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SVEP FOR MULTIPLIERS ON A FAITHFUL COMMUTATIVE BANACH ALGEBRA

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Abstract. We give some sufficient conditions that each multiplier on a faithful commutative Banach algebra has SVEP. On the other hand, we show that there exist a faithful commutative Banach algebra and a multiplier on it without SVEP. Such examples of multipliers can actually be found within the class of multiplication operators on unital commutative Banach algebras. This answers in negative a question that is stated as Open problem 6.2.1 by Laursen and Neumann, 2000.

1. Introduction

A mapping $T: \mathcal{A} \to \mathcal{A}$ on a commutative complex Banach algebra $\mathcal{A}$ is a multiplier if

$$a(Tb) = (Ta)b \qquad \text{for all} \quad a,\ b \in \mathcal{A}.$$

For every element $a \in \mathcal{A}$, the multiplication operator $L_a: \mathcal{A} \to \mathcal{A}$ that is given by $L_a b = ab$ ($b \in \mathcal{A}$) is clearly a multiplier. When $\mathcal{A}$ has a unit, say e, the converse holds. Namely, if T is a multiplier on $\mathcal{A}$, then $Tb = e(Tb) = (Te)b$, for all $b \in \mathcal{A}$, which means that T is a multiplication operator induced by Te on $\mathcal{A}$.

Let $L(\mathcal{A})$ be the set of all multiplication operators on $\mathcal{A}$. For a non-unital commutative Banach algebra $\mathcal{A}$, the set $M(\mathcal{A})$ of all multipliers on $\mathcal{A}$ may be larger than $L(\mathcal{A})$. However, if $\mathcal{A}$ is faithful in the sense that, for $a \in \mathcal{A}$, the products ab ($b \in \mathcal{A}$) are all zero if and only if $a = 0$, then $M(\mathcal{A})$ is a closed commutative Banach subalgebra of $B(\mathcal{A})$, the algebra of all bounded linear operators on $\mathcal{A}$, that contains I, the identity operator on $\mathcal{A}$, and all multiplication operators. Every multiplier T on a faithful commutative Banach algebra satisfies $T(ab) = aT(b)$ for all $a, b \in \mathcal{A}$. Note that any unital algebra is faithful. Moreover, any semi-prime commutative Banach algebra, i.e. a commutative Banach algebra without non-zero nilpotent elements, is faithful. The reader is referred to [2] for details about multipliers.

Let $\mathcal{X}$ be a complex Banach space and let $T \in B(\mathcal{X})$. Then T is an operator with SVEP (single-valued extension property) if, for every open set $U \subseteq \mathbb{C}$, the only analytic function $f: U \to \mathcal{X}$ that satisfies $(T - \lambda)f(\lambda) = 0$ ($\lambda \in U$) is the zero function on U ([3], Definition 1.2.9). We are concerned with the question whether all multipliers on a faithful commutative Banach algebra do have SVEP. The question is stated as an open problem in [3] (Problem 6.2.1). In Section 3 we

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shall answer the question in the negative. Before that, in Section 2, we give some sufficient conditions for a multiplier on a faithful commutative Banach algebra to have SVEP.

2. Sufficient conditions for SVEP

During this section, $\mathcal{A}$ will always mean a faithful commutative Banach algebra and $M(\mathcal{A})$ will be the algebra of all multipliers on $\mathcal{A}$. We shall give some sufficient conditions for $T \in M(\mathcal{A})$ to have SVEP.

The dual space $\mathcal{A}^*$ of the algebra $\mathcal{A}$ has a natural structure of a left Banach $\mathcal{A}$-module. Namely, for $a \in \mathcal{A}$ and $\xi \in \mathcal{A}^*$, the product $a \cdot \xi \in \mathcal{A}^*$ is given by $\langle a \cdot \xi, b\rangle = \langle \xi, ab\rangle$, for all $b \in \mathcal{A}$. For $T \in M(\mathcal{A})$, we shall denote by T^* the adjoint operator of T on $\mathcal{A}^*$.

Proposition 2.1. *If, for a multiplier $T \in M(\mathcal{A})$, the adjoint operator $T^* \in B(\mathcal{A}^*)$ has* SVEP*, then T has* SVEP*, as well.*

Proof. Let $U \subseteq \mathbb{C}$ be an arbitrary non-empty open set and let $f : U \to \mathcal{A}$ be an analytic function such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. Choose an arbitrary $\xi \in \mathcal{A}^*$ and denote $F_\xi(\lambda) = f(\lambda) \cdot \xi$. It is obvious that $F_\xi : U \to \mathcal{A}^*$ is an analytic function. Since

$$\langle (T^* - \lambda)F_\xi(\lambda), a\rangle = \langle \xi, f(\lambda)\big((T - \lambda)a\big)\rangle = \langle \xi, \big((T - \lambda)f(\lambda)\big)a\rangle = 0 \qquad (\lambda \in U)$$

holds for all $a \in \mathcal{A}$ we conclude that $(T^* - \lambda)F_\xi(\lambda) = 0$, for $\lambda \in U$. By the assumption, T^* has SVEP, therefore $F_\xi \equiv 0$ on U. Thus, $f(\lambda) \cdot \xi = 0$, for all $\xi \in \mathcal{A}^*$ and all $\lambda \in U$. This gives $\langle \xi, f(\lambda)a\rangle = 0$, for all $a \in \mathcal{A}$, $\xi \in \mathcal{A}^*$, and $\lambda \in U$. By the faithfulness of $\mathcal{A}$, we conclude $f \equiv 0$ on U. □

Let $\mathcal{X}$ be a Banach space, $T \in B(\mathcal{X})$, and let $\mathcal{B} \subseteq B(\mathcal{X})$ be a closed subspace such that $TB \in \mathcal{B}$, for any $B \in \mathcal{B}$. It is not hard to see that the multiplication operator L_T that is induced by T on $\mathcal{B}$ has SVEP whenever T has SVEP. The following proposition asserts that the converse holds if $\mathcal{X} = \mathcal{A}$, $T \in M(\mathcal{A})$, and $\mathcal{B}$ contains $L(\mathcal{A})$, the space of all multiplication operators.

Proposition 2.2. *Let $T \in M(\mathcal{A})$ and assume that $\mathcal{B} \subseteq B(\mathcal{A})$ is a closed subspace such that $L(\mathcal{A}) \subseteq \mathcal{B}$ and $TB \in \mathcal{B}$, for any $B \in \mathcal{B}$. Then T has* SVEP *whenever the multiplication operator L_T that is induced by T on $\mathcal{B}$ has* SVEP*.*

Proof. If T does not have SVEP, then there exist a non-empty open set $U \subseteq \mathbb{C}$ and a non-zero analytic function $f : U \to \mathcal{A}$ such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. Define $F : U \to \mathcal{B}$ by $F(\lambda) = L_{f(\lambda)}$. Since $\mathcal{A}$ is faithful F is a non-zero analytic function. Of course, $(L_T - \lambda)F(\lambda) = 0$, for all $\lambda \in U$, which shows that L_T is without SVEP. □

Let T be a multiplier on a faithful commutative Banach algebra $\mathcal{A}$. If α and β are distinct eigenvalues of T and $a, b \in \mathcal{A}$ are corresponding eigenvectors, then $ab = 0$. This follows from $\alpha ab = (Ta)b = a(Tb) = \beta ab$. Similarly, if α is an eigenvalue of T, $\beta \neq \alpha$ is an eigenvalue of T^*, and $a \in \mathcal{A}$, $\xi \in \mathcal{A}^*$ are corresponding eigenvectors, then $a \cdot \xi = 0$. Namely, it is easily seen that $(Ta) \cdot \xi = a \cdot (T^*\xi)$ holds for all $a \in \mathcal{A}$ and $\xi \in \mathcal{A}^*$. Thus, the assertion follows from the equality $\alpha a \cdot \xi = (Ta) \cdot \xi = a \cdot (T^*\xi) = \beta a \cdot \xi$. Also, a similar assertion holds if T^* is replaced by multiplication operator L_T on $\mathcal{B}$, where $\mathcal{B} \subseteq B(\mathcal{A})$ is a closed subspace

that contains $L(\mathcal{A})$ and is invariant for multiplication by T (i.e. $TB \in \mathcal{B}$, for any $B \in \mathcal{B}$).

Proposition 2.3. *Let $T \in M(\mathcal{A})$. Let $U, V \subseteq \mathbb{C}$ be open sets and let $f : U \to \mathcal{A}$ and $g : V \to \mathcal{A}$ be analytic functions such that $(T - \lambda)f(\lambda) = 0$ $(\lambda \in U)$ and $(T-\mu)g(\mu) = 0$ $(\mu \in V)$. Then $f^{(m)}(\lambda)g^{(n)}(\mu) = 0$, for all $\lambda \in U$ and $\mu \in V$, where $f^{(m)}$ and $g^{(n)}$ are derivatives of f and g of orders $m \geq 0$ and $n \geq 0$, respectively.*

Proof. If $\lambda \in U$ and $\mu \in V$ are distinct, then $f(\lambda)g(\mu) = 0$, by the observation above. Since f and g are continuous we conclude that $f(\lambda)g(\lambda) = 0$ holds as well (for $\lambda \in U \cap V$). By double induction, $f^{(m)}(\lambda)g^{(n)}(\mu) = 0$. □

The following corollary was obtained, with a different proof, as Proposition 4.2.1 in [3].

Corollary 2.4. *Any multiplier on a semi-prime commutative Banach algebra has* SVEP.

Proof. Let $T \in M(\mathcal{A})$. If $U \subseteq \mathbb{C}$ is a non-empty subset and $f : U \to \mathcal{A}$ is an analytic function such that $(T-\lambda)f(\lambda) = 0$, for all $\lambda \in U$, then $f(\lambda)^2 = 0$ $(\lambda \in U)$, by Proposition 2.3. It follows that $f \equiv 0$ on U, by semi-primeness. □

Remark 2.5. Let T be a multiplier on $\mathcal{A}$ and let $T^* \in B(\mathcal{A})$ be its adjoint operator. If $U, V \subseteq \mathbb{C}$ are non-empty open sets and $f : U \to \mathcal{A}$, $G : V \to \mathcal{A}^*$ are analytic functions such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$, and $(T^* - \mu)G(\mu) = 0$, for all $\mu \in V$, then $f^{(m)}(\lambda) \cdot G^{(n)}(\mu) = 0$, for all $\lambda \in U$, $\mu \in V$ and all non-negative integers m and n. The proof of this assertion is a simple modification of the proof of Proposition 2.3. A similar assertion holds if T^* is replaced by L_T, where L_T is the multiplication operator induced by T on $\mathcal{B}$ (a closed subspace of $B(\mathcal{A})$ that contains $L(\mathcal{A})$ and is invariant for multiplication by T).

Let $\mathcal{B} \subseteq \mathcal{A}$ be a subalgebra. We shall say that the multiplication on $\mathcal{B}$ is trivial if $ab = 0$, for all $a, b \in \mathcal{B}$.

Proposition 2.6. *If each subalgebra $\mathcal{B}$ of $\mathcal{A}$ on which the multiplication is trivial is of finite dimension or of finite co-dimension, then every multiplier on $\mathcal{A}$ has* SVEP.

Proof. Assume towards a contradiction that there is a multiplier $T \in M(\mathcal{A})$ without SVEP. Then there exist a non-empty open set $U \subseteq \mathbb{C}$ and a non-zero analytic function $f : U \to \mathcal{A}$ such that $(T - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. There is no loss of generality if we assume that $U = \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$. Thus, there exist elements $a_n \in \mathcal{A}$, not all zero, such that

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \qquad (\lambda \in U).$$

Of course, we may assume that $a_0 \neq 0$. Since $Tf(\lambda) = \lambda f(\lambda)$, for all $\lambda \in U$, we get, by comparing the coefficients, $Ta_0 = 0$ and $Ta_n = a_{n-1}$, for $n \geq 1$. It follows that

$$a_n a_m = \big(T^{m+1} a_{n+m+1}\big) a_m = a_{n+m+1}\big(T^{m+1} a_m\big) = 0,$$

for all non-negative integers m and n. Denote by $\mathcal{B}$ the subalgebra of $\mathcal{A}$ that is generated by $\{a_n;\ n \geq 0\}$. It is obvious that the multiplication on $\mathcal{B}$ is trivial. For every $n \geq 0$, the set $\{a_0, \ldots, a_n\}$ is linearly independent. Namely, assume that this is not the case. Then there exists a minimal $n \geq 1$ such that $a_n =$

$\alpha_0 a_0 + \cdots + \alpha_{n-1} a_{n-1}$, for some complex numbers $\alpha_0, \ldots, \alpha_{n-1}$. However, then we also have $a_{n-1} = Ta_n = \alpha_0 Ta_0 + \cdots + \alpha_{n-1} Ta_{n-1} = \alpha_1 a_0 + \cdots + \alpha_{n-1} a_{n-2}$, which is impossible, by the minimality of n. We conclude that $\mathcal{B}$ is not of finite dimension.

Since f is a non-zero analytic function on U there exists $\xi \in \mathcal{A}^*$ such that $F(\lambda) := f(\lambda) \cdot \xi$ is a non-zero analytic function on U with values in $\mathcal{A}^*$. Of course, $(T^* - \lambda)F(\lambda) = 0$, for all $\lambda \in U$. Let $\xi_n := a_n \cdot \xi$ $(n \geq 0)$. Then $T^*\xi_n = \xi_{n-1}$, for $n \geq 1$, and $T^*\xi_0 = 0$. A similar argument as before shows that the set $\{\xi_n;\ n \geq k\}$ is linearly independent, where k is the smallest non-negative integer such that $\xi_k \neq 0$. Since $\langle \xi_n, b\rangle = \langle \xi, a_n b\rangle = 0$, for all $b \in \mathcal{B}$ and all $n \geq k$, we conclude that $\mathcal{B}^\perp := \{\eta \in \mathcal{A}^*;\ \langle \eta, b\rangle = 0,\ \text{for all } b \in \mathcal{B}\}$ is not of finite dimension. This is in contradiction with the assumption that $\mathcal{B}$ or $\mathcal{B}^\perp$ is of finite dimension. □

Remark 2.7. By simple modifications of some proofs in this section it is possible to show that $T \in M(\mathcal{A})$ has SVEP if each subalgebra of $M(\mathcal{A})$ with trivial multiplication is of finite dimension or finite co-dimension. Also, if each submodule of $\mathcal{A}^*$ with trivial multiplication has finite dimension or co-dimension, every multiplier on $\mathcal{A}$ has SVEP.

3. A counterexample

In this section we shall show that there exist a unital commutative Banach algebra and a multiplication operator without SVEP on it. This answers the question 6.2.1 in [3] in the negative.

Let $\mathcal{A}$ be a Banach algebra (not necessarily commutative) and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule.

Proposition 3.1. *The direct sum $\mathcal{A} \oplus \mathcal{X}$ is a Banach algebra through the norm and multiplication that are given by*

$$\|a \oplus x\|_1 = \|a\| + \|x\|, \qquad (a \oplus x)(b \oplus y) = ab \oplus (a \cdot y + x \cdot b) \qquad (a \oplus x,\ b \oplus y \in \mathcal{A} \oplus \mathcal{X}).$$

If $\mathcal{A}$ has unit, say e, then $e \oplus 0$ is unit for $\mathcal{A} \oplus \mathcal{X}$.

Proof. Straightforward; see the proof of Theorem 5 in [4]. □

Corollary 3.2. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{A} \subset B(\mathcal{X})$ be a closed commutative subalgebra; then $\mathcal{A} \oplus \mathcal{X}$ is a commutative Banach algebra for the multiplication*

$$(A \oplus x)(B \oplus y) = AB \oplus (Ay + Bx) \qquad (A \oplus x,\ B \oplus y \in \mathcal{A} \oplus \mathcal{X}).$$

If $\mathcal{A}$ contains the identity operator I, then $\mathcal{A} \oplus \mathcal{X}$ is unital with unit $I \oplus 0$.

Proof. If $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{X}$ is a left Banach $\mathcal{A}$-module, then $\mathcal{X}$ can be considered as a Banach $\mathcal{A}$-bimodule if the right module structure on $\mathcal{X}$ is given by $x \cdot a := a \cdot x$. □

The following proposition may be deduced from the results in [1]. However, we shall include a more transparent and direct proof of it.

Proposition 3.3. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{A} \subset B(\mathcal{X})$ be a closed commutative subalgebra that contains I. Then, for an operator $S \in \mathcal{A}$, the following assertions are equivalent:*

(a) S has SVEP*;*

(b) for every $x \in \mathcal{X}$, the multiplication operator $L_{S \oplus x}$ on $\mathcal{A} \oplus \mathcal{X}$ has SVEP*;*

(c) there exists $x \in \mathcal{X}$ such that $L_{S \oplus x}$ has SVEP.

Proof. Assume that there exists $x \in \mathcal{X}$ such that $L_{S\oplus x}$ is without SVEP. Let $U \subseteq \mathbb{C}$ be a non-empty open set and $F : U \to \mathcal{A} \oplus \mathcal{X}$ a non-zero analytic function such that $(L_{S\oplus x} - \lambda)F(\lambda) = 0$, for $\lambda \in U$. Of course, $F(\lambda) = \varphi(\lambda) \oplus f(\lambda)$, for analytic functions $\varphi : U \to \mathcal{A}$ and $f : U \to \mathcal{X}$. It follows from $(L_{S\oplus x} - \lambda)F(\lambda) = 0$ that $(S - \lambda)\varphi(\lambda) \oplus (\varphi(\lambda)x + (S - \lambda)f(\lambda)) = 0$ and therefore

$$(L_S - \lambda)\varphi(\lambda) = 0 \qquad \text{and} \qquad \varphi(\lambda)x + (S - \lambda)f(\lambda) = 0 \qquad (\lambda \in U),$$

where L_S is the multiplication operator induced by S on $\mathcal{A}$. If φ is non-zero, then L_S does not have SVEP and consequently S does not have SVEP. On the other hand, if $\varphi \equiv 0$ on U, then $(S - \lambda)f(\lambda) = 0$, for $\lambda \in U$, and f is a non-zero function. Thus, again, S does not have SVEP. This proves the implication (a)$\Rightarrow$(b). The implication (b)$\Rightarrow$(c) is obvious.

Assume that $S \in \mathcal{A}$ does not have SVEP and let $x \in \mathcal{X}$ be an arbitrary vector. Then there exists a non-empty open subset $U \subseteq \mathbb{C}$ and a non-zero analytic function $f : U \to \mathcal{X}$ such that $(S - \lambda)f(\lambda) = 0$, for all $\lambda \in U$. Define $F(\lambda) = 0 \oplus f(\lambda)$, for $\lambda \in U$. Then $F : U \to \mathcal{A} \oplus \mathcal{X}$ is a non-zero analytic function such that $(L_{S\oplus x} - \lambda)F(\lambda) = 0$, for all $\lambda \in U$. Thus (a) follows from (c). $\square$

Let S be an operator without SVEP on a Banach space $\mathcal{X}$ and let $\mathcal{A} \subseteq B(\mathcal{X})$ be the closed subalgebra that is generated by S and I. Then $\mathcal{A} \oplus \mathcal{X}$ is a faithful commutative Banach algebra such that the multiplication operator $L_{S\oplus 0}$ on $\mathcal{A} \oplus \mathcal{X}$ does not have SVEP, by Proposition 3.3.

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