LOG-LOG CONVEXITY AND BACKWARD UNIQUENESS

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Abstract. We study backward uniqueness properties for equations of the form

\[ u' + Au = f. \]

Under mild regularity assumptions on \( A \) and \( f \), it is shown that \( u(0) = 0 \) implies \( u(t) = 0 \) for \( t < 0 \). The argument is based on \( \alpha \)-log and log-log convexity. The results apply to mildly nonlinear parabolic equations and systems with rough coefficients and the 2D Navier-Stokes system.

1. Introduction

Backward uniqueness for evolution partial differential equations is a classical problem initiated by Lax [L], and minimal regularity requirements under which backward uniqueness holds are not known for many important partial differential equations and systems. A basic question for an evolution equation written in a form

\[ u' + Au = f(u) \]

is under which conditions \( u(T) = 0 \) implies \( u(t) = 0 \) for \( t < T \). Backward uniqueness is substantially more difficult than forward uniqueness due to ill-posedness (in general) of the backward evolution problem. Since it is impossible to survey the large literature on this topic, we provide a short description of relevant work. There are basically two methods addressing this problem. The first is based on logarithmic convexity [AN1, AN2, A, BT, G, O] and the second on time-weighted inequalities [P, LP, S]. The approach, based on logarithmic convexity and second-order inequalities, was developed in [AN1]. The approach was substantially simplified by Ogawa [O], who reduced a proof of backward uniqueness to establishing upper bounds on the Dirichlet quotient \( Q(t) = (Au, u)/\|u\|^2 \). Further simplifications and applications were given in [BT, CFNT, G, CFKM]; in particular, the important identity (5) was established in [CFNT, G]. In summary, the most general known situation for the backward uniqueness property is

\[ \|f\|^2 \leq K(Au, u) + K\|u\|^2 \]

with a certain integrability assumption on \( K \) if \( K \) depends on \( t \) [G]. In our previous paper [K] we have shown the connection between backward uniqueness and unique

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continuation. The paper [CFKM] used Dirichlet quotients extensively to study backward behavior and Eulerian dynamics for the 2D Navier-Stokes equations.

In this paper, we introduce a log-Dirichlet quotient

$$\tilde{Q}(t) = \frac{(Au(t), u(t))}{\|u(t)\|^2 \left(\log \frac{M_0^2}{\|u(t)\|^2}\right)},$$

where $M_0$ is a suitably large constant. While the classical Dirichlet quotient measures exponential decay of $\|u(t)\|$, the log-Dirichlet quotient quantifies $\exp(-C|t|^{1/(1-\alpha)})$ type decay of $\|u(t)\|$ if $\alpha \in (0, 1)$ and $\exp(-Ce^{1/2})$ type decay of $\|u(t)\|$ if $\alpha = 1$. The advantage of this quotient is that the differential inequality for $\tilde{Q}$ contains an extra positive term on the left-hand side which is even quadratic in $Q$ (see (6) below). Exploring this fact, we are able to treat nonlinear equations with much rougher coefficients than allowed before (see Section 3 below for applications). Moreover, we are able to obtain a sharper result even in the classical range (that is, a slight sublinearity is allowed when coefficients are not too irregular). We emphasize that the generality of the presentations allows applications to higher-order evolution equations as well as systems. Section 2 contains the statement and the proof of the main results. Section 3 contains three applications—a parabolic nonlinear equation, a parabolic system and a theorem on boundedness of log-Dirichlet quotients for differences of solutions of the 2D periodic Navier-Stokes equations on the global attractor.

2. The main result

Let $H$ be a real or complex Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let $A$ be a symmetric operator with the domain $D(A) \subseteq H$. Assume that $(Au, u) \geq 0$ for $u \in D(A)$. Let $u \in C([T_0, 0], D(A)) \cap C^1([T_0, 0], H)$ be a solution of

$$u' + Au = f$$

with $f \in C([T_0, 0], H)$, where the requirement $u \in C([T_0, 0], D(A))$ means $Au \in C([T_0, 0], H)$. Denote

$$L(\|u\|) = \log \frac{M_0^2}{\|u\|^2}$$

(that is, $L(x) = \log(M_0^2/x^2)$), where $M_0$ is any constant such that

$$M_0 \geq 2 \sup_{t \in [T_0, 0]} \|u(t)\|.$$ 

Note that $L(\|u(t)\|) \geq 1$ for all $t \in [T_0, 0]$. Our assumptions do not imply existence of $A^{1/2}$; however, it will be convenient to use the notation

$$\|A^{1/2}v\| = (Au, v)^{1/2}, \quad v \in D(A).$$

On $f$, which in applications depends on $u$, we assume

1. $\|f\| \leq \frac{K_1}{L(\|u\|)^{\beta/2}} \|A^{1/2}u\|^{1-\beta}\|Au\|^\beta + K_2L(\|u\|)^{\alpha/2}\|u\|$ and

2. $\Re(f, u) \geq -K_3L(\|u\|)^{\alpha(2-\beta_0)/2}\|u\|^{2-\beta_0}\|A^{1/2}u\|^{\beta_0} - K_4L(\|u\|)^{\alpha}\|u\|^2$
for some \( \alpha, \beta \in [0, 1] \), \( \beta_0 \in [0, 2] \), and \( K_1, K_2, K_3, K_4 \geq 0 \). Additionally, we assume
\[
K_1^2 \leq \frac{\alpha}{8}
\]
if \( \beta = 1 \). Note that the classical case corresponds to \( \beta = 0, \beta_0 = 1, \alpha = 0 \).

**Theorem 2.1.** Let \( u : [T_0, 0] \rightarrow H \) be as above. Then \( u(0) = 0 \) implies \( u(t) = 0 \) for all \( t \in [T_0, 0] \).

**Proof.** For \( t \in [T_0, 0] \), denote \( \bar{L}(t) = L(\|u(t)\|) \). By continuity, it is sufficient to assume \( \|u(t)\| \neq 0 \) for \( t \in [T_0, 0] \) and prove that \( \|u(0)\| \neq 0 \). For this, we introduce the log-Dirichlet quotient
\[
\tilde{Q}(t) = \frac{Q(t)}{L(\|u\|)^\alpha} = \frac{\|A^{1/2}u\|^2}{\|u\|^2L(\|u\|)^\alpha} \leq \frac{\|A^{1/2}u(t)\|^2}{\|u(t)\|^2L(t)^\alpha},
\]
where \( Q(t) = \|A^{1/2}u\|^2/\|u\|^2 \) is the classical Dirichlet quotient \([\text{Q}][\text{BT}]\). Note that while \( Q(t) \) controls the exponential decay of \( \|u(t)\| \), we shall show that the log-Dirichlet quotient controls \( \exp(-C|t|^{1/(1-\alpha)}) \) type decay if \( \alpha \in (0, 1) \), and \( \exp(-C_0|t|) \) type decay if \( \alpha = 1 \).

Note that our assumptions imply
\[
1 \frac{d}{dt}(u, u) = \Re(u, u) = -(Au, u) + \Re(f, u)
\]
and
\[
1 \frac{d}{dt}(Au, u) = \Re(u_t, Au) = -(Au, Au) + \Re(f, Au).
\]
From here, we obtain the identity \([\text{CFNT}[\text{G}]\]
\[
(\tilde{Q}(t) + \|\alpha(A - Q(t)I)w\|^2) = \frac{\Re\left(\frac{f}{\|u\|} |(A - Q(t)I)w\|\right)}{\|u\|^2L(\|u\|)^\alpha} \leq \frac{\Re(f, (A - Q(t)I)w)}{\|u\|^2L(t)^\alpha} + \frac{\alpha \tilde{Q}(t) \Re(f, u)}{\|u\|^2L(t), t \in [T_0, 0]).
\]
Using \( f, (A - Q(t)I)w \leq \|f\| \|(A - Q(t)I)w\| \) on the first term on the right-hand side and
\[
\|\tilde{Q}(t) \Re(f, u) \leq \frac{\alpha \tilde{Q}(t) \Re(f, u)}{\|u\|^2L(t)^\alpha} \leq \frac{\alpha \tilde{Q}(t)^2}{2L(t)^{1-\alpha}} + \frac{\alpha \|f\|^2}{2L(t)^{1+\alpha} \|u\|^2} \leq \frac{\alpha \tilde{Q}(t)^2}{2L(t)^{1-\alpha}} + \frac{\|f\|^2}{2L(t)^{1+\alpha} \|u\|^2}
\]
on the second, we obtain
\[
\tilde{Q}(t) + \frac{\alpha \tilde{Q}(t)^2}{L(t)^{1-\alpha}} + \frac{\|(A - Q(t)I)w\|^2}{L(t)^\alpha} \leq \frac{2\|f\|^2}{\|u\|^2L(t)^\alpha}.
\]
Therefore, by squaring \( \|Au\| \leq \|(A - Q(t)I)u\| + Q(t)\|u\| \),

\[
\frac{2\|f\|^2}{\|u\|^2 L(t)^\alpha} \leq \frac{4K_1^2}{L(t)^{\alpha + \beta}} \frac{\|A^{1/2}u\|^{2-2\beta}\|Au\|^{2\beta}}{\|u\|^2} + 4K_2^2 \\
\leq \frac{8K_1^2}{L(t)^{\alpha + \beta}} Q(t)^{1-\beta} \|(A - Q(t)I)w\|^{2\beta} + \frac{8K_1^2 Q(t)^{1+\beta}}{L(t)^{\alpha + \beta}} + 4K_2^2.
\]

If \( \beta \in (0, 1) \), we use \( ab \leq \alpha^p/p + \beta^q/q \) with \( p = 1/\beta \) and \( q = 1/(1 - \beta) \). We obtain

\[
\frac{2\|f\|^2}{\|u\|^2 L(t)^\alpha} \leq \frac{(1 - \beta)(8K_1^2)^{1/(1-\beta)} \tilde{Q}(t)}{L(t)^{\beta/(1-\beta)}} + \beta \frac{\|(A - Q(t)I)w\|^2}{L(t)^\alpha} \\
+ \frac{8K_1^2 Q(t)^{1+\beta}}{L(t)^{\alpha + \beta}} + 4K_2^2.
\]

The third term equals

\[
\frac{8K_1^2 \tilde{Q}(t)^{1+\beta}}{L(t)^{\beta/(1-\alpha)}} = \left( \frac{\alpha \tilde{Q}(t)^2}{L(t)^{1-\alpha}} \right)^\beta \frac{8K_1^2 \tilde{Q}(t)^{1-\beta}}{\alpha^\beta} \\
\leq \frac{\alpha \beta \tilde{Q}(t)^2}{L(t)^{1-\alpha}} + (1 - \beta) \left( \frac{8K_1^2}{\alpha^\beta} \right)^{1/(1-\beta)} \tilde{Q}(t).
\]

Since \( \tilde{L}(t) \geq 1 \), we get \( \tilde{Q}^\prime(t) \leq K_5 \tilde{Q}(t) + K_6 \), where \( K_5 = (1 - \beta)(8K_1^2)^{1/(1-\beta)}(1 + \alpha^{-\beta/(1-\beta)}) \) and \( K_6 = 4K_2^2 \), which implies

(7) \( \sup_{t \in [T_0, 0)} \tilde{Q}(t) < \infty. \)

If \( \beta = 0 \), all the above inequalities hold trivially. If \( \beta = 1 \), then we have

\[
\tilde{Q}^\prime(t) + \frac{\alpha \tilde{Q}(t)^2}{L(t)^{1-\alpha}} + \frac{\|(A - Q(t)I)w\|^2}{L(t)^\alpha} \\
\leq \frac{8K_1^2}{L(t)^{1-\alpha}} \|(A - Q(t)I)w\|^2 + \frac{8K_1^2 \tilde{Q}(t)^2}{L(t)^{1-\alpha}} + 4K_2^2.
\]

Since (8) holds if \( \beta = 1 \), we get \( \tilde{Q}^\prime \leq K_5 \tilde{Q} + K_6 \) with \( K_5 = 0 \) and \( K_6 = 4K_2^2 \), and (7) follows also in this case.

It remains to be checked that (7) implies that \( \|u(0)\| \) is nonzero. From (4), we get

\[
\frac{1}{2\pi} \frac{\|u\|^2}{L(t)^\alpha \|u\|^2} + \tilde{Q}(t) = \frac{\Re(f, u)}{L(t)^\alpha \|u\|^2}, \quad t \in [T_0, 0).
\]

Using (2), we get

\[
\frac{1}{2\pi} \frac{\|u\|^2}{L(t)^\alpha \|u\|^2} + \tilde{Q}(t) \geq -\frac{K_3 \tilde{Q}(t)^{\beta_0/2}}{L(t)^{\alpha \beta_0/2}} - K_4 = -K_3 \tilde{Q}(t)^{\beta_0/2} - K_4.
\]

Therefore,

(8) \( \frac{1}{2\pi} \frac{\|u\|^2}{L(t)^\alpha \|u\|^2} + K_7 \tilde{Q}(t) \geq -K_8 \),

where \( K_7 = 1 + \beta_0 K_3/2 \) and \( K_8 = K_4 + (2 - \beta_0) K_3/2. \)
3.1. A second-order evolution equation. First, consider a solution of
\[ \frac{\partial u}{\partial t} - \Delta u + W_j(x,t)|u|^{\theta} \partial_j u + V(x,t)|u|^{\theta} u = 0, \]
where \( u \in C^1([0,T], H^2(\mathbb{R}^n)) \cap C([0,T], L^2(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times [0,T]) \) and where \( \theta > 0 \) is arbitrary. By Theorem 2.1 if \( u(0) \neq 0 \) and \( V \in L^\infty([0,T], L^p(\mathbb{R}^n)) \) and \( W_j \in L^\infty([0,T], L^q(\mathbb{R}^n)) \) for \( j = 1, \ldots, n \) where \( p > n/2 \) and \( q > n \), with an additional condition \( p, q \geq 2 \), then
\[ u(t) \neq 0, \quad t \in (0,T]. \]
A similar statement can be obtained for higher-order parabolic equations. Let \( n \geq 4 \) (the cases \( n = 1, 2, 3 \) are similar). It is sufficient to check the assumptions under the restriction \( p \in (n/2, n) \) since other cases can be covered by a standard logarithmic convexity argument. As \( n \geq 4 \) and \( p \in (n/2, n) \), we may choose \( \epsilon \in (0, \theta) \) such that
\[ \frac{1}{n} < \frac{1}{p} + \frac{\epsilon}{2} < \frac{2}{n} \]
and estimate for every \( t \in [0,T] \),
\[ \|V\|^{1+\theta}_{L^2} \leq M^{\theta-\epsilon} \|V\|_{L^p} \|u\|_{L^2}^{\theta} \|u\|_{L^p}. \]
where $M = \|u\|_{L^\infty\left(\mathbb{R}^n \times [0,T]\right)}$ and $1/p + \epsilon/2 + 1/p = 1/2$ and then interpolate

$$\|u\|_{L^\infty} \leq C\|\nabla u\|_{L^2}^{-1}\|\Delta u\|_{L^2},$$

where $\gamma = n/p + cn/2 - 1$, which gives (1). Other conditions and cases are checked in a similar manner. We note that the equation $W_j = 0$, $θ = 0$ was treated in [K] by reducing the backward uniqueness to a unique continuation theorem in $E$ [EV] (however, the assumption $n \geq 5$ should be added to the assumptions or a requirement $p \geq 2$ added for $v \in C([T_0, 0], L^p(\mathbb{R}^n))$ in [K, Corollary 2.3]).

### 3.2. A second-order parabolic system.

Consider a solution of

$$\frac{\partial u}{\partial t} - \partial_k(a_{ijkl}(x,t)\partial_l u_i) + W_{ijkl}(x,t)|u|^\gamma \partial_j u_k + V_{ik}(x,t)|u|^\theta u_k = 0, \quad i, j, k, l, m = 1, \ldots, m,$$

where $u = (u_1, u_2, \ldots, u_m) \in C^1([0, T], H^2(\mathbb{R}^n)^m) \cap C([0, T], L^2(\mathbb{R}^n)^m) \cap L^\infty(\mathbb{R}^n \times [0, T])$ and where $θ > 0$ is arbitrary. We assume that the coefficient tensor $a_{ijkl}(x,t)$ is bounded, $C^1$, symmetric, and uniformly strictly elliptic. By Theorem 2.1 if $u(0) \neq 0$ and $V \in L^\infty([0, T], L^p(\mathbb{R}^n))$ and $W_j \in L^\infty([0, T], L^q(\mathbb{R}^n))$ for $j = 1, \ldots, n$, where $p > n/2$ and $q > n$ with an additional assumption $p, q \geq 2$, then

$$u(t) \neq 0, \quad t \in (0, T].$$

Strictly speaking, Theorem 2.1 gives the statement if $a_{ijkl}$ depends only on $x$. It is easy to extend the statement to the general case as well.

### 3.3. 2D Navier-Stokes equation.

Consider the Navier-Stokes system

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f,$$

$$\nabla \cdot u = 0$$

with periodic boundary conditions on $Ω = [0, 2\pi]^2$. Let $H$ be the closure of

$$\left\{v \in L^2_{\text{per}}(Ω)^2 : v \text{ is an } Ω-\text{periodic trigonometric polynomial}, \quad \nabla \cdot v = 0 \text{ in } Ω, \int_Ω v = 0 \right\}$$

in the (real) Hilbert space $L^2_{\text{per}}(Ω)^2$. Then, under the condition $f \in H$ ($f$ is time-independent), the equation possesses a global attractor

$$A = \left\{u_0 \in H : S(t)u_0 \text{ exists for all } t \in \mathbb{R}, \sup_{t \in \mathbb{R}} \|S(t)u_0\|_{L^2_{\text{per}}(Ω)} < \infty \right\},$$

where $S(t)u_0$ denotes a solution starting at $u_0$ on its maximal interval of existence (cf. [CF]). It is an open problem whether

$$\sup_{u_1, u_2 \in A, u_1 \neq u_2} \frac{\|\nabla(u_1 - u_2)\|_{L^2_{\text{per}}(Ω)}}{\|u_1 - u_2\|_{L^2_{\text{per}}(Ω)}} < \infty.$$

By the proof of Theorem 2.1 we get the following statement.

**Theorem 3.1.** With the above assumptions, we have

$$\sup_{u_1, u_2 \in A, u_1 \neq u_2} \frac{\|\nabla(u_1 - u_2)\|_{L^2_{\text{per}}(Ω)}}{\|u_1 - u_2\|_{L^2_{\text{per}}(Ω)} \log(\|u_1 - u_2\|_{L^2_{\text{per}}(Ω)}/\|u_0\|_{L^2_{\text{per}}(Ω)})} < \infty,$$

where $M_0 = 4\sup_{u_0 \in A} \|u_0\|_{L^2_{\text{per}}(Ω)}$. 

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The upper bound can be explicitly estimated in terms of \( \|f\|_{L^2} \). A similar statement holds for the 2D Navier-Stokes equations in a smooth bounded domain.

References


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