A SMOOTH COUNTEREXAMPLE TO NORI’S CONJECTURE ON THE FUNDAMENTAL GROUP SCHEME

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Abstract. We show that Nori’s fundamental group scheme $\pi(X,x)$ does not base change correctly under extension of the base field for certain smooth projective ordinary curves $X$ of genus 2 defined over a field of characteristic 2.

1. Introduction

In the paper [N] Madhav Nori introduced the fundamental group scheme $\pi(X,x)$ for a reduced and connected scheme $X$ defined over an algebraically closed field $k$ as the Tannaka dual group of the Tannakian category of essentially finite vector bundles over $X$. In characteristic zero $\pi(X,x)$ coincides with the étale fundamental group, but in positive characteristic it does not (see, e.g., [MS]). By analogy with the étale fundamental group, Nori conjectured that $\pi(X,x)$ base changes correctly under extension of the base field. More precisely:

Nori’s conjecture (see [MS], page 144 or [N], page 89). If $K$ is an algebraically closed extension of $k$, then the canonical homomorphism

$$h_{X,K} : \pi(X_K,x) \longrightarrow \pi(X,x) \times_k \text{Spec}(K)$$

is an isomorphism.

In [MS] V.B. Mehta and S. Subramanian show that Nori’s conjecture is false for a projective curve with a cuspidal singularity. In this note (Corollary 4.2) we show that certain smooth projective ordinary curves of genus 2 defined over a field of characteristic 2 also provide counterexamples to Nori’s conjecture.

The proof has two ingredients: the first is an equivalent statement of Nori’s conjecture in terms of $F$-trivial bundles due to V.B. Mehta and S. Subramanian (see section 2), and the second is the description of the action of the Frobenius map on rank-2 vector bundles over a smooth ordinary curve $X$ of genus 2 defined over a field of characteristic 2 (see section 3). In section 4 we explicitly determine the set of $F$-trivial bundles over $X$.

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2. Nori’s conjecture and F-trivial bundles

Let $X$ be a smooth projective curve defined over an algebraically closed field $k$ of characteristic $p > 0$. Let $F : X \to X$ denote the absolute Frobenius of $X$ and $F^n$ its $n$-th iterate for some positive integer $n$.

2.1. Definition. A rank-$r$ vector bundle $E$ over $X$ is said to be $F^n$-trivial if $E$ is stable and $F^{\ast}E \cong \mathcal{O}_X^r$.

2.2. Proposition ([MS] Proposition 3.1). If the canonical morphism $h_{X,K}$ (see ([LP1] Proposition 6.1 (4))) is an isomorphism, then any $F^n$-trivial vector bundle $E_K$ over $X_K := X \times_k \text{Spec}(K)$ is isomorphic to $E_k \otimes_k K$ for some $F^n$-trivial vector bundle $E_k$ over $X$.

3. The action of the Frobenius map on rank-2 vector bundles

We briefly recall some results from [LP1] and [LP2].

Let $X$ be a smooth projective ordinary curve of genus 2 defined over an algebraically closed field $k$ of characteristic 2. By [LP2] section 2.3 the curve $X$ equipped with a level-2 structure can be uniquely represented by an affine equation of the form

\begin{equation}
(3.1) \quad y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c),
\end{equation}

for some scalars $a, b, c \in k$. Let $\mathcal{M}_X$ denote the moduli space of $S$-equivalence classes of semistable rank-2 vector bundles with trivial determinant over $X$; see, e.g., [LeP]. We identify $\mathcal{M}_X$ with the projective space $\mathbb{P}^3$ (see [LP1] Proposition 5.1). We denote by $V : \mathbb{P}^3 \to \mathbb{P}^3$ the rational map induced by pull-back under the absolute Frobenius $F : X \to X$. There are homogeneous coordinates $(x_{00} : x_{01} : x_{10} : x_{11})$ on $\mathbb{P}^3$ such that the equations of $V$ are given as follows (see [LP2] section 5):

\begin{equation}
(3.2) \quad V(x_{00} : x_{01} : x_{10} : x_{11}) = (\sqrt{abca}P^2_{00}(x) : \sqrt{b}P^2_{01}(x) : \sqrt{c}P^2_{10}(x) : \sqrt{abc}P^2_{11}(x)),
\end{equation}

with

\begin{align*}
P_{00}(x) &= x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2, & P_{10}(x) &= x_{00}x_{10} + x_{01}x_{11}, \\
P_{01}(x) &= x_{00}x_{01} + x_{10}x_{11}, & P_{11}(x) &= x_{00}x_{11} + x_{10}x_{01}.
\end{align*}

Given a semistable rank-2 vector bundle $E$ with trivial determinant, we denote by $[E] \in \mathcal{M}_X \cong \mathbb{P}^3$ its $S$-equivalence class. The semistable boundary of $\mathcal{M}_X$ equals the Kummer surface $\text{Kum}_X$ of $X$. Given a degree 0 line bundle $N$ on $X$, we also denote the point $[N \oplus N^{-1}] \in \mathbb{P}^3$ by $N$.

3.1. Proposition ([LP3] Proposition 6.1 (4)). The preimage $V^{-1}(N)$ of the point $N \in \text{Kum}_X \subset \mathcal{M}_X \cong \mathbb{P}^3$ with coordinates $(x_{00} : x_{01} : x_{10} : x_{11})$

- is a projective line if $x_{00} = 0$;
- consists of the 4 square roots of $N$ if $x_{00} \neq 0$.

4. Computations

In this section we prove the following.

4.1. Proposition. Let $X = X_{a,b,c}$ be the smooth projective ordinary curve of genus 2 given by the affine model (3.1). Suppose that

\begin{equation}
(4.1) \quad a^2 + b^2 + c^2 + a + c = 0.
\end{equation}
Then there exists a nontrivial family $E \to X \times S$ parametrized by a 1-dimensional variety $S$ (defined over $k$) of $F^n$-trivial rank-2 vector bundles with trivial determinant over $X$. Moreover any $F^n$-trivial rank-2 vector bundle $E$ with trivial determinant appears in the family $E$, i.e., is of the form $(\text{id}_X \times s)^*E$ for some $k$-valued point $s : \text{Spec}(k) \to S$.

We therefore obtain a counterexample to Nori’s conjecture.

4.2. Corollary. Let $X = X_{a,b,c}$ be a curve satisfying (4.1). Then for any algebraically closed extension $K$, the morphism $h_{X,K}$ is not an isomorphism.

Proof. Since $S$ is 1-dimensional, there exists a $K$-valued point $s : \text{Spec}(K) \to S$ that is not a $k$-valued point. Then the bundle $E_K = (\text{id}_X \times s)^*E$ over $X_K$ is not of the form $E_k \otimes_k K$. Now apply Proposition 2.2.

Proof of Proposition 4.1. The method of the proof is to determine explicitly all $F^n$-trivial rank-2 vector bundles $E$ over $X$ for $n = 1, 2, 3, 4$. Taking tensor products of $E$ with $2^{n+1}$-torsion line bundles allows us to restrict attention to $F^n$-trivial vector bundles with trivial determinant.

We first compute the preimage under iterates of $V$ of the point $A_0 \in \mathbb{P}^3$ determined by the trivial rank-2 vector bundle over $X$. We recall (see, e.g., [LP1], Lemma 2.11 (i)) that the coordinates of $A_0 \in \mathbb{P}^3$ in the coordinate system $(x_0 : x_1 : x_2 : x_3)$ are $(1 : 0 : 0 : 0)$. It follows from Proposition 3.1 and equations (3.2) that $V^{-1}(A_0)$ consists of the 4 points

\[(4.2) \quad (1 : 0 : 0 : 0), \quad (0 : 1 : 0 : 0), \quad (0 : 0 : 1 : 0) \quad \text{and} \quad (0 : 0 : 0 : 1),\]

which correspond to the 2-torsion points of the Jacobian of $X$. Abusing notation we denote by $A_1$ both the 2-torsion line bundle on $X$ and the point $(0 : 1 : 0 : 0) \in \mathbb{P}^3$.

Both points $A_0$ and $A_1$ correspond to $S$-equivalence classes of semistable rank-2 vector bundles. The set of isomorphism classes represented by the two $S$-equivalence classes $A_0$ and $A_1$ equal $\mathbb{P} \text{Ext}^1(A_1,A_1) \cup \{0\}$ and $\mathbb{P} \text{Ext}^1(O_X,O_X) \cup \{0\}$, respectively, where 0 denotes the trivial extensions $A_1 \oplus A_1$ and $O_X \oplus O_X$. Note that the two cohomology spaces $\text{Ext}^1(A_1,A_1)$ and $\text{Ext}^1(O_X,O_X)$ are canonically isomorphic to $H^1(O_X)$. The pull-back by the absolute Frobenius $F$ of $X$ induces a rational map

\[F^* : \mathbb{P} \text{Ext}^1(A_1,A_1) \to \mathbb{P} \text{Ext}^1(O_X,O_X),\]

which coincides with the projectivized $p$-linear map on the cohomology $H^1(O_X) \to H^1(O_X)$ induced by the Frobenius map $F$. Since we have assumed $X$ ordinary, this $p$-linear map is bijective. Hence we obtain that there is only one (strictly) semistable bundle $E$ such that $[E] = A_1$ and $F^*E \cong O_X^3$, namely $E = A_1 \oplus A_1$. In particular there are no $F^1$-trivial rank-2 vector bundles over $X$.

By Proposition 3.1 and using the equations (3.2), we easily obtain that the preimage $V^{-1}(A_1)$ is a projective line $L \cong \mathbb{P}^1$, which passes through the two points

\[(1 : 1 : 1 : 1) \quad \text{and} \quad (0 : 0 : 1 : 1).\]

We now determine the bundles $E$ satisfying $F^*E \cong A_1 \oplus A_1$. Given $E$ with $[F^*E] = A_1 \in \mathbb{P}^3$ we easily establish the equivalence

\[F^*E \cong A_1 \oplus A_1 \iff \dim \text{Hom}(F^*E, A_1) = \dim \text{Hom}(E, F_*A_1) = 2.\]

Suppose that $E$ is stable and $F^*E \cong A_1 \oplus A_1$. The quadratic map

\[\det : \text{Hom}(E, F_*A_1) \to \text{Hom}(\det E, \det F_*A_1) = H^0(O_X(w))\]
has nontrivial fibre over 0, since dim Hom(E, F, A_1) = 2. Hence there exists a nonzero \( f \in \text{Hom}(E, F, A_1) \) not of maximal rank. We consider the line bundle \( N = \text{im} \ f \subset F, A_1 \). Since \( F, A_1 \) is stable (see [LaP], Proposition 1.2), we obtain the inequalities

\[
0 = \mu(E) < \deg N < \frac{1}{2} = \mu(F, A_1),
\]
a contradiction. Therefore \( E \) is strictly semistable and \( [E] = [A_2 \oplus A_2^{-1}] \) for some 4-torsion line bundle \( A_2 \) with \( A_2 \otimes 2 = A_1 \). The \( S \)-equivalence class \( [A_2 \oplus A_2^{-1}] \) contains three isomorphism classes, and a standard computation shows that only the decomposable bundle \( A_2 \oplus A_2^{-1} \) is mapped by \( F^* \) to \( A_1 \oplus A_1 \). In particular there are no \( F^2 \)-trivial rank-2 bundles.

We now determine the coordinates of \( A_2 \) by intersecting the line \( L \), which can be parametrized by \( (r : r : s : s) \) with \( r, s \in k \), with the Kummer surface, whose equation is (see [LP2], Proposition 3.1)

\[
c(x_0^2x_{10}^2 + x_{01}^2x_{11}^2) + b(x_0x_{10}^2 + x_{01}x_{11}^2) + a(x_0^2x_{11} + x_{10}x_{01}) + x_0x_1x_10x_{11} = 0.
\]

The computations are straightforward and will be omitted. Let \( u \in k \) be a root of the equation

\[
u^2 + u = b.
\]

Then \( u + 1 \) is the other root. The coordinates of the two 4-torsion line bundles (modulo the canonical involution of the Jacobian of \( X \)) \( A_2 \) such that \( A_2 \otimes 2 = A_1 \) are

\[
(u : u : \sqrt{u} : \sqrt{b}) \quad \text{and} \quad (u + 1 : u + 1 : \sqrt{u} : \sqrt{b}).
\]

Now the equation \( u = 0 \) (resp. \( u + 1 = 0 \)) implies by (4.3) that \( b = 0 \), which is excluded because we have assumed \( X \) smooth. So by Proposition 3.1 the preimage \( V^{-1}(A_2) \) consists of the 4 line bundles \( A_3 \) such that \( A_3 \otimes 2 = A_2 \). In particular there are no \( F^3 \)-trivial rank-2 bundles.

One easily verifies that the image under the rational map \( V \) given by (3.2) of the hyperplane \( x_{00} = 0 \) is the quartic surface given by the equation

\[
bx_{11}^2x_{10}^2 + cx_{11}x_{01}^2 + ax_{10}x_{01}^2 + x_0x_1x_{10}x_{11} = 0.
\]

When we replace \( (x_{00} : x_{01} : x_{10} : x_{11}) \) with \( (u : u : \sqrt{u} : \sqrt{b}) \) in (4.1) we obtain the equation

\[
b^2 + u^2(1 + a + c) = 0.
\]

Similarly replacing \( (x_{00} : x_{01} : x_{10} : x_{11}) \) with \( (u + 1 : u + 1 : \sqrt{u} : \sqrt{b}) \) in (4.1) we obtain the equation

\[
b^2 + (u^2 + 1)(1 + a + c) = 0.
\]

Finally the product of \( \text{Lm} \) with \( \text{Lm} \) equals (here one uses [4.3]) equation (4.1) up to a factor \( b^2 \), which we can drop since \( b \neq 0 \) — note that we have assumed \( X \) smooth, hence \( b \neq 0 \) by [LP2], Lemma 2.1. To summarize we have shown that if (4.1) holds, then by Proposition 3.1 there exists an 8-torsion line bundle \( A_3 \) with \( A_3 \otimes 2 = A_1 \) and such that the preimage \( V^{-1}(A_3) \) is a projective line \( \Delta \subset \mathbb{P}^3 \).

Consider a point \( [E] \in \Delta \) away from the Kummer surface — note that \( \Delta \) is not contained in the Kummer surface \( \text{Kum} \) because its intersection is contained in the set of 16-torsion points. Then \( E \) is stable and \( [F^*E] = [A_3 \oplus A_3^{-1}] \). There are three isomorphism classes represented by the \( S \)-equivalence class \( [A_3 \oplus A_3^{-1}] \), namely the trivial extension \( A_3 \oplus A_3^{-1} \) and two nontrivial extensions (for the details
Since $E$ is invariant under the hyperelliptic involution we obtain $F^*E = A_3 \oplus A_3^{-1}$ and finally that $E$ is $F^4$-trivial. Hence any stable point on $\Delta$ is $F^4$-trivial.

Therefore, assuming (4.1), there exists a 1-dimensional subvariety $\Delta_0 \subset M_X \setminus \text{Kum}_X$ parametrizing all $F^4$-trivial rank-2 bundles. Passing to an étale cover $S \to \Delta_0$ ensures existence of a “universal” family $E \to X \times S$ and we are done. □

Remark. Note that equation (4.1) depends on the choice of a nontrivial 2-torsion line bundle $A_1$. If one chooses the 2-torsion line bundle $(0 : 0 : 1 : 0)$ or $(0 : 0 : 0 : 1)$ (see (4.2)), then the corresponding equations are

$$a^2 + b^2 + c^2 + a + b = 0 \quad \text{or} \quad a^2 + b^2 + c^2 + b + c = 0.$$