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RATE OF L^2-CONCENTRATION OF BLOW-UP SOLUTIONS FOR CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. This paper concerns the rate of L^2-concentration of the blow-up solutions for the critical nonlinear Schrödinger equation. The result of Tsutsumi is improved in terms of Merle and Raphaël's recent arguments.

1. Introduction

This paper is concerned with the Cauchy problem of the following nonlinear Schrödinger equation:

$$iu_t + \triangle u + |u|^{4/N}u = 0, \qquad t \geq 0, x \in \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $u = u(t,x)$: $[0,T) \times R^N \to C$ and $0 < T \leq \infty$; $i = \sqrt{-1}$; $\triangle$ is the Laplace operator on $\mathbb{R}^N$. If we replace the nonlinear term by $|u|^{p-1}u$, it is known that the exponent $p = p_c = 1 + 4/N$ in dimension N is the critical value for nonexistence of global solutions (see [2], [20]).

For the Cauchy problem (1.1), (1.2), Ginibre and Velo [3] established the local existence in $H^1(\mathbb{R}^N)$. Glassey [4], Weinstein [20], Ogawa and Tsutsumi [17], Zhang [21] proved that for some initial data, the solutions of the Cauchy problem (1.1), (1.2) blow up in finite time.

In particular, let $Q(x)$ be the ground state (see [5, 18]), which is the unique, positive, radially symmetric solution of the following nonlinear elliptic equation:

$$-\triangle u + u - |u|^{4/N}u = 0, \quad u \in H^1(\mathbb{R}^N). \tag{1.3}$$

Weinstein [20] proved that if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solutions of the Cauchy problem (1.1), (1.2) globally exist. On the other hand, if $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, then the solutions of the Cauchy problem (1.1), (1.2) may blow up.

The ground state Q of equation (1.3) plays an important role in the formation of singularities for the solutions of the Cauchy problem (1.1), (1.2). Merle and

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Tsutsumi [12] studied the behavior of the radially symmetric blow-up solutions of the Cauchy problem (1.1), (1.2) and obtained the following result:

Suppose that $u_0 = u_0(|x|)$, $N \geq 2$ and that the solution $u(t) \in C([0,T), H^1)$ of the Cauchy problem (1.1), (1.2) blows up at T. Then for all $R > 0$, one has

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<R)} \geq \|Q\|_2, \tag{1.4}$$

where Q is the ground state solution of (1.3).

Under the same conditions, Tsutsumi [19] further showed that for any $\varepsilon > 0$, there exists a $K > 0$ such that

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<K(T-t)^{1/2}))} \geq (1-\varepsilon)\|Q\|_2. \tag{1.5}$$

(1.4) shows that under the condition that the initial datum is radially symmetric, the L^2-density of the blow-up solution of the Cauchy problem (1.1), (1.2) $u(t,x)$ concentrates at the origin 0 as $t \to T$ (blow-up time). We call this phenomenon *L^2-concentration* or *mass concentration*. Furthermore, (1.5) gives the *rate of L^2-concentration*.

In recent years, Merle and Raphaël studied the blow-up solutions with small super critical mass, i.e.,

$$u_0 \in B_{\alpha^*} = \{u_0 \in H^1 \quad \text{with} \quad \int Q^2 \leq \int u_0^2 < \int Q^2 + \alpha^*\},$$

for some parameter α^* small enough. They obtained a series of profound results on the qualitative properties such as the blow-up rate and L^2-concentration ([7]–[11]). In particular, Merle and Raphaël established the sharp upper and lower bound of the blow-up rate in [7, 11]. In terms of the sharp lower bound of the blow-up rate we studied the rate of L^2-concentration for the blow-up solutions, and (1.5) is improved; that is, under some conditions, we can show that for any $\varepsilon > 0$, there exists a $K > 0$ such that:

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<K(\frac{T-t}{\ln|\ln(T-t)|})^{\frac{1}{2}})} > (1-\varepsilon)\|Q\|_{L^2}. \tag{1.6}$$

This paper is organized as follows. In Section 2, we recall the lower bound of the blow-up rate recently obtained by Merle and Raphaël. In Section 3, we investigate the rate of L^2-concentration.

We conclude this section with some notation. We denote $L^q(\mathbb{R}^N)$ and $\|\cdot\|_{L^q(\mathbb{R}^N)}$ by L^q and $\|\cdot\|_q$, respectively. The various positive constants will be simply denoted by C.

2. Blow-up rate

From [2] or [3] one has the following proposition.

Proposition 2.1. *For any $u_0 \in H^1$, there exists a unique solution $u(t,x)$ of the Cauchy problem* (1.1), (1.2) *in $C([0,T); H^1)$ for some $T \in (0,\infty]$* (maximal existence time)*, and $u(t)$ satisfies the two conservation laws of mass and energy:*

$$\|u(t)\|_2 = \|u_0\|_2, \tag{2.1}$$

$$E(u(t)) := \|\nabla u(t)\|_2^2 - \frac{1}{1+2/N}\|u(t)\|_{2+4/N}^{2+4/N} = E(u_0) \tag{2.2}$$

for $t \in [0,T)$. Furthermore, we have the following alternatives: $T = \infty$ (global existence) *or else $T < \infty$ and $\lim_{t\to T}\|u(t)\|_{H^1} = \infty$* (blow up).

The blow-up rate of the Cauchy problem (1.1), (1.2) is very complicated:

• There exist in dimension $N = 2$ solutions with blow-up rate $\|\nabla u(t)\|_2 \sim \frac{1}{T-t}$ (see [1]).

• Another fact suggested by numerical simulations is the existence of solutions blowing up as $\|\nabla u(t)\|_2 \sim (\frac{\ln|\ln(T-t)|}{T-t})^{\frac{1}{2}}$ (see [6]).

For the blow-up solutions with small super critical data ($u_0 \in B_{\alpha^*}$), Merle and Raphaël established the shape lower bound for the blow-up rate in [11]. The proof of Merle and Raphaël [11] depends on the following property.

Spectral Property. *Let $N \geq 1$. Consider the Schrödinger operators*

$$Ł_1 = -\Delta + \frac{2}{N}(\frac{4}{N}+1)Q^{\frac{4}{N}-1}y\cdot\nabla Q, \quad Ł_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q,$$

and the real-valued quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$:

$$H(\varepsilon,\varepsilon) = (Ł_1\varepsilon_1,\varepsilon_1) + (Ł_2\varepsilon_2,\varepsilon_1).$$

Then there exists a universal constant $\delta > 0$ such that $\forall\varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0$, then

$$H(\varepsilon,\varepsilon) \geq \delta(\int |\nabla\varepsilon|^2 dx + \int |\varepsilon|^2 e^{-|y|}dx),$$

where $Q_1 = \frac{N}{2}Q + y\cdot\nabla Q$ and $Q_2 = \frac{N}{2}Q_1 + y\cdot\nabla Q_1$.

Remark 2.2. This property has been proved in [9] for dimension $N = 1$ using the explicit formula for the ground state Q. In dimensions $N = 2, 3, 4$, this property accords with the numerical computation.

Proposition 2.3 (Merle and Raphaël [11])**.** *Let $N = 1$ or $N \geq 2$ assuming that the Spectral Property holds true. There exists $\alpha^* > 0$ and a universal constant $C^* > 0$ such that the following is true. Let*

$$\int Q^2 \leq \int u_0^2 < \int Q^2 + \alpha^*,$$

and assume that the corresponding solution $u(t)$ of the Cauchy problem (1.1), (1.2) *blows up in finite time, $0 < T < +\infty$. Then one has the following lower bound on the blow-up rate for t close to T:*

$$\|\nabla u(t)\|_2 \geq C^*(\frac{\ln|\ln(\tau-t)|}{\tau-t})^{\frac{1}{2}}. \tag{2.3}$$

3. Rate of L^2-concentration

Now, we state the main theorem of this paper:

Theorem 3.1. *Suppose all the assumptions in Proposition* 2.3 *are satisfied. Let $N \geq 2$, and let $u(t)$ be a radially symmetric solution of the Cauchy problem* (1.1), (1.2) *in $C([0,T);H)$ such that $u(t)$ blows up at a finite time T.*

(i) *If $a(t)$ is a decreasing function from $[0,T)$ to R^+ such that $a(t) \longrightarrow 0$ $(t \to T)$ and $\frac{(\frac{\ln|\ln(T-t)|}{T-t})^{\frac{1}{2}}}{a(t)} \longrightarrow 0$ $(t \to T)$, then*

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<a(t))} \geq \|Q\|_{L^2}, \tag{3.1}$$

where Q is a ground state solution of (1.3).

(ii) *For any $\varepsilon > 0$, there exists a $K > 0$ such that*

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<K(\frac{T-t}{\ln|\ln(T-t)|})^{\frac{1}{2}})} > (1-\varepsilon)\|Q\|_{L^2}. \tag{3.2}$$

Remark 1. From Tsutsumi [19], we can get that

$$\liminf_{t\to T} \|u(t)\|_{L^2(|x|<K(T-t)^{1/2})} > (1-\varepsilon)\|Q\|_{L^2}.$$

Obviously the result of this paper is superior to that of [19].

Before the proof of Theorem 3.1, we give several lemmas.

Lemma 3.2 (Strauss [18])**.** *Assume that $N \geq 2$. Let $V(x)$ be a radially symmetric function in $H^1(\mathbb{R}^N)$. Then for any $R > 0$,*

$$\|V\|^2_{L^\infty(|x|>R)} \leq C_0 R^{-(N-1)} \|\nabla V\|_{L^2(|x|>R)} \|V\|_{L^2(|x|>R)}, \tag{3.3}$$

where C_0 does not depend on R and $V(x)$.

Lemma 3.3 (Weinstein [20])**.** *For any $f \in H^1(\mathbb{R}^N)$,*

$$\|f\|^{2+4/N}_{2+4/N} \leq (1+2/N)(\frac{\|f\|_2}{\|Q\|_2})^{4/N} \| \bigtriangledown f\|_2^2,$$

where Q is the unique ground state solution of equation (1.3).

In order to state our proof precisely, we need an auxiliary function $\rho(x)$: let $\rho(x) = \rho(|x|)$ be a radially symmetric nonnegative function in $C_0^1(\mathbb{R}^N)$ such that

$$\rho(r) = \begin{cases} 1, & r = |x| < 1/2, \\ 0, & r = |x| > 1, \end{cases} \tag{3.4}$$

and $0 \geq \rho_r(r) \geq -8$.

Lemma 3.4. *Let $N \geq 2$, and let $u(t)$ be a radially symmetric solution of the Cauchy problem* (1.1), (1.2) *in $C([0,T);H)$ such that $u(t)$ blows up at a finite time T. We put $\lambda(t) = \|\nabla u(t)\|_2$. Then, there exist two positive constants M_1 and M_2 such that*

$$\limsup_{t\to T} \frac{\|\nabla u(t)\|_2}{\|\nabla u(t)\|_{L^2(|x|<M_1/\lambda(t))}} \leq M_2. \tag{3.5}$$

Proof. Let M_1 be a large positive constant to be determined later. From (2.2), we have

$$\begin{aligned} \lambda(t)^2 &\leq \tfrac{N}{N+2}(\|\rho(\tfrac{\lambda(t)}{M_1}x)u(t)\|_{2+4/N} + \|\{1-\rho(\tfrac{\lambda(t)}{M_1}x)\}u(t)\|_{2+4/N})^{2+4/N} + E(u_0) \\ &\leq C\|\rho(\tfrac{\lambda(t)}{M_1}x)u(t)\|^{2+4/N}_{2+4/N} + C\|\{1-\rho(\tfrac{\lambda(t)}{M_1}x)\}u(t)\|^{2+4/N}_{2+4/N} + E(u_0), \end{aligned} \tag{3.6}$$

where $\rho(x)$ is defined as in (3.4).

The Gagliardo–Nirenberg inequality implies that

$$
\begin{aligned}
& \|\rho(\tfrac{\lambda(t)}{M_1}x)u(t)\|_{2+4/N}^{2+4/N} \\
\le\ & C\|\rho(\tfrac{\lambda(t)}{M_1}x)u(t)\|_2^{4/N}\|\nabla\{\rho(\tfrac{\lambda(t)}{M_1}x)u(t)\}\|_2^2 \\
\le\ & C\|u(t)\|_2^{4/N}\|\nabla\rho(\tfrac{\lambda(t)}{M_1}x)\cdot u(t)+\rho(\tfrac{\lambda(t)}{M_1}x)\cdot\nabla u(t)\|_2^2 \\
\le\ & C\|u(t)\|_2^{4/N}\|\nabla\rho(\tfrac{\lambda(t)}{M_1}x)\cdot u(t)\|_2^2+C\|u(t)\|_2^{4/N}\|\rho(\tfrac{\lambda(t)}{M_1}x)\cdot\nabla u(t)\|_2^2 \\
\le\ & C\|u(t)\|_2^{4/N}\|\tfrac{\lambda(t)}{M_1}\cdot\nabla\rho(x)\cdot u(t)\|_2^2+C\|u(t)\|_2^{4/N}\|\nabla u(t)\|^2_{L^2(|x|<\frac{M_1}{\lambda(t)})} \\
\le\ & C\|u(t)\|_2^{4/N}\tfrac{\lambda(t)^2}{M_1^2}\|u(t)\|_2^2+C\|u(t)\|_2^{4/N}\|\nabla u(t)\|^2_{L^2(|x|<\frac{M_1}{\lambda(t)})} \\
\le\ & \tfrac{C\|u(t)\|_2^{2+4/N}}{M_1^2}\lambda(t)^2+C\|u(t)\|_2^{4/N}\|\nabla u(t)\|^2_{L^2(|x|<\frac{M_1}{\lambda(t)})}.
\end{aligned}
\tag{3.7}
$$

By Lemma 3.2, we have

$$
\begin{aligned}
& C\|\{1-\rho(\tfrac{\lambda(t)}{M_1}x)\}u(t)\|_{2+4/N}^{2+4/N} \\
\le\ & \|u(t)\|^{2+4/N}_{L^{2+4/N}(|\frac{\lambda(t)}{M_1}x|>\frac{1}{2})} \\
\le\ & C\|u(t)\|^{4/N}_{L^\infty(|x|>\frac{M_1}{2\lambda(t)})}\|u(t)\|^2_{L^2(|x|>\frac{M_1}{2\lambda(t)})} \\
\le\ & C(\tfrac{\lambda(t)}{M_1})^{2(N-1)/N}\|\nabla u(t)\|^{2/N}_{L^2(|x|>\frac{M_1}{2\lambda(t)})}\|u(t)\|^{2+2/N}_{L^2(|x|>\frac{M_1}{2\lambda(t)})} \\
\le\ & C(\tfrac{\lambda(t)}{M_1})^{2(N-1)/N}\lambda(t)^{2/N}\|u(t)\|^{2+2/N}_{L^2(|x|>\frac{M_1}{2\lambda(t)})} \\
\le\ & \tfrac{C\|u(t)\|_2^{2+2/N}}{M_1^{2(N-1)/N}}\lambda(t)^2.
\end{aligned}
\tag{3.8}
$$

From (3.6)–(3.8), we have

$$
\begin{aligned}
\lambda(t)^2 \le\ & C_1\|u(t)\|_2^{4/N}\|\nabla u(t)\|^2_{L^2(|x|<\frac{M_1}{\lambda(t)})} \\
& +\{\frac{C_2\|u(t)\|_2^{2+4/N}}{M_1^2}+\frac{C_3\|u(t)\|_2^{2+2/N}}{M_1^{2(N-1)/N}}\}\lambda(t)^2+E(u_0).
\end{aligned}
\tag{3.9}
$$

If we choose M_1 so large that

$$
\frac{C_2\|u(t)\|_2^{2+4/N}}{M_1^2}+\frac{C_3\|u(t)\|_2^{2+2/N}}{M_1^{2(N-1)/N}}\le\frac{1}{2},
\tag{3.10}
$$

then we have by (3.10),

$$
\lambda(t)^2\le 2C_1\|u(t)\|_2^{4/N}\|\nabla u(t)\|^2_{L^2(|x|<\frac{M_1}{\lambda(t)})}+2E(u_0).
\tag{3.11}
$$

Since $\lambda(t)\to\infty$ $(t\to\infty)$, (3.11) and (2.1) imply that

$$
\|\nabla u(t)\|_{L^2(|x|<M_1/\lambda(t))}\longrightarrow\infty\ (t\to T).
$$

This fact and (3.11) show (3.5). □

Now, we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\rho(x)$ be defined as in (3.4). We put $\rho_a(x)=\rho(\frac{x}{a(t)})$ and $\lambda_a(t)=\|\nabla(\rho_a u_b(t))\|_2$.

(i) By (2.2) and Lemma 3.2, we have

$$\begin{aligned}\lambda(t)^2 - \frac{N}{N+2}\|u(t)\|^{2+4/N}_{L^{2+4/N}(|x|<\frac{a(t)}{2})} &= \frac{N}{N+2}\|u(t)\|^{2+4/N}_{L^{2+4/N}(|x|>\frac{a(t)}{2})} + E(u_0)\\ &\le \frac{N}{N+2}\|u(t)\|^{4/N}_{L^{\infty}(|x|>\frac{a(t)}{2})}\|u(t)\|^2_{L^2(|x|>\frac{a(t)}{2})} + E(u_0)\\ &\le Ca(t)^{-2(N-1)/N}\|u(t)\|^{2+2/N}_{L^2(|x|>\frac{a(t)}{2})}\lambda(t)^{2/N} + E(u_0).\end{aligned} \tag{3.12}$$

A simple calculation gives us

$$-\frac{N}{N+2}\|\rho_a u(t)\|^{2+4/N}_{2+4/N} \le -\frac{N}{N+2}\|u(t)\|^{2+4/N}_{L^{2+4/N}(|x|<\frac{a(t)}{2})}, \tag{3.13}$$

$$\begin{aligned}\lambda_a(t)^2 &\le (\|\rho_a\nabla u_0\|_2 + \|\nabla\rho_a\cdot u_0\|_2)^2\\ &\le (\lambda(t) + \frac{C\|u(t)\|_2}{a(t)})^2\\ &\le \lambda(t)^2 + \frac{C\|u(t)\|_2\lambda(t)}{a(t)} + \frac{C\|u(t)\|_2^2}{a(t)^2}.\end{aligned} \tag{3.14}$$

On the other hand, from Lemma 3.3, we have

$$\frac{N}{N+2}\|\rho_a u_b(t)\|^{2+4/N}_{2+4/N} \le (\frac{\|\rho_a u_b(t)\|_2}{\|Q\|_2})^{4/N}\lambda_a(t)^2. \tag{3.15}$$

By (3.12)–(3.15), we obtain

$$\begin{aligned}1 - (\frac{\rho_a u_b(t)}{\|Q\|_2})^{4/N} \le{}& C\|u(t)\|_2^{2+2/N}\frac{\lambda(t)^{2/N}}{\lambda_a(t)^{2/N}(a(t)\lambda_a(t))^{2(N-1)/N}}\\ &+ \frac{E(u_0)}{\lambda_a(t)^2} + \frac{C\|u(t)\|_2^2\lambda(t)}{a(t)\lambda_a(t)^2} + \frac{C\|u(t)\|_2^2}{(a(t)\lambda_a(t))^2}.\end{aligned} \tag{3.16}$$

Since $\frac{(\frac{\ln|\ln(T-t)|}{T-t})^{\frac{1}{2}}}{a(t)} \longrightarrow 0$ $(t\to T)$, by Proposition 2.3, we know that

$$\frac{1}{\lambda(t)a(t)} \to 0 \quad \text{as} \quad t\to T. \tag{3.17}$$

Letting $t\to T$ in (3.16), we obtained by (2.1), Lemma 3.4 and (3.17) that

$$\limsup_{t\to T}\{1 - (\frac{\|\rho_a u_b(t)\|_2}{\|Q\|_2})^{4/N}\} \le 0, \tag{3.18}$$

which proves Theorem 3.1(i).

(ii) We use the same argument as in the proof of Theorem 3.1(i) to prove Theorem 3.1(ii).

Let K be a sufficiently large constant to be determined later. If we insert $a(t) = \frac{K}{\lambda(t)}$ into (3.16), we have by Lemma 3.4,

(3.19)
$$1-(\|\rho(\frac{\lambda(t)}{K}x)u(t)\|_2/\|Q\|_2)^{4/N} \leq C\|u(t)\|_2^{2+2/N}K^{-2(N-1)/N} + E(u_0)\lambda(t)^{-2} + C\|u(t)\|_2K^{-1} + C\|u(t)\|_2^2K^{-2}.$$

Letting $t \to T$ in (3.19), we obtain

$$\limsup_{t\to T}\{1-(\|\rho(\frac{\lambda(t)}{K}x)u(t)\|_2/\|Q\|_2)^{4/N}\} \leq C\|u(t)\|_2^{2+2/N}K^{-2(N-1)/N}+C\|u(t)\|_2K^{-1} + C\|u(t)\|_2^2K^{-2}. \tag{3.20}$$

(3.20) implies that if we choose K sufficiently large, then (3.2) holds. □

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