BANACH-STONE THEOREM
FOR BANACH LATTICE VALUED CONTINUOUS FUNCTIONS

Z. ERCAN AND S. ÖNAL

(Communicated by Joseph A. Ball)

ABSTRACT. Let \( X \) and \( Y \) be compact Hausdorff spaces, \( E \) be a Banach lattice and \( F \) be an AM space with unit. Let \( \pi : C(X, E) \to C(Y, F) \) be a Riesz isomorphism such that \( 0 \not\in f(X) \) if and only if \( 0 \not\in \pi(f)(Y) \) for each \( f \in C(X, E) \). We prove that \( X \) is homeomorphic to \( Y \) and \( E \) is Riesz isomorphic to \( F \). This generalizes some known results.

INTRODUCTION

In this paper we use the standard terminology and notation of the Riesz spaces theory. For unexplained definitions and notation we refer to [1]. Throughout this paper \( X \) and \( Y \) stand for compact Hausdorff spaces. For a compact Hausdorff space \( Z \) and a Banach lattice \( E \), the Banach lattice (under pointwise algebraic operations and order) of continuous functions from \( Z \) into \( E \) is denoted by \( C(Z, E) \). If \( E = \mathbb{R} \) we write \( C(Z) \) instead of \( C(Z, E) \). \( 1_Z \in C(Z) \) is defined by \( 1_Z(x) = 1 \).

One of the versions of the Banach-Stone Theorem states that \( X \) and \( Y \) are homeomorphic if and only if \( C(X, \mathbb{R}) \) and \( C(Y, \mathbb{R}) \) are Riesz isomorphic. More precisely, if \( \pi : C(X, \mathbb{R}) \to C(Y, \mathbb{R}) \) is a Riesz homomorphism, then there exists a homeomorphism \( \sigma : Y \to X \) such that \( \pi(f) = \pi(1_X)(f\sigma) \). A simple and direct proof of this theorem can be found in [5]. This theorem is generalized in [2] as follows:

**Theorem 1.** Let \( E \) be a Banach lattice and \( \pi : C(X, E) \to C(Y, F) \) be a Riesz isomorphism such that \( \pi(f)(y) \neq 0 \) for each \( y \in Y \) whenever \( f(x) \neq 0 \) for each \( x \in X \). Then \( X \) is homeomorphic to \( Y \) and \( E \) is Riesz isomorphic to \( F = \mathbb{R} \).

A simple and direct proof of the above theorem is given in [3], also providing a positive answer to a conjecture in [2]. By using Theorem 2 of [4] it can also be proved that the above theorem is still true for \( d \) - isomorphism \( \pi \); that is, \( x \perp y \) if and only if \( \pi(x) \perp \pi(y) \).

THE MAIN RESULT

The aim of this paper is to generalize the above theorem as follows:

Received by the editors June 16, 2005 and, in revised form, May 21, 2006.
2000 Mathematics Subject Classification. Primary 46E40; Secondary 46B42.
Key words and phrases. Riesz isomorphism, Banach lattices, Banach-Stone Theorem.
Theorem 2. Let $E$ be a Banach lattice and $F$ be an AM space with unit. Let $\pi : C(X,E) \to C(Y,F)$ be a Riesz isomorphism. If

$$f(x) \neq 0 \quad \text{for each } x \in X \iff \pi f(y) \neq 0 \quad \text{for each } y \in Y,$$

then $X$ is homeomorphic to $Y$ and $E$ is Riesz isomorphic to $F$.

To prove this theorem we will need the following lemmas. The proofs of the first two lemmas are clear.

Lemma 3. Let $M$ and $N$ be compact Hausdorff spaces and $\sigma : X \times M \to Y \times N$ be a homeomorphism. If for each $x \in X$ there exists a unique $y \in Y$ such that $\sigma(\{x\} \times M) = \{y\} \times N$, then $X$ and $M$ are homeomorphic to $Y$ and $N$, respectively.

Lemma 4. Let $f : X \to Y$ be a continuous open surjective map. If $K$ is a compact subset of $Y$ and $O$ is an open subset of $X$ with $K \subset f(O)$, then there is a closed subset $F$ of $X$ such that $F \subset O$ and $K \subset f(F)$.

Lemma 5. Let $M$ and $N$ be compact Hausdorff spaces and $\pi : C(X \times M) \to C(Y \times N)$ be a Riesz isomorphism with $\pi(1_{X \times M}) = 1_{Y \times N}$. If

$$f(\{x\} \times M) \neq 0 \quad \text{for each } x \in X \iff \pi f(\{y\} \times N) \neq 0 \quad \text{for each } y \in Y,$$

then $X$ and $M$ are homeomorphic to $Y$ and $N$, respectively.

Proof. From the Banach-Stone theorem there exists a homeomorphism $\sigma : Y \times N \to X \times M$ such that

$$\pi(f) = f \sigma \quad \text{and} \quad \pi^{-1}(g) = g \sigma^{-1}.$$

Let $P_X$ be the projection of $X \times M$ onto $X$; that is, $P_X : X \times M \to X$ is defined by $P_X(x,m) = x$.

Claim $a$: For each $y_0 \in Y$, $P_X \sigma((Y - \{y_0\}) \times N) \neq X$. Indeed, suppose it does not hold. Note that $P_X \sigma : Y \times N \to X$ is a surjective open map and $(Y - \{y_0\}) \times N$ is open in $Y \times N$. Since $X \subset P_X \sigma((Y - \{y_0\}) \times N)$ there exists a closed set $F$ in $Y \times N$ such that

$$F \subset Y - \{y_0\} \times N \quad \text{and} \quad X \subset P_X \sigma(F).$$

Let $A = \sigma(F)$ and $B = \sigma(\{y_0\} \times N)$. Clearly $A$ and $B$ are closed in $Y \times N$ and $A \cap B = \emptyset$. Then there exists $f \in C(Y \times N)$ such that

$$f(A) = \{1\} \quad \text{and} \quad f(B) = \{0\}.$$

Let $x \in X$ be given. Since $X = P_X \sigma(F)$, there exists $m \in M$, $(y,n) \in F$ such that

$$x = P_X(x,m) = P_X \sigma(y,n).$$

Then $\sigma(y,n) \in A$ and so $1 = f(\sigma(y,n)) = f(x,m)$. From the hypothesis, $\pi f(\{y\} \times N) \neq \{0\}$ for each $y \in Y$, in particular,

$$\{0\} = f(B) = \pi f(\{y_0\} \times N) \neq \{0\}.$$

This is a contradiction.

Claim $b$: For each $n \in N$, $P_X \sigma(Y \times \{n\}) = X$. To see this, suppose that $x_0 \notin P_X \sigma(Y \times \{n\})$ for some $x_0 \in X$ and $n \in N$. Since $\{x_0\} \times M$ and $\sigma(Y \times \{n\})$ are nonempty disjoint closed sets, there exists $f \in C(X \times M)$ such that

$$f \sigma(Y \times \{n\}) = \{1\} \quad \text{and} \quad f(\{x_0\} \times N) = \{0\}.$$

That implies that $\pi f(Y \times \{n_0\}) = \{1\}$, so $f(\{x_0\} \times N) \neq \{0\}$. This contradiction shows that Claim $b$ is also true.
Claim c: For each $y \in Y$ there exists a unique $x \in X$ such that $\sigma^{-1}([y] \times N) = \{x\} \times M$. Let $y \in Y$ be given. Since for each $n \in N$,

$$P_X \sigma((Y - \{y\}) \times N) \neq P_X \sigma(Y \times \{n\}) = X,$$

there exists $x \in X$ such that $(x, m) \notin \sigma((Y - \{y\}) \times N)$ for each $m \in M$, so $\{x\} \times M \subset \sigma([y] \times N)$. Hence $\sigma^{-1}([x] \times M) \subseteq \{y\} \times N$. Similarly there exists $y_1 \in Y$ such that $\sigma([y_1] \times N) \subseteq [x] \times M$. From

$$\{y_1\} \times N \subset \sigma^{-1}([x] \times M) \subseteq \{y\} \times N

we have $y_1 = y$. Now, it is clear that $\sigma^{-1}([x] \times M) = \{y\} \times N$ and $y$ must be unique. Now Lemma 3 is applied to complete the proof. \hfill \Box

Now we are ready to give the proof of the theorem.

Proof of Theorem 2. From the Kakutani Representation Theorem there exists a compact Hausdorff space $N$ such that $F$ is isometrically Riesz isomorphic to $C(N)$. Since for each $x \in X$, $\pi^{-1}(1_Y \otimes e)(x)$ is a strong order unit for $E$ ($1_Y \otimes e$ is defined by $1_Y \otimes e(y) = e$), $E$ is (norm) isomorphic and Riesz isomorphic to $C(M)$ for some compact Hausdorff space. Since for compact Hausdorff spaces $Y$ and $Z$, the Banach lattices $C(Y, C(Z))$ and $C(Y \times Z)$ are Riesz isomorphic, we have that $C(X \times M)$ and $C(Y \times N)$ are Riesz isomorphic spaces. Moreover, it is clear that from the hypotheses of the theorem there exists a Riesz isomorphism $\pi_1 : C(X \times M) \to C(Y \times N)$ such that $\pi_1(1_{X \times M}) = 1_{Y \times N}$ and

$$f([x] \times M) \neq 0 \quad \text{for each } x \in X \iff \pi_1f([y] \times N) \neq 0 \quad \text{for each } y \in Y.

Hence from Lemma 5, $X$ and $M$ are homeomorphic to $Y$ and $N$ respectively, so $E$ is (norm) isomorphic and Riesz isomorphic to $F$. \hfill \Box

Let $X$ and $Y$ be compact Haudorff spaces, and $E$ and $F$ be Banach lattices. Let $\pi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$. We conjecture that $X$ and $Y$ are homeomorphic and $E$ and $F$ are Riesz isomorphic.

REFERENCES


DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, 06531 ANKARA, TURKEY
E-mail address: zercan@metu.edu.tr

DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, 06531 ANKARA, TURKEY
E-mail address: osul@metu.edu.tr

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use