the above argument can be repeated. □

**Remark 3.3.** The proof of Theorem 3.2 shows that chordal graphs are sequentially Cohen-Macaulay regardless of the characteristic of $k$ because the linear quotients property is independent of $k$. Faridi [5] showed that if $I$ is any monomial ideal that is sequentially Cohen-Macaulay, then the polarization of $I$, a squarefree monomial ideal associated to $I$, is also sequentially Cohen-Macaulay. Thus, if $I$ is any monomial ideal whose polarization is the edge ideal of a chordal graph, $I$ must be sequentially Cohen-Macaulay.

Recall that a graph $G$ is a forest if it has no cycles. A forest, therefore, is an example of a chordal graph, so we get:

**Corollary 3.4.** If $G$ is a forest, then $G$ is sequentially Cohen-Macaulay.

**Remark 5.** In [3], Faridi proved that if $\mathcal{I}(\Delta)$ is the facet ideal of simplicial forest $\Delta$, then $R/\mathcal{I}(\Delta)$ is sequentially Cohen-Macaulay. When the simplicial forest has dimension 1, then $\mathcal{I}(\Delta)$ is simply the edge ideal of a forest. So, our result can be viewed as a partial generalization of Faridi’s result.

We close by describing how our Theorem 3.2 implies Herzog, Hibi, and Zheng’s result characterizing Cohen-Macaulay chordal graphs. We begin with a lemma.

**Lemma 3.6.** Let $I$ be a squarefree monomial ideal in $R = k[x_1, \ldots, x_n]$. Then $R/I$ is Cohen-Macaulay if and only if $R/I$ is sequentially Cohen-Macaulay and $I$ is unmixed.

**Proof.** When $R/I$ is Cohen-Macaulay, the result is obvious, so assume that $R/I$ is sequentially Cohen-Macaulay and that $I$ is unmixed. Let $I'$ be the Alexander dual of $I$. Then by Theorem 3.1 $I'$ is componentwise linear since $R/I$ is sequentially Cohen-Macaulay. Moreover, since $I$ is unmixed, $I'$ is generated in a single degree,
meaning that \( \mathcal{I}^\vee \) actually has a linear resolution. By \([3] \) Theorem 3], \( R/\mathcal{I}^\vee = R/I \) is Cohen-Macaulay. 

Herzog, Hibi, and Zheng’s result now follows.

**Corollary 3.7.** A chordal graph is Cohen-Macaulay if and only if its edge ideal is unmixed.  

**Proof.** All chordal graphs are sequentially Cohen-Macaulay, so the corollary is an immediate consequence of Lemma \([8,6] \). □

4. **Sequential Cohen-Macaulayness and nonchordal graphs**

In the previous section we showed that if \( G \) is a chordal graph, then \( R/\mathcal{I}(G) \) is sequentially Cohen-Macaulay. We now explore the situation in which \( G \) is not chordal. As we show, \( R/\mathcal{I}(G) \) may or may not be sequentially Cohen-Macaulay.

We begin with a classification of the sequentially Cohen-Macaulay \( n \)-cycles. Villarreal shows in \([16] \) Corollary 6.3.6] that the only Cohen-Macaulay cycles have three or five vertices. We prove that these are the only sequentially Cohen-Macaulay cycles as well. Note that this does not follow immediately from Villarreal’s result because cycles need not be unmixed (in fact, Exercise 6.2.15 of \([16] \) implies that an \( n \)-cycle is unmixed if and only if \( n = 3, 4, 5, 7 \).

**Proposition 4.1.** Let \( G \) be an \( n \)-cycle for some \( n \geq 3 \). Then \( G \) is sequentially Cohen-Macaulay if and only if \( n = 3 \) or \( 5 \). In fact, when \( n = 3, 5 \), the \( n \)-cycle is Cohen-Macaulay.  

**Proof.** Since a 3-cycle is chordal, the result for \( n = 3 \) follows from Theorem \([8,2] \) and the Cohen-Macaulayness is easy to see. When \( n = 5 \), \( \mathcal{I}(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5) \), and \( k[x_1, \ldots, x_5]/\mathcal{I}(G) \) is Gorenstein.

Now suppose \( n = 2r \) for \( r \geq 2 \). We have \( 2r \) edges to cover, and each vertex is incident to exactly two edges. Therefore the minimum cardinality of a vertex cover is \( r \), and \( \{x_1, x_3, \ldots, x_{2r-1}\} \) (odd indices) and \( \{x_2, x_4, \ldots, x_{2r}\} \) (even indices) are the two minimal vertex covers. Thus \( (\mathcal{I}(G)^\vee) = (x_1x_3\cdots x_{2r-1}, x_2x_4\cdots x_{2r}) \), which is a complete intersection of monomials of degree \( r \geq 2 \), and therefore it does not have a linear resolution. Hence \( \mathcal{I}(G)^\vee \) is not componentwise linear, and \( G \) is not sequentially Cohen-Macaulay.

Suppose next that \( n = 2r+1 \) for some \( r \geq 3 \). A minimal vertex cover of \( G \) consists of alternating vertices plus one additional vertex since alternating vertices leaves a single edge uncovered; hence the lowest degree in which \( \mathcal{I}(G)^\vee \) is generated is degree \( r+1 \). Therefore there are \( 2r+1 \) minimal generators of degree \( r + 1 \), one for each edge that gets double-covered when we add an adjacent vertex. Let \( J = (\mathcal{I}(G)^\vee)_{r+1} \). We show that \( J \) does not have a linear resolution. This implies that \( \mathcal{I}(G)^\vee \) is not componentwise linear, and hence \( G \) is not sequentially Cohen-Macaulay.

To compute the Betti numbers of \( J \), we use simplicial homology. Define a square-free vector to be a vector with its entries in \( \{0, 1\} \). Let \( M \) be a monomial ideal, and let

\[
K^b(M) = \{ \text{squarefree vectors } c \in \{0, 1\}^{2r+1} \text{ such that } \frac{x^c}{x^b} \in M \}.
\]

This is the upper Koszul simplicial complex of \( M \), defined, for example, in \([13] \). We can compute the \( \mathbb{N}^n \)-graded Betti numbers of \( M \) with the relation

\[
\beta_{i,b}(M) = \dim_k \tilde{H}_{i-1}(K^b(M), k)
\]
from \cite{[13]} Theorem 1.34. Summing over all squarefree \(b\) with degree \(j\) gives \(\beta_{i,j}(M)\).

We show that \(\beta_{2,2r+1}(J) \neq 0\), which proves that \(J\) does not have a linear resolution when \(r \geq 3\). There is a single squarefree vector corresponding to degree \(2r+1\), \(b = (1, \ldots, 1)\), which is associated to the monomial \(m = x_1 \cdots x_{2r+1}\). We have a chain complex

\[
\cdots \rightarrow C_2(K^b(J)) \xrightarrow{\partial_2} C_1(K^b(J)) \xrightarrow{\partial_1} C_0(K^b(J)) \xrightarrow{\partial_0} C_{-1}(K^b(J)) \rightarrow 0.
\]

Below, we shall use the following notation: If \((i_1, \ldots, i_n)\) is a vector with entries in \(\{0, 1\}\) corresponding to a face in our simplicial complex, we shall often write the face as \([x_{j_1}, \ldots, x_{j_p}]\), where the \(j_i\) are exactly the nonzero entries of \((i_1, \ldots, i_n)\). For example, the face \((1, 0, 0, 1, 0, 1)\) is written as \([x_1, x_4, x_6]\).

Note that the basis of \(C_s(K^b(J))\) consists of the \(s\)-dimensional faces \([x_{i_0}, \ldots, x_{i_s}]\) of \(K^b(J)\), and

\[
\partial_s([x_{i_0}, \ldots, x_{i_s}]) = \sum_{t=0}^{s} (-1)^t [x_{i_0}, \ldots, \hat{x}_{i_t}, \ldots, x_{i_s}].
\]

All the faces with which we work have dimension at most two; we orient the faces so that if \(i_0 < i_1 < i_2\), we traverse \([x_{i_0}, x_{i_1}]\) and \([x_{i_1}, x_{i_2}]\) in the positive direction and \([x_{i_0}, x_{i_2}]\) in the negative direction. Similarly, we direct edges so that going from \(x_{i_0}\) to \(x_{i_1}\) is in the positive direction.

To find \(\beta_{2,2r+1}(J)\), we need to compute \(\dim_k \tilde{H}_i(K^b(J), k) = \dim_k(\ker \partial_1 / \im \partial_2)\).

If we can produce an element in \(\ker \partial_1\) that is not in \(\im \partial_2\), we will have shown that \(\beta_{2,2r+1}(J) > 0\). We shall refer to vertex covers and the corresponding monomials interchangeably below.

Initially, suppose that \(2r + 1 > 7\); we handle the case \(2r + 1 = 7\) separately. We claim first that \(m/x_1x_4x_7 \notin J\). If it were, then there would be a minimal vertex cover \(m'\) that divided it. But then \(x_2x_3x_5x_6x_8x_{2r+1}\) divides \(m'\) since \(x_1\), \(x_4\), and \(x_7\) are missing, and \(m'\) is a cover. If \(2r + 1 > 9\), then to cover the remaining \(2r - 9\) edges not covered, we need at least \(r - 4\) vertices. This means that \(\deg m' \geq 6 + r - 4 = r + 2\), but all the minimal vertex covers in \(J\) have degree \(r + 1\) since \(J = (E(G))^{r+1}\). Also, when \(2r + 1 = 9\), the minimal generators of \(J\) have degree five, and \(x_2x_3x_5x_6x_8x_{2r+1}\) is a minimal vertex cover of degree six and hence is not divisible by an element of \(J\). Thus in either case, \(m/x_1x_4x_7 \notin J\).

Next we show that \(m/x_1x_4, m/x_4x_7, \) and \(m/x_1x_7\) are in \(J\). To prove this, we need to show that a minimal vertex cover divides each of these monomials. In the first case, use \(x_2x_3x_5x_7 \cdots x_{2r+1}\); in the second, \(x_2x_3x_5x_6x_8x_{10} \cdots x_{2r}\) works; and in the last, use \(x_2x_4x_6x_8x_{10} \cdots x_{2r}x_{2r+1}\).

Hence \([x_1, x_4]\), \([x_4, x_7]\), and \([x_1, x_7]\) are edges of \(K^b(J)\), but \([x_1, x_4, x_7]\) is not a face of \(K^b(J)\). Thus \(f = [x_1, x_4] + [x_4, x_7] - [x_1, x_7] \in C_1(K^b(J))\) is not in the image of \(\partial_2\). However,

\[
\partial_1(f) = [x_4] - [x_1] + [x_7] - [x_4] - ([x_7] - [x_1]) = 0.
\]

Thus \(f\) is in the kernel of \(\partial_1\), and \(\beta_{2,2r+1}(J) \neq 0\), so \(J\) does not have a linear resolution.

When \(2r + 1 = 7\), we need a slightly different argument. One can compute that in this case, the Alexander dual of \(\mathcal{I}(G)\) is

\[
\mathcal{I}(G)^\vee = (x_1x_2x_4x_6, x_1x_3x_4x_6, x_1x_3x_5x_6, x_1x_3x_5x_7, x_2x_3x_5x_7, x_2x_4x_5x_7, x_2x_4x_6x_7),
\]
and it has minimal graded free resolution

\[ 0 \rightarrow R(-7) \rightarrow R(-5) \rightarrow R(-4) \rightarrow I(G)^\vee \rightarrow 0. \]

Because of the second syzygy in degree seven, \( I(G)^\vee = (I(G))_4 \) does not have a linear resolution. Therefore \( G \) is not sequentially Cohen-Macaulay.

**Remark 4.2.** Proposition 4.1 is independent of the characteristic of \( k \). Note that if \( k \) has prime characteristic, the graded Betti numbers of \( R/J \) are either the same as in characteristic zero, or they go up since the behavior is the same for the dimensions of the homology groups we computed. The dimensions of the homology groups in characteristic \( p > 0 \) are either the same as in characteristic zero, or they may increase if there is a \( p \)-torsion part introduced. See, for example, the latter part of the discussion of Universal Coefficients in [14, Chapter 9]. Thus we have \( \beta_{2,2r+1}(J) > 0 \) for \( r > 2 \) over all \( k \).

The case of a 5-cycle shows that the converse of Theorem 3.2 is false. There are many nonchordal sequentially Cohen-Macaulay graphs. We present two simple examples here to demonstrate that small changes in a graph that is not sequentially Cohen-Macaulay can give a graph with the property. For further investigation of this idea, see [6].

**Example 4.3.** Let \( G \) be a 4-cycle, and let \( H \) be the graph \( G \) with a fifth vertex, connected to \( G \) by a single edge. Thus \( I(G) = (x_1x_2, x_2x_3, x_3x_4, x_1x_4) \), and \( I(H) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5) \). By Proposition 4.1, \( G \) is not sequentially Cohen-Macaulay. The Alexander dual of \( I(H) \) is

\[ I(H)^\vee = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_1, x_4) \cap (x_4, x_5) = (x_2x_4, x_1x_3x_5, x_1x_3x_4). \]

It is easy to check that \( I(H)^\vee \) is componentwise linear since it has a single generator in degree two and regularity three. Hence \( H \) is sequentially Cohen-Macaulay.

**Example 4.4.** For a slightly more complicated example, suppose that \( G \) is a 6-cycle, and we obtain the graph \( H \) by adding a seventh vertex and connecting it to two adjacent vertices of \( G \). Thus

\[ I(H) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6, x_1x_7, x_6x_7), \]

and

\[ I(H)^\vee = (x_2x_4x_6x_7, x_1x_3x_5x_7, x_1x_3x_5x_6, x_1x_3x_4x_6, x_1x_2x_4x_6, x_2x_3x_5x_6x_7, x_1x_2x_4x_5x_7). \]

One can check in Macaulay 2 that \( I(H)^\vee \) is componentwise linear, so \( H \) is sequentially Cohen-Macaulay. We remark that tests in Macaulay 2 suggest that adding a triangle in this way to a cycle that is not sequentially Cohen-Macaulay may always produce a sequentially Cohen-Macaulay graph.

We round out this paper with a sufficient condition for a graph to fail to be sequentially Cohen-Macaulay. This condition makes use of another characterization of sequential Cohen-Macaulayness of quotients by monomial ideals due to Duval [2].

Recall that an element \( F \in \Delta \), where \( \Delta \) is a simplicial complex, is called a face of \( \Delta \). The dimension of a face \( F \) is \( \dim F = |F| - 1 \). The dimension of \( \Delta \) is then \( \dim \Delta = \max_{F \subseteq \Delta} \{ \dim F \} \). We write \( \Delta_i \) to denote the subcomplex of \( \Delta \) whose maximal faces (the facets) are all the faces of \( \Delta \) of dimension \( i \).
Theorem 4.5 ([12, Theorem 3.3]). Let $I$ be a squarefree monomial ideal, and let $\Delta$ be the simplicial complex defined by $I$ via the Stanley-Reisner correspondence. Let $\Delta_{i}$ be the pure $i$-dimensional subcomplex of $\Delta$. Then $R/I$ is sequentially Cohen-Macaulay if and only if for every $i$, $-1 \leq i \leq \dim \Delta$, $R/I_{\Delta_{i}}$ is Cohen-Macaulay.

We also need the following definition [15].

Definition 4.6. If $\Delta$ is a simplicial complex of dim $d - 1$, then the $f$-vector $f(\Delta) = (f_{-1}, f_{0}, f_{1}, \ldots, f_{d-1})$, where $f_{i}$ is the number of faces of dimension $i$ (where $f_{-1} = 1$). If

$$H_{R/(I_{\Delta})}(t) = \frac{h_{0} + h_{1}t + h_{2}t^{2} + \cdots + h_{d}t^{d}}{(1-t)^{d}}$$

is the Hilbert-Poincare series of $R/I_{\Delta}$, then the $h$-vector of $\Delta$ is $h(\Delta) = (h_{0}, \ldots, h_{d})$.

The complement of a simple graph $G$, denoted $G^{c}$, is the graph with the same vertex set as $G$, but with edge set $E_{G^{c}} = \{ \{x_{i}, x_{j}\} \mid \{x_{i}, x_{j}\} \notin E_{G}\}$, and the clique-complex (sometimes called the flag complex) of a simple graph $H$, denoted $\Delta(H)$, is the simplicial complex whose faces are the subsets of vertices on which the induced subgraph of $H$ is a clique.

Theorem 4.7. Let $G$ be a simple graph. Let $H_{2}$ be the set of isolated vertices of $G^{c}$, and set $H_{1} = G^{c}\setminus H_{2}$ (so $G^{c}$ is the disjoint union of $H_{1}$ and $H_{2}$). If $\#E_{H_{1}} - \#V_{H_{1}} + 1 < 0$, then $I(G)$ is not sequentially Cohen-Macaulay.

Proof. Since $I(G)$ is a squarefree monomial ideal, $I(G)$ also corresponds to a simplicial complex via the Stanley-Reisner correspondence. In particular, $I(G) = I(\Delta(G^{c}))$ where $\Delta(G^{c})$ is the clique-complex associated to $G^{c}$. Let $\Delta(G^{c})_{1}$ denote the pure 1-dimensional subcomplex of $\Delta(G^{c})$. Now $\Delta(G^{c})_{1}$ is simply the 1-skeleton of $G^{c}$, i.e., it is a graph. Specifically, $\Delta(G^{c})_{1} = H_{1}$. Since $H_{1}$ is a graph, the $f$-vector of $H_{1}$ is

$$f(H_{1}) = (1, \#V_{H_{1}}, \#E_{H_{1}}).$$

Using the relation between the $f$-vectors and $h$-vectors as given on page 58 of Stanley’s book [15], we have

$$h(H_{1}) = (1, \#V_{H_{1}} - 2, \#E_{H_{1}} - \#V_{H_{1}} + 1).$$

If $\#E_{H_{1}} - \#V_{H_{1}} + 1 < 0$, then $h(H_{1})$ has negative values. So $R/I_{\Delta(G^{c})_{1}}$ is not Cohen-Macaulay by [15, Corollary 3.2] because the $h$-vector of a Cohen-Macaulay Stanley-Reisner ring must contain only nonnegative values (in fact, must be an O-sequence). Thus, by Theorem 4.5, $I(G) = I(\Delta(G^{c}))$ is not sequentially Cohen-Macaulay.

Example 4.8. The above result gives an alternative justification for why the 4-cycle is not sequentially Cohen-Macaulay. Since $G^{c} = \\{\{x_{1}, x_{3}\}, \{x_{2}, x_{4}\}\}$, the graph $G^{c}$ has two edges, but 4 vertices, so $I(G)$ cannot be sequentially Cohen-Macaulay since $2 - 4 + 1 < 0$.

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