ON THE CHARACTERISTIC POLYNOMIAL
OF THE ALMOST MATHIEU OPERATOR

MICHAEL P. LAMOUREUX AND JAMES A. MINGO

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ABSTRACT. Let $A_\theta$ be the rotation C*-algebra for angle $\theta$. For $\theta = p/q$ with $p$ and $q$ relatively prime, $A_\theta$ is the sub-C*-algebra of $M_q(C(T^2))$ generated by a pair of unitaries $u$ and $v$ satisfying $uv = e^{2\pi i \theta} vu$. Let $h_{\theta, \lambda} = u + u^{-1} + \lambda/2(v + v^{-1})$ be the almost Mathieu operator. By proving an identity of rational functions we show that for $q$ even, the constant term in the characteristic polynomial of $h_{\theta, \lambda}$ is $(-1)^{q/2}(1 + (\lambda/2)^q) - (z^q + z_1^{-q} + (\lambda/2)^q(z^q + z_2^{-q}))$.

1. Introduction

Let $\theta$, $\lambda$, and $\psi$ be real numbers with $\lambda$ positive. The second order difference operator $H_{\theta, \lambda, \psi}$ on $\ell^2(\mathbb{Z})$ given by

$$H_{\theta, \lambda, \psi}(\xi)(n) = \xi(n + 1) + \xi(n - 1) + \lambda \cos(2\pi n \theta + \psi)\xi(n)$$

for $\xi \in \ell^2(\mathbb{Z})$ is called the almost Mathieu operator. $H_{\theta, \lambda, \psi}$ is a discrete Schrödinger operator which models an electron moving in a crystal lattice in a plane perpendicular to a magnetic field.

An object of much study has been the spectrum $\sigma(\theta, \lambda) = \bigcup_{\psi} \sigma(H_{\theta, \lambda, \psi})$. In [H], Hofstadter calculated $\sigma(\theta, 2)$ for $\theta = p/q$ and $1 \leq p < q \leq 50$. The remarkable pattern he found is called Hofstadter’s butterfly. For irrational $\theta$, a longstanding concern has been the connectedness and Lebesgue measure of $\sigma(\theta, \lambda)$ and the labelling of the gaps, about which quite a bit is now known (see [AJ], [AK], and [P] for spectacular recent advances as well as [AVMS], [BS], [B], [CEY], [LT] for earlier work). In addition there has been numerical work on computing the spectrum to high accuracy for large $q$.

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Let $A_\theta$ be the rotation $C^*$-algebra (see [B]). For $\theta = p/q$ with $p$ and $q$ relatively prime and $\rho = e^{2\pi i \theta}$ let

$$u_\theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & \ddots & 1 \end{pmatrix} \quad \text{and} \quad v_\theta = \begin{pmatrix} \rho & \rho^2 & \ddots \\ \rho & \rho^2 & \ddots \\ \rho^{q-1} & \ddots & \ddots & \ddots \end{pmatrix},$$

i.e. $u_\theta$ cyclically permutes the elements of the standard basis and $v_\theta$ is a diagonal operator. Then define $u, v : \mathbb{T}^2 \to M_q(\mathbb{C})$ by $u(z_1, z_2) = z_1 u_\theta$ and $v(z_1, z_2) = z_2 v_\theta$. Then $u v = \rho v u$ and $A_\theta$ is the $C^*$-algebra generated by $u$ and $v$ (see [B]). The operator $h_{\theta, \lambda} = u + u^{-1} + \lambda/2(v + v^{-1})$ contains all the spectral information of $H_{\theta, \lambda, \psi}$ in that $\text{Sp}(h_{\theta, \lambda}) = \sigma_{\theta, \lambda} := \bigcup \text{Sp}(H_{\theta, \lambda, \psi})$.

The main tool in the analysis of $\sigma_{\theta, \lambda}$ is $\Delta_{\theta, \lambda}$, the discrete analogue of the discriminant. For $\theta = p/q$, $\Delta_{\theta, \lambda}(x) = \text{Tr}(A_1(x) \cdots A_q(x))$ where

$$A_k(x) = \begin{pmatrix} x - \lambda \cos(2\pi k p/q + \pi/(2q)) & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Below are the first few values of this polynomial. Note that the form of $\Delta_{\theta, \lambda}$ so displayed depends only on the denominator $q$; however, $\xi_0 = 2 \cos(2\pi p/q)$ depends on the numerator $p$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\Delta_{\theta, \lambda}(x)$ for $\theta = p/q$ and $\xi_0 = 2 \cos(2\pi \theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$x^2 - 4$</td>
</tr>
<tr>
<td>3</td>
<td>$x^3 - 6x$</td>
</tr>
<tr>
<td>4</td>
<td>$x^4 - 8x^2 + 4$</td>
</tr>
<tr>
<td>5</td>
<td>$x^5 - 10x^3 + 5(3 - \xi_0)x$</td>
</tr>
<tr>
<td>6</td>
<td>$x^6 - 12x^4 + 6(5 - \xi_0)x^2 - 4$</td>
</tr>
<tr>
<td>7</td>
<td>$x^7 - 14x^5 + 7(7 - \xi_0)x^3 - 7(6 - 2\xi_0 + 2\xi_{2\theta})x$</td>
</tr>
<tr>
<td>8</td>
<td>$x^8 - 16x^6 + 8(9 - \xi_0)x^4 - 8(12 - 4\xi_0 + 2\xi_{2\theta})x^2 + 4$</td>
</tr>
<tr>
<td>9</td>
<td>$x^9 - 18x^7 + 9(11 - \xi_0)x^5 - 9(31/3 - 6\xi_0 + 2\xi_{2\theta})x^3 + 9(14 - 8\xi_0 + 3\xi_{2\theta})x$</td>
</tr>
</tbody>
</table>

One can calculate for $k = 1, 2$ the coefficient of $x^{q-2k}$; for $k = 3$ the formula is conjectural (from numerical evidence). A deeper understanding of the structure of $\Delta_{\theta, \lambda}$ would be quite interesting.

<table>
<thead>
<tr>
<th>$k$</th>
<th>coefficient of $x^{q-2k}$ in $\Delta_{\theta, \lambda}$ ($\mu = \lambda/2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-q(1 + \mu^2)$</td>
</tr>
<tr>
<td>2</td>
<td>$q \left( \frac{1}{2}(q^{-3})\mu^4 + (q - 4 - \xi_0)\mu^2 + \frac{1}{q} \right) \left( \frac{q^{-3}}{2} \right)$</td>
</tr>
<tr>
<td>3</td>
<td>$-q \left( \frac{1}{4} \left( \frac{q^{-3}}{3} \right) \mu^6 + (1 + \left( q^{-3} \right)) - (q - 6)\xi_0 + 2\xi_{2\theta} \right) \mu^4$</td>
</tr>
<tr>
<td></td>
<td>$+ (1 + \left( q^{-3} \right)) - (q - 6)\xi_0 + 2\xi_{2\theta} \mu^2 + \frac{1}{q} \left( \frac{q^{-3}}{3} \right)$</td>
</tr>
</tbody>
</table>

The connection with the characteristic polynomial of $h_{\theta, \lambda}$ is given by

$$\det(x I_q - h_{\theta, \lambda}(z_1, z_2)) = \Delta_{\theta, \lambda}(x) + z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q})$$
and thus $\sigma_{\theta, \lambda} = \Delta_{p, \lambda}^{-1}[-2(1+(\lambda/2)^q), 2(1+(\lambda/2)^q)]$. Indeed, $\Delta_{p, \lambda}(x)$ can be written as a determinant (cf. Toda [T, §4])

$$(2) \quad \Delta_{p/q, \lambda}(x) = \det \begin{pmatrix} \alpha_1 & 1 & 1 \\ 1 & \alpha_2 & 1 \\ & & \ddots & \ddots \\ 1 & & & \alpha_q \end{pmatrix} + 2\left\{ (-1)^q + (\lambda/2)^q \right\}$$

where all the other entries are 0 and $\alpha_k = x - \lambda \cos(2\pi kp/q + \pi/(2q))$. Since

$$(3) \quad \Delta_{p/q, \lambda}(-x) = (-1)^q \Delta_{p/q, \lambda}(x)$$

the coefficient of $x^{q-(2k+1)}$ is 0 for $0 \leq k < q/2$.

The main result of the paper asserts that for $a_1 = 2\cos(2\pi lp/q)$ and $1 \leq k < q/2$ we have

$$\sum_{i_1, i_2, \ldots, i_{q-2k}} a_{i_1} a_{i_2} \cdots a_{i_{q-2k}} = 0$$

where the summation is over all subsets of $\{1, 2, 3, \ldots, q\}$ obtained by deleting $k$ pairs of adjacent elements - counting 1 and $k$ as adjacent. This is proved by establishing the following identity for $k \geq 3$ and $q \geq 2k - 1$:

$$\sum_{i_1=1}^{q-2(k-1)} \cdots \sum_{i_{k-1}+2}^{q} \prod_{j=1}^{k} \frac{(x^{-i_j} + x^{i_j})^{-1}}{(x^{-i_j-1} + x^{i_j+1})} = \frac{\left( x^{-q} - x^q \right)^{2k-2} \prod_{i=k+1}^{k-2} \left( x^{-q+i} - x^{q-i} \right)}{\prod_{i=1}^{k} \left( x^{-2i} - x^{2i} \right) \prod_{i=-1}^{k-2} \left( x^{-q+i} + x^{q-i} \right)}$$

$$+ \frac{(x^{-1} + x^1)^{-1}(x^{-q} + x^q)^{-1}}{(x^{-2} + x^2)(x^{-q+1} + x^{q+1})} \sum_{i_1=3}^{q-2(k-2)} \cdots \sum_{i_{k-1}+2}^{q-2} \prod_{j=1}^{k-2} \frac{(x^{-i_j} + x^{i_j})^{-1}}{(x^{-i_1-1} + x^{i_1+1})}.$$

We then use this to show that for $a_1 = 2\cos(2\pi lp/q)$

$$\det \begin{pmatrix} a_1 & 1 & 1 \\ 1 & a_2 & \ddots \\ & \ddots & \ddots \\ 1 & & & 1 \end{pmatrix} = \begin{cases} 0 & q \equiv 0 \pmod{4}, \\ 4 & q \equiv 1, 3 \pmod{4}, \\ -8 & q \equiv 2 \pmod{4}. \end{cases}$$

From this we show that the constant term (i.e. the coefficient of $x^0$) in

$$\det(xI_q - h_{\theta, \lambda}(z_1, z_2))$$

is

$$(-1)^{q/2}2(1 + (\lambda/2)^q) - (z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$$

when $q$ is even. When $q$ is odd it follows from (3) that the coefficient of $x^0$ is $-(-z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q})).$

Similar, though simpler, reasoning shows that the coefficient of $x^{q-2}$ is $-q(1 + \lambda/2)$ and that the coefficient of $x^{q-4}$ is

$$(\lambda/2)^4 q(q-3)/2 + (\lambda/2)^2 q(q-4 - 2\cos(2\pi\theta)) + q(q-3)/2.$$
Let us use the following notation: let $a_1, \ldots, a_n$ be elements of a commutative ring and let

\[
(a_1, a_2, \ldots, a_n) = \begin{bmatrix}
  a_1 & 1 & 0 \\
  1 & a_2 & 1 \\
  & \ddots & \ddots \\
  & & a_3 & \ddots \\
  & & & \ddots & 1 \\
  0 & & & & 1 & a_n
\end{bmatrix}
\]

and

\[
[a_1, a_2, \ldots, a_n] = \begin{bmatrix}
  a_1 & 1 & 1 \\
  1 & a_2 & 1 \\
  & \ddots & \ddots \\
  & & a_3 & \ddots \\
  & & & \ddots & 1 \\
  1 & 1 & a_n
\end{bmatrix}
\]

The first matrix is a tridiagonal matrix with 1’s on the sub and super-diagonal and 0’s elsewhere. The second matrix is the same tridiagonal matrix with in addition 1’s in the upper right and lower left corners; all other entries are 0. Expanding along the bottom row we have

\[
[a_1, a_2, \ldots, a_n] = (a_1, a_2, \ldots, a_n) - (a_2, a_3, \ldots, a_{n-1}) + 2(-1)^{n-1}
\]

and

\[
[-a_1, -a_2, \ldots, -a_n] = (-1)^n[a_1, a_2, \ldots, a_n] + 2(-1)^{n-1}.
\]

Rewriting equation (2) we have

\[
\Delta_{\lambda/2, \lambda}(x) = [a_1, \ldots, a_q] + 2((-1)^q + (\lambda/2)^q).
\]

**Notation**

(i) For $0 \leq k \leq n/2$, let $S^{[n]}_k = \{I \subset \{1, 2, \ldots, n\} \mid |I| = n - 2k\}$ and $I$ is obtained from $\{1, 2, \ldots, n\}$ by deleting $k$ pairs of adjacent elements. $S^{[2k]}_k = \emptyset$, $S^{[2k+1]}_k = \{\{1\}, \{3\}, \{5\}, \ldots, \{2k + 1\}\}$, \ldots, $S^{[n]}_n = \{\{1, 2, 3, \ldots, n\}\}$.

(ii) For $0 \leq k \leq (n-1)/2$, let $S^{[n]}_k = \{I \subset \{2, 3, \ldots, n\} \mid |I| = n - 2k - 1\}$ and $I$ is obtained from $\{2, 3, \ldots, n\}$ by deleting $k$ pairs of adjacent elements. $S^{[2k+1]}_k = \emptyset$, $S^{[2k+2]}_k = \{\{2\}, \{4\}, \{6\}, \ldots, \{2k + 2\}\}$, \ldots, $S^{[n]}_n = \{2, 3, \ldots, n\}$.

(iii) For $\mathcal{S}$ a collection of subsets of $\{1, 2, \ldots, n - 1\}$ let $\mathcal{S} \cup \{n\} = \{I \cup \{n\} \mid I \in \mathcal{S}\}$.

(iv) For $0 \leq k \leq n/2$, let $\mathcal{S}^{[n]}_k = \{I \subset \{1, 2, 3, \ldots, n\} \mid |I| = n - 2k\}$ and $I$ is obtained from $\{1, 2, \ldots, n\}$ by deleting $k$ pairs of adjacent elements, counting $\{n, 1\}$ as an adjacent pair. $\mathcal{S}^{[2k]}_k = \emptyset$, $\mathcal{S}^{[2k+1]}_k = \{\{1\}, \{2\}, \ldots, \{n\}\}$, \ldots, $\mathcal{S}^{[n]}_n = \{1, 2, 3, \ldots, n\}$.

(v) For $a_1, a_2, a_3, \ldots, a_n$ elements of a commutative ring, and $I = \{i_1, i_2, i_3, \ldots, i_k\} \subset \{1, 2, 3, \ldots, n\}$, let $a_I = a_{i_1}a_{i_2}a_{i_3}\cdots a_{i_n}$. We shall adopt the convention that $a_{\emptyset} = 1$. 

Part (ii) of the next proposition goes back to Sylvester’s original paper on continuants [S]; part (iv) is a straightforward extension of this. For the reader’s convenience we present a proof.

Proposition 2.2.  

(i) Suppose $1 \leq k < n/2$; then $S_{[k]}^n = (S_{[k]}^{n-1} \cup \{n\}) \cup S_{[k-1]}^{n-2}$.

(ii) \[
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S_{[k]}^n} a_I = a_n \sum_{I \in S_{[n]}^{n-1}} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I \in S_{[k-1]}^{n-2}} a_I.
\]

(iii) $\tilde{S}_{[k]}^n = S_{[k]}^n \cup S_{[k-1]}^{n-1}$ for $1 \leq k \leq n/2$.

(iv) When $n$ is odd,

\[
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}_{[k]}^n} a_I = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S_{[k]}^n} a_I + \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I.
\]

When $n$ is even,

\[
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}_{[k]}^n} a_I = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S_{[k]}^n} a_I + \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I.
\]

Proof.  

(i) Let $I \in S_{[k]}^n$. If $n \notin I$, then $n-1 \notin I$ and so $I \in S_{[k-1]}^{n-2}$. Suppose $n \in I$. Let $K = \{1, 2, 3, \ldots, n\} \setminus I$ and $\bar{I} = I \setminus \{n\}$. Then $\bar{I} = \{1, 2, 3, \ldots, n-1\} \setminus K$; so $\bar{I} \in S_{[k-1]}^{n-1}$. Hence $I = I \cup \{n\} \in S_{[k]}^{n-1} \setminus \{n\}$.

(ii) Let us assume that $n = 2m$ is even. The same idea works for odd $n$, but the proof is slightly simpler. Observe

\[
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in S_{[k]}^n} a_I = \sum_{k=0}^{m} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-2}} a_I + (-1)^m
\]

\[
= \sum_{I \in S_{[0]}^n} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-2}} a_I + (-1)^m
\]

\[
= a_n \left\{ \sum_{I \in S_{[0]}^{n-1}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I \right\} + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in S_{[k]}^{n-2}} a_I + (-1)^m
\]

\[
= a_n \sum_{k=0}^{m-1} \sum_{I \in S_{[k]}^{n-1}} a_I - \sum_{k=0}^{m-1} \sum_{I \in S_{[k]}^{n-2}} a_I + (-1)^m
\]

\[
= a_n \sum_{k=0}^{[(n-1)/2]} (-1)^k \sum_{I \in S_{[k]}^{n-1}} a_I - \sum_{k=0}^{[(n-2)/2]} (-1)^k \sum_{I \in S_{[k]}^{n-2}} a_I.
\]

(iii) For $I \in \tilde{S}_{[k]}^n$ let $K_1 = \{1, 2, 3, \ldots, n\} \setminus I$ and $K_2 = \{2, 3, \ldots, n\} \setminus I$. Also, $\min\{i \mid i \in K_1\}$ is odd if and only if $I \in S_{[k]}^{n-1}$ and $\min\{i \mid i \in K_2\}$ is even if and only if $I \in S_{[k-1]}^{n-1}$. 

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(iv) Suppose \( n = 2m \). Then
\[
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I = \sum_{I \in \tilde{S}^{[n/k]}} a_I + \sum_{k=1}^{m-1} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I + (-1)^m \sum_{I \in \tilde{S}^{[n/k]}} a_I
\]
\[
= \left( \sum_{I \in \tilde{S}^{[n/k]}} a_I + \sum_{k=1}^{m} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I \right) + \sum_{k=1}^{m} (-1)^k \sum_{I \in \tilde{S}^{[n/k-1]}} a_I + (-1)^m
\]
\[
= \sum_{k=0}^{m} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I - \sum_{k=0}^{m-1} (-1)^k \sum_{I \in \tilde{S}^{[n/k-1]}} a_I + (-1)^m.
\]
The case of \( n \) odd is similar. \( \square \)

**Corollary 2.3.** Let \( a_1, a_2, \ldots, a_n \) be elements of a commutative ring.

(i) \( (a_1, \ldots, a_n) = \sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I \).

(ii) \( [a_1, \ldots, a_n] = \begin{cases} 
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I + 2, & n \text{ odd,} \\
\sum_{k=0}^{[n/2]} (-1)^k \sum_{I \in \tilde{S}^{[n/k]}} a_I - 2 + (-1)^{n/2}, & n \text{ even.}
\end{cases} \)

**Proof.**

(i) For \( n = 1 \) the left hand side and the right hand side equal \( a_1 \). Both sides satisfy the same recurrence relation.

(ii) By equation (i)
\[
[a_1, \ldots, a_n] = (a_1, \ldots, a_n) - (a_2, \ldots, a_{n-1}) - (-1)^n 2
\]
so the result now follows from (i) and Proposition 2.2 (iii). \( \square \)

**Proposition 2.4.** Let \( 1 \leq p < q \) be relatively prime, \( \theta = p/q \), and \( a_k = 2 \cos(2\pi k\theta) \).

Then
\[
a_1a_2\cdots a_q = \begin{cases} 
0 & q \equiv 0 \pmod{4}, \\
2 & q \equiv 1, 3 \pmod{4}, \\
-4 & q \equiv 2 \pmod{4}.
\end{cases}
\]

**Proof.** Let \( T_q \) be the \( q \)th Chebyshev polynomial of the first kind. The constant term of \( T_q \) is 0 for \( q \) odd and \((-1)^{q/2}\) for \( q \) even. The result now follows from the identity (see e.g. [R §1.2])
\[
\prod_{i=1}^{q} (x - a_i) = 2(T_q(x/2) - 1).
\]

The statement of the main theorem follows. Its proof will be given at the end of the next section.
Theorem 2.5. Let $1 \leq p < q$ be relatively prime, $a_k = 2\cos(2\pi k\theta)$, and $\theta = p/q$. For $1 \leq k < q/2$, 
$$
\sum_{I \in \mathcal{S}[q]} a_I = 0.
$$

Corollary 2.6. Let $1 \leq p < q$ be relatively prime, $\theta = p/q$, $\lambda > 0$, and $a_k = \lambda \cos(2\pi k\theta)$. Then

$$
[a_1, a_2, \ldots, a_q] = \begin{cases}
0 & q \equiv 0 \pmod{4}, \\
2(1 + (\lambda/2)^q) & q \equiv 1, 3 \pmod{4}, \\
-4(1 + (\lambda/2)^q) & q \equiv 2 \pmod{4}
\end{cases}
$$

and $\Delta_{\theta, \lambda}(0) = (-1)^{q/2}2(1 + (\lambda/2)^q)$ for $q$ even.

Proof. Suppose $q$ is even. By Theorem 2.5 all the terms of
$$
\sum_{k=0}^{[q/2]} (-1)^k \sum_{I \in \mathcal{S}[q]} a_I
$$
are zero except the terms for $k = 0$ and $k = q/2$. The term for $k = 0$ is $a_1a_2 \cdots a_q$. The term for $k = q/2$ is $(-1)^{q/2}$. Thus when $q = 4m$ we have by Proposition 2.4
$$
[a_1, a_2, \ldots, a_q] = a_1a_2 \cdots a_q - (-1)^q2 + (-1)^{q/2}2 = 0,
$$
and when $q = 4m + 2$,
$$
[a_1, a_2, \ldots, a_q] = a_1a_2 \cdots a_q - (-1)^q2 + (-1)^{q/2}2 = -4(1 + (\lambda/2)^q).
$$

To obtain the final claim we apply equation (10). \square

From the corollary and equation (10) we have the theorem which corrects an error in [CPI] p. 232.

Theorem 2.7. The coefficient of $x^0$ in $\det(xI_q - h_{\theta, \lambda}(z_1, z_2))$ is

$$
-(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q})) + (-1)^{q/2}2(1 + (\lambda/2)^q)
$$

when $q$ is even and $-(z_1^q + z_1^{-q} + (\lambda/2)^q(z_2^q + z_2^{-q}))$ when $q$ is odd.

3. Proof of the main theorem

Theorem 3.1. Suppose $a_1, a_2, \ldots, a_q$ are elements in a commutative ring and let $a_{q+1} = a_1$. For $I \subset \{1, 2, \ldots, q\}$, let $I^c = \{1, 2, \ldots, q\} \setminus I$ be the complement of $I$ in $\{1, 2, \ldots, q\}$. Then

$$
\sum_{I \in \mathcal{S}[q]} a_I = \sum_{i_1=1}^{q-2(k-1)} \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q} \prod_{j=1}^{k} a_{i_j} a_{i_j+1} - a_1a_2 \left[ \sum_{i_1=3}^{q-2(k-2)} \sum_{i_2=i_1+2}^{q-2(k-3)} \cdots \sum_{i_{k-2}=i_{k-3}+2}^{q-2} \prod_{j=1}^{k-2} a_{i_j} a_{i_j+1} \right] a_q a_{q+1}.
$$
Proof. Recall that elements of $S[q]$ are obtained by deleting $k$ adjacent pairs $\{i, i+1\}$ from $\{1, 2, \ldots, q\}$, counting $q$ and $1$ as adjacent. So if $I^c \in S[q]$, then $I = \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}$ with $1 \leq i_1, j_1 = i_1 + 1 < i_2, \ldots, j_k-1 = i_k-1 + 1 < i_k \leq q$ and either $j_k = i_k + 1$ if $i_k < q$ or $j_k = 1$ if $i_k = q$.

Now let $T'[q] = \{ \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\} | 1 \leq i_1, j_1 = i_1 + 1 < i_2, \ldots, j_k-1 = i_k-1 + 1 < i_k \leq q, j_k = i_k + 1 \}$. Define $\phi : \{1, 2, \ldots, q, q+1\} \rightarrow \{1, 2, \ldots, q\}$ by $\phi(q+1) = 1$ and $\phi(i) = i$ for $i \leq q$. Then $a_{\phi(I)} = a_I$ for $I \in T'[q]$.

If $I = \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}$ and $i_1 = 1$ and $i_k = q$, then $\phi(I)^c \not\in S[q]$ because $\phi(j_k) = \phi(i_1) = 1$ and the pairs must be disjoint. So let $T''[q] = \{ \{1, 2, i_1, j_1, \ldots, i_{k-1}, j_{k-1}, q, q+1\} | 3 \leq i_1, j_1 = i_1 + 1 < i_2, \ldots, i_k-1 \leq q - 2, j_k-1 = i_k-1 + 1 \}$. For $I \in T'[q] \setminus T''[q]$, $\phi(I)^c \in S[q]$ and $\phi : T'[q] \setminus T''[q] \rightarrow S[q]$ is a bijection. This with the identity $a_{\phi(I)} = a_I$ proves the theorem.

Lemma 3.2. (i) For $q \geq 1$,
\[
\sum_{i=1}^{q} (x^{-i} + x^{i})^{-1}(x^{-i-1} + x^{i+1})^{-1} = \frac{x^{-q} - x^{q}}{(x^{-2} - x^{2})(x^{-q-1} + x^{q+1})}.
\]

(ii) For $k \geq 1$
\[
\prod_{i=1}^{2k} (x^{-i} + x^{i})^{-1} = \prod_{i=1}^{2k} \frac{1}{(x^{-2i} - x^{2i})(x^{-2i-1} + x^{2i+1})}.
\]

Proof. (i) One checks directly that the formula holds when $q = 1$; then (i) follows by induction on $q$.

(ii) follows from the identity
\[
\frac{x^{-i} - x^{i}}{(x^{-2i} - x^{2i})(x^{-k-i} + x^{k+i})} = \frac{1}{(x^{-i} + x^{i})(x^{-k-i} + x^{k+i})}.
\]

Corollary 3.3. For $q \geq 5$
\[
\frac{(x^{-1} + x)^{-1}(x^{-2} + x^{2})^{-1}}{(x^{-q} + x^{q})(x^{-q-1} + x^{q+1})} \sum_{i=3}^{q-2} (x^{-i} + x^{i})^{-1}(x^{-i-1} + x^{i+1})^{-1}
\]
\[
= \frac{(x^{-3} - x^{3})(x^{-q+4} - x^{q-4})}{(x^{-4} - x^{4})(x^{-6} - x^{6})(x^{-q+1} + x^{q-1})(x^{-q} + x^{q})(x^{-q-1} + x^{q+1})}.
\]

Proof. By Lemma 3.2 (i)
\[
\sum_{i=3}^{q-2} (x^{-i} + x^{i})^{-1}(x^{-i-1} + x^{i+1})^{-1}
\]
\[
= \frac{x^{-q+2} - x^{q-2}}{(x^{-2} - x^{2})(x^{-q+1} + x^{q-1})} - \frac{x^{-2} - x^{2}}{(x^{-2} - x^{2})(x^{-3} + x^{3})}
\]
\[
= \frac{(x^{-q+4} - x^{q-4})(x^{-1} + x)}{(x^{-2} - x^{2})(x^{-3} + x^{3})(x^{-q+1} + x^{q-1})}.
\]

The result then follows by multiplying both sides by
\[
(x^{-1} + x)(x^{-2} + x^{2})(x^{-3} + x^{3})(x^{-q+1} + x^{q-1}).
\]
Theorem 3.4. For $k \geq 1$ and $q \geq 2k - 1$,

\[
\sum_{i_1 = 1}^{q-2(k-1)} \sum_{i_2 = i_1 + 2}^{q-2(k-2)} \cdots \sum_{i_k = i_{k-1} + 2}^{q-2(k-2)} \prod_{j=1}^{k} (x^{-i_j} + x^{i_j} - 1)(x^{-i_j} - 1 + x^{i_j + 1})^{-1}
\]

\[= \frac{\prod_{i=k-1}^{2k-2} (x^{-(q-i)} - x^{q-i})}{\prod_{i=1}^{k} (x^{-2i} - x^{2i}) \prod_{i=1}^{k-2} (x^{-(q-i)} + x^{q-i})}.\]

Proof. We prove the equation by induction on $k$. When $k = 1$ the equation holds by Lemma 3.2 (i). Lemma 3.2 (ii) shows that for arbitrary $k$ the formula holds for $q = 2k - 1$; so we fix $k$ and proceed by induction on $q$. Let $S_{k,q}$ and $T_{k,q}$ denote respectively the left hand and right hand sides of equation (7).

If we write $S_{k,q}$ as a sum of two terms, the first in which $i_k < q$ and the second when $i_k = q$, we see that $S_{k,q}$ satisfies the recurrence relation

\[S_{k,q} = S_{k,q-1} + (x^{-q} + x^q)^{-1}(x^{-q-1} + x^{q+1})^{-1} S_{k-1,q-2}.\]

Thus we have only to show that $T_{k,q}$ satisfies the same relation. Now

\[T_{k,q-1} = \frac{\prod_{i=k}^{2k-2} (x^{-(q-i)} - x^{q-i})}{\prod_{i=1}^{k} (x^{-2i} - x^{2i}) \prod_{i=1}^{k-1} (x^{-(q-i)} + x^{q-i})}\]

and

\[T_{k-1,q-2} = \frac{\prod_{i=k-1}^{2k-2} (x^{-(q-i)} - x^{q-i})}{\prod_{i=1}^{k-1} (x^{-2i} - x^{2i}) \prod_{i=1}^{k-2} (x^{-(q-i)} + x^{q-i})}.\]

The proof of the recurrence relation for $T_{k,q}$ is thus reduced to verifying that

\[
\frac{(x^{-(q-(k-1)}) - x^{q-(k-1)})(x^{-(q-(k-1))} + x^{q-(k-1)})}{(x^{-q-1} + x^{q+1})(x^{-q} + x^q)} = \frac{x^{-(q-(2k-1))} - x^{q-(2k-1)}}{x^{-q} + x^q} + \frac{x^{-2k} - x^{2k}}{(x^{-q} + x^q)(x^{-q-1} + x^{q+1})}.\]

\[= \frac{(x^{-(q-k+1)} - x^{k-1})(x^{-k} - x^k) \prod_{i=k+1}^{2k-2} (x^{-q+i} - x^{-i})}{\prod_{i=1}^{k} (x^{-2i} - x^{2i}) \prod_{i=1}^{k-2} (x^{-q+i} + x^{q-i})}.\]

Theorem 3.5. For $k \geq 3$ and $q \geq 2k - 1$,

\[\sum_{i_1 = 1}^{q-2(k-1)} \sum_{i_2 = i_1 + 2}^{q-2(k-3)} \cdots \sum_{i_k = i_{k-1} + 2}^{q-2(k-2)} \prod_{j=1}^{k-2} (x^{-i_j} + x^{i_j} - 1)(x^{-i_j} - 1 + x^{i_j + 1})^{-1}
\]

\[= \frac{(x^{-(q-k-1)} - x^{k-1})(x^{-k} - x^k) \prod_{i=k+1}^{2k-2} (x^{-q+i} - x^{-i})}{\prod_{i=1}^{k} (x^{-2i} - x^{2i}) \prod_{i=1}^{k-2} (x^{-q+i} + x^{q-i})}.\]

Proof. Let us denote the left and right hand sides of the identity by $S_{k,q}$ and $T_{k,q}$ respectively. By Corollary 3.3 $S_{3,q} = T_{3,q}$. We write $S_{k,q}$ as the sum of two terms: in the first $i_{k-2} < q - 2$ and in the second $i_{k-2} = q - 2$. As in the proof of the previous theorem we obtain a recurrence relation, in this case:

\[S_{k,q} = S_{k,q-1}(x^{-q-1} + x^{q+1})^{-1}(x^{-q+1} + x^{q-1}) + S_{k-1,q-2}(x^{-q} + x^q)^{-1}(x^{-q-1} + x^{q+1})^{-1}.\]

It is routine to verify that $T_{k,q}$ satisfies the same recurrence relation. \[\square\]
Corollary 3.6.

\[
\begin{align*}
\sum_{i_1=1}^{q-2(k-1)} & \sum_{i_2=i_1+2}^{q-2(k-2)} \cdots \sum_{i_k=i_{k-1}+2}^{q-2(k-2)} \prod_{j=1}^{k} (x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\
& - \prod_{j=1}^{k} (x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\
\times & \sum_{i_1=3}^{q-2-k} \sum_{i_2=i_1+2}^{q-2-k} \cdots \sum_{i_k-3=i_{k-2}+2}^{q-2-k} \prod_{j=1}^{k-2} (x^{-i_j} + x^{i_j})^{-1}(x^{-i_j-1} + x^{i_j+1})^{-1} \\
& = \frac{\prod_{i=1}^{2k-2} (x^{2i} - x^{-2i}) \prod_{i=1}^{k-2} (x^{-q+i} + x^{q-i})}{\prod_{k=1}^{q} (x^{2i} - x^{-2i})}.
\end{align*}
\]  

(9)

Proof of Theorem [2,4]. We recall that 1 ≤ p < q and p and q are relatively prime. We set θ = p/q and \(a_j = 2\cos(2\pi j\theta)\). We shall split the proof into two cases.

Case 1: \(q \equiv 0 \pmod{4}\). When \(q \equiv 0 \pmod{4}\) and \(a_j \neq 0\) for all j, moreover when \(x = e^{2\pi i\theta}\), \(x^{4j} \neq 1\) and \(x^{2(q-j)} \neq -1\) for all i. Thus the denominator on the right hand side of (9) does not vanish but the numerator does. Hence by Theorem [3,1]

\[
\sum_{I \in S[i]} (a_I)^{-1} = 0.
\]

Upon multiplying by \(a_1a_2 \cdots a_q\) we obtain that

\[
\sum_{I \in S[i]} a_I = a_1a_2 \cdots a_q \sum_{I \in S[i]} (a_I)^{-1} = 0.
\]

Case 2: \(q \equiv 0 \pmod{4}\). Again we wish to show that \(\sum_{I \in S[i]} a_I = 0\) and so we must multiply both sides of equation (9) by \(\prod_{i=1}^{q} (x^{-i} + x^i)\) and evaluate at \(x = e^{2\pi i\theta}\).

The denominator of the right hand side of (9) is zero when \(x^{4q} = 1\) or \(x^{2(q-j)} = -1\), i.e. when \(i = j = q/4\); the corresponding factors are \(x^{-q/2} - x^{q/2}\) and \(x^{-3q/4} + x^{3q/4}\) respectively.

Apart from the factor \(x^{-q} - x^q\), the numerator of the right hand side of equation (9) is zero only when \(x^{2(q-j)} = 1\), i.e. when \(i = q/2\). This produces the factor \(x^{-q/2} - x^{q/2}\) which cancels one of the zeros in the denominator. The other zero is cancelled when we multiply by \(\prod_{i=1}^{q} (x^{-i} + x^i)\). Hence the product of \(\prod_{i=1}^{q} (x^{-i} + x^i)\) and the right side of (9) is zero when \(x = e^{2\pi i\theta}\). □

References


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