4 PLANES IN $\mathbb{R}^4$

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This paper is dedicated to Julia.

ABSTRACT. We establish a homeomorphism between the moduli space $A_{4,k}^{\text{ord}}(\mathbb{R})$ of ordered $k$-tuples $(H_1, \ldots, H_k)$ of 2-dimensional linear subspaces $H_i \subset \mathbb{R}^4$ (mod $GL_4(\mathbb{R})$) and the quotient by simultaneous conjugation of a certain open subset $(GL_2(\mathbb{R}))^{k-3} \subset (GL_2(\mathbb{R}))^k$. For $k = 4$, this leads to an explicit computation of the moduli space $A_{4,4}(\mathbb{R})$ of central 2-arrangements in $\mathbb{R}^4$ mod $GL_4(\mathbb{R})$ and its subspace $A_{2,4}(\mathbb{C})$ of those classes that contain a complex hyperplane arrangement.

1. INTRODUCTION

In [4], Goresky and MacPherson propose the investigation of so-called 2-arrangements, which they define to be finite collections $A$ of affine $\mathbb{R}$-subspaces of codimension 2 in some $\mathbb{R}^{2n}$ with the additional property that all intersections $\bigcap_{H \in I} H$, $I \subset A$ are even-dimensional. Clearly, complex hyperplane arrangements can be viewed as such. It is natural to ask whether for 2-arrangements there exist more homotopy types of the complement $M(A) := \mathbb{R}^{2n} \setminus \bigcup_{H \in A} H$ than for complex hyperplane arrangements.

In [8], Ziegler presents two central (i.e., the planes are linear) 2-arrangements of 4 planes in $\mathbb{R}^4$ which differ in cohomology algebras of their complements. One of these is a complex line arrangement in $\mathbb{C}^2$, while the other is not. A classical result of Orlik and Solomon [6] states that the cohomology algebra of the complement of a complex hyperplane arrangement is uniquely determined by the intersection lattice. Since Ziegler’s arrangements obviously are lattice-equivalent, the non-complex one does not have the homotopy type of a complex hyperplane arrangement.

So far, the class of central 2-arrangements of $k$ planes in $\mathbb{R}^4$ is not understood in general. The state of the art is Matei and Suciu’s classification [5] of central 2-arrangements in $\mathbb{R}^4$ with $k \leq 6$ up to homotopy type of the complement.

In this paper a natural approach to the problem is presented and carried out in detail for the case $k = 4$: We study the moduli space $A_{4,k}(\mathbb{R})$ of central 2-arrangements of $k$ planes in $\mathbb{R}^4$, identifying two of them if they can be transformed into one another by a general linear transformation of $\mathbb{R}^4$. By this approach, $k = 4$ is the first non-trivial case (for $k = 3$, we get a one-point-space, since any 3 trivially...
intersecting planes can be transformed into a standard configuration; see Section 2).

In computing the moduli space $A_{4,4}(R)$ (and the subspace $A_{2,4}(C) \subset A_{4,4}(R)$ of those classes of 2-arrangements mod $GL_4 := GL_4(R)$ that contain a complex hyperplane arrangement), we not only rediscover Ziegler’s result of finding exactly two homotopy types of complements for $k = 4$, but also identify $A_{4,4}(R)$ as an interesting topological space:

Let $\mathbb{P}R_2$ denote the two-dimensional real projective space, that is, the space of all lines $[x, y, z] := \{(\lambda x, \lambda y, \lambda z) | \lambda \in \mathbb{R}\}$, $(x, y, z) \neq (0, 0, 0)$, in $\mathbb{R}^3$, endowed with the quotient topology with respect to the canonical projection $\mathbb{R}^3 \to \mathbb{P}R_2$.

Let $\mathbb{P}R_2^*$ be the subspace of $\mathbb{P}R_2$ consisting of all lines in $\mathbb{R}^3$ that do not lie on any coordinate plane, that is,

$$\mathbb{P}R_2^* := \{[x, y, z] \in \mathbb{P}R_2 | xyz \neq 0\}.$$ 

Furthermore, let $Q \subset \mathbb{P}R_2$ be the following quadric in $\mathbb{P}R_2$:

$$Q := \{[x, y, z] \in \mathbb{P}R_2 | x^2 + y^2 + z^2 - 2(xy + xz + yz) = 0\}.$$

With $Q^*$ a duplicate of $\mathbb{P}R_2^* \cap Q$, and $\pi: \mathbb{P}R_2^* \cup Q^* \to \mathbb{P}R_2^*$ the canonical projection, we topologize $\mathbb{P}R_2^* \cup Q^*$ by requiring $Q^*$ to be closed in $\mathbb{P}R_2^* \cup Q^*$ and that for all $q \in Q^*$,

$$B_q := \{\pi^{-1}(U) | U \text{ a neighborhood of } \pi(q)\}$$

is a neighborhood basis of $q$. Let

$$Q_< := \{[x, y, z] \in \mathbb{P}R_2 | x^2 + y^2 + z^2 - 2(xy + xz + yz) < 0\} \cup Q^*.$$

Figure 1 illustrates the situation: Identify the four open quarters of the upper hemisphere in the figure with $\mathbb{P}R_2^*$. Then the dark area represents $Q_<$, its boundary $Q$.

![Figure 1. The moduli space $A_{4,4}(R)$](image-url)
Theorem 1.  
\[ A_{4,4}(\mathbb{R}) \cong \mathbb{R}^*_2 \cup Q^*/ \text{permutations of coordinates } x,y,z, \]
\[ A_{2,4}(\mathbb{C}) \cong Q_</ \text{permutations of coordinates } x,y,z. \]

Corollary 2. There exist exactly 2 homotopy types of complements of central 2-arrangements of 4 planes in \( \mathbb{R}^4 \).

For general \( k \) we show the following: Let \( A_{4,k}^\text{ord}(\mathbb{R}) \) be the moduli space mod \( \text{GL}_4 \) of all \( k \)-tuples \((H_1,\ldots,H_k)\) of planes in \( \mathbb{R}^4 \) such that \( \{H_1,\ldots,H_k\} \in A_{4,k}(\mathbb{R}) \). Let 
\[ (\text{GL}_4^k)^* := \{(\alpha_1,\ldots,\alpha_i) \in \text{GL}_4^i(\mathbb{R}) | \det(\alpha_i - id) \neq 0, \ det(\alpha_i - \alpha_j) \neq 0 \ \forall i \neq j \}. \]

For \( \alpha = (\alpha_1,\ldots,\alpha_i) \), \( \alpha' = (\alpha'_1,\ldots,\alpha'_i) \in \text{GL}_4^i \) we say that \( \alpha \) and \( \alpha' \) are simultaneously conjugate if there exists \( \delta \in \text{GL}_4^i \) such that \( \delta \alpha_1 = \alpha'_1 \delta,\ldots,\delta \alpha_i = \alpha'_i \delta \).

Theorem 3.  
\[ A_{4,k}^\text{ord}(\mathbb{R}) \cong (\text{GL}_2^{k-3})^*/\text{simultaneous conjugation}. \]

2. \( k \)-tuples of planes in \( \mathbb{R}^4 \)

In this section we prove Theorem 3. For calculations it is convenient to describe the planes as kernels of linear maps: Let \((\mathcal{L}_{4,2}^k)^*\) consist of all \( k \)-tuples \((\gamma_1,\ldots,\gamma_k)\) \in \text{Hom}(\mathbb{R}^4,\mathbb{R}^2)^k \) such that \( \{\ker \gamma_1,\ldots,\ker \gamma_k\} \) is a central 2-arrangement in \( \mathbb{R}^4 \), that is:

- \( \text{rk} \gamma_i = 2 \) for all \( i = 1,\ldots,k \),
- \( \ker \gamma_i \cap \ker \gamma_j = \{0\} \) for all \( 1 \leq i < j \leq k \).

We endow \( A_{4,k}^\text{ord}(\mathbb{R}) \) with the quotient topology with respect to the map
\[ (\mathcal{L}_{4,2}^k)^* \longrightarrow A_{4,k}^\text{ord}(\mathbb{R}), \]
\[ (\gamma_1,\ldots,\gamma_k) \longmapsto (\ker \gamma_1,\ldots,\ker \gamma_k). \]

Denote by \( \pi \in \text{Hom}(\mathbb{R}^4,\mathbb{R}^2) \) the orthogonal projection onto the first two coordinates, by \( \rho \) the one onto the third and fourth. For any \((\gamma_1,\gamma_2,\gamma_3) \in (\mathcal{L}_{4,2}^3)^* \) there is a linear transformation \( \Phi \in \text{GL}_4 \), such that \( (\gamma_1 \circ \Phi^{-1},\gamma_2 \circ \Phi^{-1},\gamma_3 \circ \Phi^{-1}) = (-\alpha^{-1}\pi,\beta^{-1}\rho,\rho - \pi) \) for some \( \alpha,\beta \in \text{GL}_2 \), namely \( \Phi := \begin{pmatrix} -\alpha \gamma_1 \\ \beta \gamma_2 \end{pmatrix} \) \in \text{GL}_4 \), where \((\alpha,\beta) \in \text{GL}_2^2 \) is the unique pair such that \( \gamma_3 = \alpha \gamma_1 + \beta \gamma_2 \). Geometrically, this means that mod \( \text{GL}_4 \) any 3 pairwise trivially intersecting planes in \( \mathbb{R}^4 \) are equivalent to \((\ker \rho, \ker \rho, \ker (\rho - \pi)) \).

Note that for any additional plane \( \ker \gamma \) the set \( \{\ker \pi, \ker \rho, \ker (\rho - \pi), \ker \gamma\} \) is a central 2-arrangement if and only if \( \gamma = \beta \rho - \alpha \pi \) for some \( \alpha,\beta \in \text{GL}_2 \) such that \( \alpha - \beta \in \text{GL}_2 \), or, equivalently, if and only if \( \ker \gamma = \ker (\rho - \alpha' \pi) \) with \( \alpha' \in \text{GL}_2^5 := (\text{GL}_4^2)^* \).

Furthermore, for any two additional planes \( \ker (\rho - \alpha_4 \pi), \ker (\rho - \alpha_5 \pi) \), such that \((\pi,\rho,\rho - \pi,\rho - \alpha_4 \pi), (\pi,\rho,\rho - \pi,\rho - \alpha_5 \pi) \in (\mathcal{L}_{4,2}^5)^* \), we have:
\[ \det(\alpha_4 - \alpha_5) \neq 0 \Leftrightarrow (\pi,\rho,\rho - \pi,\rho - \alpha_4 \pi,\rho - \alpha_5 \pi) \in (\mathcal{L}_{4,2}^5)^*. \]

Therefore, it is natural to consider the following maps:
\[ r : (\mathcal{L}_{4,2}^5)^* \longrightarrow (\text{GL}_2^{k-3})^*, \]
\[ (\gamma_1,\ldots,\gamma_k) \longmapsto (\beta_4^{-1}\alpha_4,\ldots,\beta_k^{-1}\alpha_k), \]
such that \( \Phi = \begin{pmatrix} -\alpha \gamma_1 \\ \beta \gamma_2 \end{pmatrix} \), \( \gamma_3 = \alpha \gamma_1 + \beta \gamma_2 \) and \( \gamma_j \circ \Phi^{-1} = \beta_j \rho - \alpha_j \pi \) for \( j = 4, \ldots, k \), and

\[
\begin{align*}
(2.2) \quad i : (GL_2^{k-3})^* & \longrightarrow (L_4^k)^* \\
(\alpha_4, \ldots, \alpha_k) & \longrightarrow (\pi, \rho - \pi, \rho - \alpha_4 \pi, \ldots, \rho - \alpha_k \pi).
\end{align*}
\]

Obviously, \( r \) is a retraction with section \( i \). The final step is to show that \( r \) induces an isomorphism

\[
r_* : A_{4,k}^{\text{ord}}(\mathbb{R}) \cong (L_4^k)^*/GL_2^k \times GL_4 \longrightarrow (GL_2^{k-3})^*/\text{simconj}.
\]

**Proof.**

(1) \( r_* \) is well defined:

Let \((\gamma_1, \ldots, \gamma_k) \in (L_4^k)^*^\times GL_4 \), \( \gamma_3 = \alpha \gamma_1 + \beta \gamma_2 \),

\[
\Phi = \begin{pmatrix} -\alpha \gamma_1 \\ \beta \gamma_2 \end{pmatrix}, \quad \gamma_j = \beta_j \rho - \alpha_j \pi, \quad j = 4, \ldots, k.
\]

(a) Let \( \Psi \in GL_4 \), \( (\gamma_1', \ldots, \gamma_k') = (\gamma_1 \circ \Psi^{-1}, \ldots, \gamma_k \circ \Psi^{-1}) \).

It follows that

\[
\begin{align*}
\gamma_3' &= \alpha \gamma_1' + \beta \gamma_2', \\
\gamma_j' &= \gamma_j \circ \Phi^{-1} = \gamma_j \circ \Psi \circ \Phi^{-1} = \gamma_j \circ \Phi^{-1}
\end{align*}
\]

and therefore

\[
r(\gamma_1', \ldots, \gamma_k') = r(\gamma_1, \ldots, \gamma_k).
\]

(b) Let \((\delta_1, \ldots, \delta_k) \in GL_2 \times GL_4 \), \( (\gamma_1', \ldots, \gamma_k') = (\delta_1 \gamma_1, \ldots, \delta_k \gamma_k) \).

It follows that

\[
\begin{align*}
\gamma_3' &= \delta_3 \gamma_3 = \delta_3 (\alpha \gamma_1 + \beta \gamma_2) = \delta_3 \alpha \gamma_1 + \delta_3 \beta \gamma_2, \\
\gamma_j' &= \delta_3 \gamma_j = \delta_3 \gamma_j \circ \Phi^{-1} = \delta_3 \gamma_j \circ \Phi^{-1}
\end{align*}
\]

so

\[
\Phi' = \begin{pmatrix} -\delta_3 \alpha \gamma_1 \\ \delta_3 \beta \gamma_2 \end{pmatrix} = \begin{pmatrix} -\delta_3 \alpha \gamma_1 \\ \delta_3 \beta \gamma_2 \end{pmatrix} = \begin{pmatrix} \delta_3 \\ \delta_3 \end{pmatrix} \circ \Phi.
\]

This implies that

\[
\begin{align*}
\gamma_j' \circ \Phi^{-1} &= \delta_3 \gamma_j \circ \Phi^{-1} \circ \begin{pmatrix} \delta_3^{-1} \\ 0 \\ 0 \\ \delta_3^{-1} \end{pmatrix} \\
&= \delta_3 \beta_j \rho - \alpha_j \pi \circ \begin{pmatrix} \delta_3^{-1} \\ 0 \\ 0 \\ \delta_3^{-1} \end{pmatrix} \\
&= \delta_3 \beta_j \delta_3^{-1} \rho - \alpha_j \delta_3^{-1} \pi;
\end{align*}
\]

thus

\[
r(\gamma_1', \ldots, \gamma_k') \sim r(\gamma_1, \ldots, \gamma_k)
\]

modulo simultaneous conjugation.

(2) The section \( i \) induces a section \( i_* : (GL_2^{k-3})^*/\text{simconj} \longrightarrow A_{4,k}^{\text{ord}}(\mathbb{R}) \):

Let \((\alpha_1, \ldots, \alpha_k) \in (GL_2^{k-3})^* \), \( \delta \in GL_2 \), \( (\alpha_1', \ldots, \alpha_k') = (\delta \alpha_1 \delta^{-1}, \ldots, \delta \alpha_k \delta^{-1}) \).

We calculate:

\[
i(\alpha_1, \ldots, \alpha_k) = (\pi, \rho - \pi, \rho - \delta \alpha_1 \delta^{-1} \pi, \ldots, \rho - \delta \alpha_k \delta^{-1} \pi)
\]

\[
= (\delta \pi, \delta \rho, \delta (\rho - \pi), \delta \rho - \delta \alpha_1 \pi, \ldots, \delta \rho - \delta \alpha_k \pi) \circ \begin{pmatrix} \delta^{-1} \\ 0 \\ 0 \\ \delta^{-1} \end{pmatrix}
\]
Lemma 4.

\[ A_{4,4}^{\text{ord}}(\mathbb{R}) \cong \text{GL}_2^*/\text{conjugation} \cong \mathbb{P}\mathbb{R}_2^* \cup \mathbb{Q}^*. \]

**Proof.** The first isomorphism is from Theorem 3. For the second, consider \( d : \text{GL}_2^* \to \mathbb{P}\mathbb{R}_2^* \), \( \alpha \mapsto [\det \alpha, \det(\alpha - \text{id}), \det \text{id}] \) and the restriction of the characteristic map on \( M(2 \times 2; \mathbb{R}) \) to \( \text{GL}_2^* : \chi : \text{GL}_2^* \to X := \{(x, y) \in \mathbb{R}^2 \mid x + y + 1 \neq 0, y \neq 0\}, \alpha \mapsto \chi((x, y) \to [y, x + y + 1, 1]) \) such that \( d = \Phi \circ \chi \).

It is known (see [3], pp. 161ff) that under a change of coordinates the fibers of \( \chi \) are hyperboloids, which are singular if and only if the discriminant \( \Delta = \det \alpha - (\text{trace} \alpha)^2/4 \) vanishes. Furthermore, regular fibers are equivalence classes modulo conjugation, while singular fibers consist of two of them: the singular point of the cone is one and the rest makes up another. The assertion follows from the equivalence \( \det \alpha - (\text{trace} \alpha)^2/4 = 0 \iff d(\alpha) \in \mathbb{Q} \subset \mathbb{P}\mathbb{R}_2^* \).

A structure of a \( \mathbb{C} \)-vector space on \( \mathbb{R}^4 \) that extends the \( \mathbb{R} \)-vector space structure is uniquely determined by an automorphism \( J \in \text{GL}_4 \) such that \( J^2 = -\text{id} \), representing scalar multiplication with \( i \in \mathbb{C} \). The standard \( \mathbb{C} \)-vector space structure on \( \mathbb{R}^4 \) is given by the matrix \( J = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \), where \( I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

We call a \( k \)-tuple \( \mathcal{H} = (H_1, \ldots, H_k) \) of planes complex if there exists a \( \mathbb{C} \)-vector space structure on \( \mathbb{R}^4 \) extending the \( \mathbb{R} \)-vector space structure, such that \( \mathcal{H} \) is complex with respect to this \( \mathbb{C} \)-vector space structure, i.e., \( J(H_i) = H_i, \ i = 1, \ldots, k \).

Note that an equivalence class \([ (H_1, \ldots, H_k) ] \in A_{4,4}^{\text{ord}}(\mathbb{R}) \) contains a complex element if and only if it contains an element that is complex with respect to any fixed complex structure. This is because for any \( J_i \in \text{GL}_4 \) such that \( J_i^2 = -\text{id}, \ i = 1, 2 \), there exists \( \phi \in \text{GL}_4 \) such that \( J_2 = \phi J_1 \phi^{-1} \), so that if \( \mathcal{H} \) is complex with respect to \( J_1 \), so is \( \phi(\mathcal{H}) \) with respect to \( J_2 \).

We define \( A_{2,4}^{\text{rel}}(\mathbb{C}) \) to be the set of equivalence classes in \( A_{4,4}^{\text{ord}}(\mathbb{R}) \) that contain a complex element.

**Lemma 5.** The isomorphism from Lemma 4 induces an isomorphism
\[ A_{2,4}^{\text{rel}}(\mathbb{C}) \cong \mathbb{Q}_< \subset \mathbb{P}\mathbb{R}_2^* \cup \mathbb{Q}^*. \]

**Proof.** Let \( \mathcal{H} = (H_1, \ldots, H_4) \) be complex with respect to the standard \( \mathbb{C} \)-vector space structure. It follows that there are \( \gamma_1, \ldots, \gamma_4 \in \text{Hom}(\mathbb{C}^2, \mathbb{C}) \) of complex rank 1 such that \( H_i = \text{ker} \gamma_i, \ i = 1, \ldots, 4 \) and that \( r(\mathcal{H}) \) represents a complex number. The assertion follows from the fact (see for example [3], p. 234) that \( \alpha \in \text{GL}_2 \) is equivalent modulo conjugation to an element representing a complex number if and only if it is either a multiple of the identity or its discriminant \( \Delta \) satisfies \( \Delta > 0 \).
We are now ready to prove Theorem 1.

Proof. Since $A_{4,4}(\mathbb{R}) = A_{4,4}(\mathbb{R})^{\text{ord}}/S_4$, where $S_4$ acts on $A_{4,4}(\mathbb{R})^{\text{ord}}$ by permutation of the 4 planes, we need to find the induced operation of $S_4$ on $\text{GL}_2^*/\text{conj}$ under the isomorphism $r_* : A_{4,4}(\mathbb{R})^{\text{ord}} \xrightarrow{\cong} \text{GL}_2^*/\text{conj}$. For this, let $[\alpha] \in \text{GL}_2^*/\text{conj}$, $\mathcal{H} = (\ker \pi, \ker \rho, \ker (\rho - \pi), \ker (\rho - \alpha \pi))$. We calculate $r_*([\sigma(\mathcal{H})])$ for $\sigma \in \{(1, 2), (2, 3), (3, 4)\}$.

1. $\sigma = (1, 2)$:
   \[ \sigma(\mathcal{H}) = (\ker \gamma_1, \ldots, \ker \gamma_4) \text{ with} \]
   \[ \gamma_1 = \rho, \gamma_2 = \pi, \gamma_3 = \rho - \pi, \gamma_4 = \rho - \alpha \pi. \]
   Definition 2.1 of $r_* : (\mathcal{L}^4_{4,2})^* \to \text{GL}_2^*$ leads to:
   \[ \gamma_3 = \id \gamma_1 + (-\id) \gamma_2, \Phi^{-1} = \Phi = \begin{pmatrix} -\rho \\ -\pi \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \alpha \rho - \pi; \]
   therefore
   \[ r((\gamma_1, \ldots, \gamma_4)) = \alpha^{-1}. \]

2. $\sigma = (2, 3)$:
   \[ \sigma(\mathcal{H}) = (\ker \gamma_1, \ldots, \ker \gamma_4) \text{ with} \]
   \[ \gamma_1 = \pi, \gamma_2 = \rho - \pi, \gamma_3 = \rho, \gamma_4 = \rho - \alpha \pi. \]
   We have
   \[ \gamma_3 = \id \gamma_1 + \id \gamma_2, \Phi^{-1} = \Phi = \begin{pmatrix} -\pi \\ \rho - \pi \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \rho - (\id - \alpha) \pi; \]
   therefore
   \[ r((\gamma_1, \ldots, \gamma_4)) = \id - \alpha. \]

3. $\sigma = (3, 4)$:
   \[ \sigma(\mathcal{H}) = (\ker \gamma_1, \ldots, \ker \gamma_4) \text{ with} \]
   \[ \gamma_1 = \pi, \gamma_2 = \rho, \gamma_3 = \rho - \alpha \pi, \gamma_4 = \rho - \pi. \]
   We have
   \[ \gamma_3 = -\alpha \gamma_1 + \id \gamma_2, \Phi = \begin{pmatrix} \alpha \pi \\ \rho \end{pmatrix}, \Phi^{-1} = \begin{pmatrix} \alpha^{-1} \pi \\ \rho \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \rho - \alpha^{-1} \pi; \]
   therefore
   \[ r((\gamma_1, \ldots, \gamma_4)) = \alpha^{-1}. \]
   The assertion now follows from (for the definition of the map $d : \text{GL}_2^* \to \mathbb{P} \mathbb{R}_2^*$, see proof of Lemma 4):

1. \[ d(\alpha^{-1}) = [\det \alpha^{-1}, \det(\alpha^{-1} - \id), \det \id] \]
   \[ = [\det \id, \det(\id - \alpha), \det \alpha] \]
   \[ = [\det \id, \det(\alpha - \id), \det \alpha] \]
   \[ = \sigma_{x,z}(d(\alpha)), \]
   where $\sigma_{x,z}$ interchanges the $x$ and $z$ coordinates in $\mathbb{P} \mathbb{R}_2^*$, and
\[ d(\text{id} - \alpha) = [\det(\text{id} - \alpha), \det((\text{id} - \alpha) - \text{id}), \det \text{id}] = [\det(\alpha - \text{id}), \det \alpha, \det \text{id}] = \sigma_{x,y}(d(\alpha)), \]

where \( \sigma_{x,y} \) interchanges the \( x \) and \( y \) coordinates in \( \mathbb{P}R^2 \). \( \square \)

Finally, we prove Corollary 2.

**Proof.** There are 4 connected components of \( \mathbb{P}R^*_2 \cup Q^* \), namely,

\[ A_{\epsilon_1,\epsilon_2} := \{(x, y, z) \in \mathbb{P}R^*_2 \cup Q^* \mid xz \epsilon_1, yz \epsilon_2 \}, \quad (\epsilon_1, \epsilon_2) \in \{\langle, \rangle\}^2. \]

Since \( A_{\epsilon_1,\epsilon_2} \) is invariant under permutations of the coordinates \( x, y, z \) while the other three are identified by them, it follows that \( A_{4,4}(\mathbb{R}) \) consists of two connected components. A result by Randell [7] asserts that the homotopy types of complements of arrangements are constant on connected components. On the other hand, Ziegler presents in [8] two such arrangements with differing cohomology algebras, and consequently, differing homotopy types of their complements. This proves the assertion. \( \square \)

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**REFERENCES**


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