A STRONG COMPARISON PRINCIPLE FOR THE $p$-LAPLACIAN

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ABSTRACT. We consider weak solutions of the differential inequality of $p$-Laplacian type

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v)$$

such that $u \leq v$ on a smooth bounded domain in $\mathbb{R}^N$ and either $u$ or $v$ is a weak solution of the corresponding Dirichlet problem with zero boundary condition. Assuming that $u < v$ on the boundary of the domain we prove that $u < v$, and assuming that $u \equiv v \equiv 0$ on the boundary of the domain we prove $u < v$ unless $u \equiv v$. The novelty is that the nonlinearity $f$ is allowed to change sign. In particular, the result holds for the model nonlinearity $f(s) = s^q - \lambda s^{p-1}$ with $q > p - 1$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Throughout this article $\Omega$ will be a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 2$. A function $w \in C^{1,\alpha}(\Omega)$ (see [6, 8, 12]) solves the equation

$$-\Delta_p w = f(w) \text{ weakly on } \Omega$$

(where $p > 1$ and $f$ is a continuous real function that is locally Lipschitz on its domain) if and only if

$$\int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \ dx = \int_{\Omega} f(w) \phi \ dx \quad \forall \ \phi \in W^{1,p}_0(\Omega).$$

In this paper we consider the following problem:

$$\begin{cases} -\Delta_p w = f(w) \text{ weakly on } \Omega, \\ w > 0 \text{ on } \Omega, \\ w = 0 \text{ on } \partial \Omega. \end{cases}$$

We restrict our attention to the case of positive solutions, and we recall that by the strong maximum principle for the $p$-Laplacian under quite general hypotheses on $f$ (see [10, 13]) any nonnegative solution is in fact strictly positive.

Two functions $u, v \in C^{1,\alpha}(\Omega)$ satisfy the inequality

$$-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \text{ weakly on } \Omega$$

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if and only if
\[ \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx - \int_{\Omega} f(u)\psi \, dx \leq \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx - \int_{\Omega} f(v)\psi \, dx \]
for every \( \psi \in W^{1,p}_0(\Omega) \) such that \( \psi \geq 0 \) a.e. Throughout this paper we will assume
\[
(A)_p \begin{cases} 
\text{both } u \text{ and } v \text{ are nonnegative on } \Omega, \\
\text{either } u \text{ or } v \text{ solves problem (1.2)}, \\
-\Delta_p u - f(u) \leq -\Delta_p v - f(v) \quad \text{weakly on } \Omega.
\end{cases}
\]

We say that a Strong Comparison Principle (SCP for short) holds for two functions \( u, v \in C^{1,\alpha}(\Omega) \) satisfying \((A)_p\) if from the inequalities
\[ u \leq v \quad \text{on } \Omega \]
we can infer the alternative
\[ u < v \quad \text{on } \Omega \quad \text{unless } u \equiv v \quad \text{on } \Omega. \]

We want to prove that, under suitable boundary conditions, such an SCP holds. The novelty of the paper is that \( f \) can be a sign changing nonlinearity. For example, our assumptions allow us to consider nonlinearities such as
\[ f(s) = s^q - \lambda s^{p-1} \quad (\text{with } q > p - 1). \]

Even when \( f \) has definite sign, it is well known that this is a hard task due to the nonlinear degenerate nature of the \( p \)-Laplace operator. In fact, comparison principles are not equivalent in this case to maximum principles as for the case of linear operators. We refer the readers to [10] and the references therein for an interesting overview on this topic, and we recall here some known results.

In [3] it is proved that, if \( f \) is locally Lipschitz, a Strong Comparison Principle holds in any connected component of \( \Omega \setminus Z_{u,v} \) where \( Z_{u,v} = \{ x \in \Omega | \nabla u(x) = 0 = \nabla v(x) \} \). In [7] it is proved that, if \( f \) is positive and nondecreasing, a Strong Comparison Principle holds assuming that \( u, v \) are both solutions of problem (1.2) or assuming as the boundary condition in (1.2) that \( u < v \) on \( \partial \Omega \). The results in [7] have been recently extended to a more general class of operators in [9], where also some interesting estimates on the set of possible touching points are proved. The assumptions of Theorem 1.3 in [9] are equivalent, in our context, to assuming that \( f \) is positive and nondecreasing. Also, we point out some interesting results in [1, 2] where the case of solutions of (1.2) is considered and a Strong Comparison Principle is proved for a particular class of problems involving nonlinearities that do not change sign.

Some details of our proofs are similar to the ones in [1, 2]. In particular, we point out that we will use a Divergence Theorem stated and proved in [2], together with some regularity results from [4]. The crucial tool anyway is a general result recently obtained in [5] where the case of positive nonlinearities is considered. Here we adapt Theorem 1.4 in [5] for future use.

\[ ^1 \]The nonlinearities considered in [1, 2] could change sign if the solutions \( u, v \) change sign. Anyway this does not occur since the authors show that the solutions are nonnegative.
We want to show that \( C \) on a domain \( \Omega \).

**Theorem 1.4.** Assume \( f \) fulfilling \( (\Omega') \subseteq \Omega \) (let us say \( f(u) = 0 \)) if \( f \) has a definite sign on \( \Omega \) (let us say \( f(u) > 0 \)); if \( u \leq \nu \) and \( u \neq \nu \) in \( \Omega \), then \( u < v \) in \( \Omega \).

The same result follows assuming that \( v \) is a solution of \( (\Omega') \) in \( \Omega \) and \( f(v) \) has a definite sign on \( \Omega \).

**Remark 1.2.** The restriction \( p > \frac{2N+2}{N+2} \) allows \( |\nabla u|^{p-2} \) to be in \( L^1(\Omega) \) (in \[3\] see Theorem 2.3). Lemma \[1.1\] follows from Theorem 1.4 in \[5\] by simple considerations. In Theorem 1.4 of \[5\] only the assumption \( f(u) > 0 \) is considered, however it is clear from its proof that the assumption \( f(u) < 0 \) is equivalent to the assumption \( f(u) > 0 \). The statement of Lemma \[1.1\] is a local version of Theorem 1.4 in \[5\] since it holds in any domain \( \Omega' \subseteq \Omega \). Looking at the proof of Theorem 1.4 in \[5\] this causes only that a local version of Theorem 2.1 in \[5\] (see also Theorem 1.1 in \[4\]) is needed. The latter can be found in \[11\].

The aim of this paper is to deal with sign changing nonlinearities. More precisely, we keep hypothesis \((A), (f_1), (f_2)\) without assuming that \( f(u) \) or \( f(v) \) has definite sign. We simply assume

\[
(f_3) \quad f(t) = \begin{cases} 
0 & \text{if } t = 0 \text{ or } t = k > 0, \\
< & \text{if } t \in (0,k), \\
> & \text{if } t \in (k,\infty),
\end{cases}
\]

\[
(f_4) \quad f \text{ is nondecreasing on some open interval } I_k \text{ containing } k.
\]

We prove the following

**Theorem 1.3.** Assume \( \frac{2N+2}{N+2} < p \leq 2 \) or \( p \geq 2 \). Let \( u, v \in C^{1,\alpha}(\Omega) \) satisfy \((A)_p\) with \( f \) fulfilling \((f_1), (f_2), (f_3), (f_4)\), and assume that \( u \leq v \) in \( \Omega \). Then, if \( u < v \) on \( \partial \Omega \), it follows

\[
u < v \text{ in } \Omega.
\]

**Theorem 1.4.** Assume \( \frac{2N+2}{N+2} < p \leq 2 \) or \( p \geq 2 \). Let \( u, v \in C^{1,\alpha} \) both satisfy \((A)_p\) with \( f \) fulfilling \((f_1), (f_2), (f_3), (f_4)\), and assume that \( u \leq v \) in \( \Omega \). Then, if \( u \equiv v \equiv 0 \) on \( \partial \Omega \), the following alternative holds:

\[
u < v \text{ in } \Omega \quad \text{or} \quad u \equiv v \text{ in } \Omega.
\]

2. **Proof of Theorem 1.3**

Let us consider the set where \( u \) and \( v \) possibly coincide:

\[
C_{u,v} = \{ x \in \Omega : u(x) = v(x) \}.
\]

We want to show that \( C_{u,v} = \emptyset \). By contradiction, we assume that the closed set \( C_{u,v} \) is not empty. This, under our hypothesis, equals \( \partial C_{u,v} \neq \emptyset \).
2.1. **Step 1.** We claim that at each \(x \in \partial C_{u,v}\) we have \(u(x) = k\). We already know that \(u \equiv v > 0\) on \(C_{u,v} \supset \partial C_{u,v}\) since either \(u\) or \(v\) is a solution of problem (1.2). Assume by contradiction that there exists some \(\bar{x} \in \partial C_{u,v}\) such that \(u(\bar{x}) \neq k\). By hypothesis (f3), we have \(f(u(\bar{x})) \neq 0\). Without loss of generality we can consider \(f(u(\bar{x})) > 0\), and as \(u\) is a solution of problem (1.2), in this case we can find an open ball \(B(\bar{x}, r_2)\) centered at \(\bar{x}\) such that \(f(u) > 0\) on \(B(\bar{x}, r_2)\). Since \(\bar{x} \in \partial C_{u,v}\), \(u\) cannot coincide with \(v\) on the whole \(B(\bar{x}, r_2)\), thus we can apply Lemma 1.1 getting \(u < v\) on \(B(\bar{x}, r_2)\), and this contradicts the hypothesis \(u(\bar{x}) = v(\bar{x})\).

2.2. **Step 2.** By assuming \(C_{u,v} \neq \emptyset\), the function \(\text{dist}(x, C_{u,v})\) is well defined at each \(x \in \Omega\) and we can consider the open set

\[
C_{u,v}^\varepsilon = \{x \in \Omega : \text{dist}(x, C_{u,v}) < \varepsilon\} \quad \text{(where } \varepsilon > 0\text{)}.
\]

Since \(u \equiv v \equiv k\) on \(\partial C_{u,v}\), we can claim that there exists a \(\bar{\varepsilon} > 0\) such that

\[
(2.1) \quad \forall x \in C_{u,v}^\varepsilon \setminus C_{u,v} \quad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k.
\]

On the contrary we would have that

\[
\forall \varepsilon > 0 \ \exists x_\varepsilon \in C_{u,v}^\varepsilon \setminus C_{u,v} \quad u(x_\varepsilon) \notin I_k \quad \text{or} \quad v(x_\varepsilon) \notin I_k.
\]

By choosing \(\varepsilon = \frac{1}{n}\) there would exist a sequence \((x_n)\) such that

\[
x_n \in C_{u,v}^\frac{1}{n} \setminus C_{u,v} \quad u(x_n) \notin I_k \quad \text{or} \quad v(x_n) \notin I_k.
\]

From this sequence we could extract a subsequence \((x_{n'})\) such that

\[
x_{n'} \in C_{u,v}^\frac{1}{n'} \setminus C_{u,v} \quad w(x_{n'}) \notin I_k
\]

where \(w\) would be either \(u\) or \(v\). As \(\Omega\) is bounded we could extract from \((x_{n'})\) a subsequence \((x_{n''})\) that would necessarily converge to some point \(z \in \partial C_{u,v}\) where \(w(z) = k\). But this would end the contradiction \(w(x_{n''}) \to k\) and \(w(x_{n''}) \notin I_k\).

2.3. **Step 3 [Contradiction].** By construction we have that \(u < v\) on \(\partial C_{u,v}^\varepsilon\). As \(\partial C_{u,v}^\varepsilon\) is compact, there exists some \(\rho > 0\) such that \(u + \rho < v\) on \(\partial C_{u,v}^\varepsilon\). Let us consider the function \(w_\rho : \Omega \to [0, +\infty)\) defined as follows:

\[
w_\rho = \begin{cases} 
(u + \rho - v)^+ & \text{on } C_{u,v}^\varepsilon, \\
0 & \text{on } \Omega \setminus C_{u,v}^\varepsilon.
\end{cases}
\]

Since \(u + \rho < v\) on \(\partial C_{u,v}^\varepsilon\), we have that \(w_\rho \in W^{1,p}_0(\Omega)\) and

\[
\nabla w_\rho = \begin{cases} 
\nabla u - \nabla v & \text{where } w_\rho > 0, \\
0 & \text{elsewhere}.
\end{cases}
\]
As \( w_\rho \) is a test function, we can use it in (1.1) obtaining:

\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w_\rho = \int_\Omega f(u)w_\rho \\
= \int_{C_{u,v}^\varepsilon} f(u)w_\rho \\
= \int_{C_{u,v}^\varepsilon \setminus C_{u,v}} f(u)w_\rho + \int_{C_{u,v}} f(u)w_\rho \\
\leq \int_{C_{u,v}^\varepsilon \setminus C_{u,v}} f(v)w_\rho + \int_{C_{u,v}} f(v)w_\rho \\
= \int_{C_{u,v}} f(v)w_\rho = \int_\Omega f(v)w_\rho \\
\langle \text{by (2.1) and (f4)} \rangle \\
\leq \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w_\rho
\]

that is,

\[
\int_\Omega \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla w_\rho \\
= \int_{\{w_\rho > 0\}} \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot (\nabla u - \nabla v) \leq 0.
\]

By recalling (see for example [3]) that there exists some positive constant \( C_p \) such that for each \( \eta, \eta' \in \mathbb{R}^N \)

\[
(|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta') \cdot (\eta - \eta') \geq C_p (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2,
\]

we get

\[
C_p \int_{\{w_\rho > 0\}} \left( |\nabla u| + |\nabla v| \right)^{p-2} |\nabla u - \nabla v|^2 \leq 0.
\]

This implies that \( u - v \) equals some constant on \( \{w_\rho > 0\} \), that is, \( w_\rho \) is a constant on \( \{w_\rho > 0\} \). By continuity of \( w_\rho \) this constant must be zero since \( w_\rho = 0 \) on \( \partial C_{u,v}^\varepsilon \). Thus, we have that \( w_\rho \equiv 0 \) in \( C_{u,v}^\varepsilon \), that is,

\[
u + \rho \leq v \quad \text{on } C_{u,v}^\varepsilon, \quad (\text{i.e. } u < v \quad \text{on } C_{u,v}^\varepsilon),
\]

and this contradicts the fact that \( C_{u,v}^\varepsilon \supset C_{u,v} \neq \emptyset \).

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\(^2\)We put comments between \( \langle \rangle \) brackets.
Lemma 1.1; therefore $u \setminus \Gamma$ and $C[5]$ it was proved that $|\nabla u| < k$ such that $0 < u < v$ on $\Omega$. Since $f(u) < 0$ on $V$, there the SCP holds by Lemma [1.1] therefore $u \equiv v$ on $V$ or $u < v$ on $V$. In the latter case we can find a set $\Gamma^c = \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \epsilon\}$ for a suitable $\epsilon > 0$ such that $u < v$ on $\partial \Gamma^c$; exploiting Theorem [1.3] we get $u < v$ on $\Gamma^c$, and therefore $u < v$ on $\Omega$. Thus, in the sequel we will consider the former case ($u \equiv v$ on $V$) and prove that $u$ must coincide with $v$ on $\Omega$. As in Theorem [1.3] we define $C_{u,v} = \{x \in \Omega : u(x) = v(x)\}$ and $C_{u,v}^c = \{x \in \Omega : \text{dist}(x, C_{u,v}) < \epsilon\}$. Let us assume by contradiction that there exists some $x_0 \in \Omega$ such that $u(x_0) < v(x_0)$. Arguing as in Section 2.2 of the proof of Theorem [1.3] we can always find an $\epsilon$ such that $0 < \epsilon < \text{dist}(x_0, \partial C_{u,v})$ and

$$\forall x \in C_{u,v}^c \setminus C_{u,v}, \quad u(x) \in I_k \quad \text{and} \quad v(x) \in I_k.$$ 

Let us observe that $(\Omega \setminus C_{u,v}) \cap C_{u,v}^c$ is a nonempty open set and $\partial C_{u,v}^c \setminus \partial \Omega \neq \emptyset$ by the assumption $0 < \epsilon < dist(x_0, \partial C_{u,v})$. Moreover at each $x \in \partial C_{u,v}^c \setminus \partial \Omega$ we have $u(x) < v(x)$. By compactness of $\partial C_{u,v}^c \setminus \partial \Omega$ and continuity of $u$ and $v$, there exists $\rho > 0$ such that $u + \rho < v$ on $\partial C_{u,v}^c \setminus \partial \Omega$. Let us define

$$w_\rho = \begin{cases} (u + \rho - v)^+ & \text{on } C_{u,v}^c, \\ 0 & \text{on } \Omega \setminus C_{u,v}. \end{cases}$$

We have that $w_\rho \in W^{1,p}(\Omega)$ and

$$\nabla w_\rho = \begin{cases} \nabla u - \nabla v & \text{where } w_\rho > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Let us observe that $\nabla w_\rho = \nabla u - \nabla v = 0$ on $\nabla$. This allows us to use $w_\rho$ “as a test function” even if $w_\rho \not\in W^{1,p}_0(\Omega)$; in fact, we will see that the boundary terms appearing in the Divergence Theorem for $u$ and $v$ coincide.

As pointed out in [5], a $C^1$ solution of (1.2), with $f$ as in our hypothesis, belongs to the class $C^2(\Omega \setminus Z)$, where $Z = \{x \in \Omega : \nabla u(x) = 0\}$; therefore the generalized derivatives of $|\nabla u|^{p-2}u_xi$ coincide with the classical ones on $\Omega \setminus Z$. Moreover in [5] it was proved that $|\nabla u|^{p-2}u_xi \in W^{1,2}(\Omega)$. Since $w_\rho \in W^{1,2}(\Omega)$ we have that $\text{div}(w_\rho|\nabla u|^{p-2}\nabla u) = L^1$. The vector field $w_\rho|\nabla u|^{p-2}\nabla u$ belongs to $[C^0(\Omega)]^N$, so we can apply the Divergence Theorem as stated in [2] pag.742, obtaining

$$\int_{\Omega} \text{div}(w_\rho|\nabla u|^{p-2}\nabla u) \, dx = \int_{\partial \Omega} w_\rho|\nabla u|^{p-2}\frac{\partial u}{\partial v} \, d\sigma.$$
Since $\text{div}(w_\rho|\nabla u|^{p-2}\nabla u) = w_\rho\text{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^{p-2}\nabla u \cdot \nabla w_\rho$ and also $-\text{div}(|\nabla u|^{p-2}\nabla u) = f(u)$ almost everywhere, we obtain (exploiting as in Theorem 1.3)

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_\rho \, dx = \int_\Omega f(u)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \, d\sigma$$

$$= \int_\Omega f(u)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma$$

$$= \int_{C_{u,v} \cap \Omega} f(u)w_\rho \, dx + \int_{C_{u,v} \setminus \Omega} f(u)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma$$

$$= \int_{C_{u,v} \cap \Omega} f(v)w_\rho \, dx + \int_{C_{u,v} \setminus \Omega} f(u)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma$$

$$\leq \int_{C_{u,v} \cap \Omega} f(v)w_\rho \, dx + \int_{C_{u,v} \setminus \Omega} f(v)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma$$

$$= \int_\Omega f(v)w_\rho \, dx + \int_{\partial \Omega} w_\rho|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} \, d\sigma$$

$$= (\ast) \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w_\rho \, dx.$$ 

Arguing as in Theorem 1.3, we conclude the contradiction $w_\rho = 0$ (that is, $u + \rho \leq v$ in $C_{u,v} \supset C_{u,v} \neq \emptyset$).

**Remark 3.1.** Further extensions are possible. For example, one may guess that in Theorem 1.4 the thesis is still valid by assuming that $u, v \in C^{1,\alpha}$ simply satisfy $(A)_p$, instead of both being solutions of (1.2). This is actually true if the function that is not a solution of (1.2) (let us say $v$) shares the same regularity as the solution $u$. In such a case the Divergence Theorem can still be applied to $v$ giving, with $(A)_p$, the inequality $\leq$ instead of the equality at the final step $(\ast)$. However, we skipped such a statement because here shortness and simplicity is our aim.

**References**


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