ON THE NUMBER OF CERTAIN GALOIS EXTENSIONS
OF LOCAL FIELDS

DA-SHENG WEI AND CHUN-GANG JI

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Abstract. In this paper, we will calculate the number of Galois extensions of local fields with Galois group $A_n$ or $S_n$.

1. Introduction

Let $p$ be a prime, $F$ a finite extension of $p$-adic field $\mathbb{Q}_p$ with $[F : \mathbb{Q}_p] = m$. Let $k$ be its residue field with $[k : \mathbb{Z}/p\mathbb{Z}] = f$. It’s easy to see that $f|m$. Let $\pi$ be a uniformizer of $F$ and $e$ the absolute ramification index of $F/\mathbb{Q}_p$. Then $m = ef$. In this paper let $\mu_l$ denote the set of $l$-th roots of unity. All notations are standard if not explained.

Since the number of the extensions of local fields with a given degree inside the fixed algebraic closure is finite, see [2], one can ask for a formula that gives the number of extensions of a given degree. Krasner [1] gave such a formula, and Serre [6] also computed the number of extensions using a different method. Pauli and Roblot [3] gave the third proof for that formula. Similarly one can also ask for a formula that gives the number of Galois extensions of a given degree. In particular, it is possible to ask for a formula that gives the number of the Galois extensions with the prescribed finite Galois group $G$. We denote this number by $\nu(F, G)$. If $G$ is a $p$-group with $\mu_p \not\subset F$, Šafarevič [4] gave an explicit formula for the number of the $G$-extensions over $F$:

$$\nu(F, G) = \frac{1}{|\text{Aut}(G)|} \left( \frac{|G|}{p^d} \right)^{m+1} \prod_{i=1}^{d-1} \left( p^{m+1} - p^i \right),$$

where $d$ is the minimal number of generators of $G$. If $G$ is a $p$-group, and $\mu_p \subset F$, Yamagishi [7] obtained a formula for $\nu(F, G)$.

In this paper, we will calculate the number of $S_n$-extensions and $A_n$-extensions over $F$, where $S_n$ is the $n$-th symmetric group and $A_n$ is the $n$-th alternating group.

The cases for $n \geq 5$ that are quickly dismissed as $S_n$ and $A_n$ are not solvable in these cases, and the Galois groups of extensions of local fields are always solvable. So we only need to handle the remaining cases, $n \leq 4$.

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Theorem 1.1. Let $F$ be a finite extension over $\mathbb{Q}_p$ with $[F : \mathbb{Q}_p] = m$, $\nu(F,G)$ the number of the Galois extensions $K/F$ with $\text{Gal}(K/F) = G$.

(1) Suppose the prime $p \neq 3$; then

$$\nu(F, S_3) = \begin{cases} 0 & \text{if } \mu_3 \subset F, \\ 3 & \text{if } \mu_3 \not\subset F. \end{cases}$$

(2) Suppose $p = 3$; then

$$\nu(F, S_3) = \begin{cases} 3^{m+1} - 3 & \text{if } \mu_3 \subset F, \\ 3^m + \frac{3^{m+1}}{2} - \frac{3}{2} & \text{if } \mu_3 \not\subset F. \end{cases}$$

Theorem 1.2. Let $F$ be a finite extension over $\mathbb{Q}_p$ with $[F : \mathbb{Q}_p] = m$, $\nu(F,G)$ the number of the Galois extensions $K/F$ with $\text{Gal}(K/F) = G$.

(1) Suppose the prime $p \geq 3$; then

$$\nu(F, S_4) = \nu(F, A_4) = 0.$$

(2) Suppose $p = 2$; then

$$\nu(F, A_4) = \begin{cases} 2^m - 1/3 & \text{if } \mu_3 \subset F, \\ (2^m - 1)/3 & \text{if } \mu_3 \not\subset F; \end{cases}$$

$$\nu(F, S_4) = \begin{cases} 0 & \text{if } \mu_3 \subset F, \\ 2^{m+1} - 1 & \text{if } \mu_3 \not\subset F \text{ and } m \text{ is even and } f = 1, \\ 2^m - 1 & \text{otherwise}. \end{cases}$$

2. Some lemmas

The number of $S_2$-extensions and $A_3$-extensions of local fields is specified by well-known results of local class field theory. So we only need to calculate the number of Galois extensions over $F$ with Galois group $S_3$, $S_4$ and $A_4$. The following lemma plays an important role in our calculation.

Lemma 1. Let $K$ be a Galois extension over $F$ with the Galois group $G$. For any subgroup $A$ of $G$, let $F_A$ be the field fixed by $A$. Then the Galois closure $\text{cl}(F_A)$ of $F_A$ is a subfield of $K$ and $\text{Gal}(K/\text{cl}(F_A)) = \bigcap_{g \in G} gA^{-1}$.

We can get some Galois extensions from some non-Galois extensions by taking their Galois closure. For example, if $G = S_3$, $D_8$, $A_4$ and $S_4$, we can choose $A$ to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $S_3$, which is the non-normal subgroup of $G$ respectively, and where $D_8$ is the 2-sylow subgroup of $S_4$. By the above lemma, the Galois extensions of $F$ with Galois group $S_3$ can be gotten by the Galois closure of extensions of degree 3 of $F$, and the Galois extensions of $F$ with Galois group $D_8$, $A_4$ and $S_4$ can be gotten by the Galois closure of extensions of degree 4 of $F$.

Let $M(n)$ denote the set of all extensions of degree $n$ of $F$. Let $Ab(n)$ denote the set of abelian extensions of degree $n$ of $F$. Also, let $M(G)$ denote the set of Galois extensions of $F$ with the Galois group $G$. Let $K$ be the Galois closure of an extension of degree $n$ of $F$. The Galois group $\text{Gal}(K/F)$ is a subgroup of $S_n$. Obviously the order of $\text{Gal}(K/F)$ must be divided by $n$. So there are the following two maps:

$$f : M(3) \to Ab(3) \cup M(S_3)$$
and
\[ g : \ M(4) \rightarrow Ab(4) \cup M(D_8) \cup M(A_4) \cup M(S_4) \]
by
\[ L \rightarrow cl(L). \]
The two maps are surjective. Any inverse image \( L \) of an element \( K \) in \( M(G) \)
is a subfield of \( K \), and \( L \) is not a Galois extension of \( F \) if \( G \) is not an abelian

\[ \text{group}. \] In these cases, the Galois group \( \text{Gal}(K/L) \) is not a normal subgroup of \( G \).
For \( G = S_3, D_8, A_4 \) and \( S_4 \), we consider respectively the number of non-normal subgroups isomorphic to \( \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \) and \( S_3 \). The subgroups are non-normal except there is an order-2 normal subgroup in \( D_8 \). So the number of inverse images of any element in \( M(S_3), M(D_8), M(A_4) \) and \( M(S_4) \) are 3, 4, 4, 4 respectively. Let \( |S| \) denote cardinality of a finite set \( S \). Let \( \nu(F,G) \) denote the number of \( M(G) \). So there is the following result.

**Lemma 2.**
\[ |M(3)| = |Ab(3)| + 3\nu(F,S_3), \]
\[ |M(4)| = |Ab(4)| + 4\nu(F,D_8) + 4\nu(F,A_4) + 4\nu(F,S_4). \]

3. **The proof of Theorem 1.1**

In the following, we denote \( m = [F: \mathbb{Q}_p] \), and \( e \) is the absolute ramification index of \( F \) and \( q = p^e \) is the number of elements of the residue field of \( F \).

**Proof.** (1) For \( p \neq 3 \),

\[ M(3) = \{ K \mid [K : F] = 3, \ K \text{ is a tamely ramified extension of } F \}. \]

(i) If 3\textsuperscript{rd} roots of unity are contained in \( F \), then \( M(3) = Ab(3) \). By Lemma 2

\[ \nu(F,S_3) = 0. \]

(ii) If 3\textsuperscript{rd} roots of unity are not contained in \( F \), then \( |M(3)| = 10, |Ab(3)| = 1. \)

By Lemma 2

\[ \nu(F,S_3) = 3. \]

(2) For \( p = 3 \), by Krasner’s theorem [3],

\[ |M(3)| = 3q^e + 6(q-1)(\sum_{a=0}^{e-1} q^a) + 1 = 9q^e - 5. \]

Suppose \( \mu_3 \nsubseteq F \); then

\[ |Ab(3)| = \frac{1}{2}(\frac{3}{3})^{m+1}(3^{m+1} - 1) = \frac{3^{m+1} - 1}{2} = \frac{3q^e - 1}{2}. \]

Suppose \( \mu_3 \subseteq F \); then

\[ |Ab(3)| = 4. \]

By Lemma 2

\[ \nu(F,S_3) = \begin{cases} \frac{5q^e - 3}{2} & \text{if } \mu_3 \nsubseteq F, \\ 3q^e - 3 & \text{if } \mu_3 \subseteq F. \end{cases} \]

\]
4. The proof of Theorem 1.2

First we give some propositions.

**Proposition 4.1.** Let the prime \( p \geq 3 \). Then
\[
\nu(F, S_4) = \nu(F, A_4) = 0.
\]

**Proof.** Suppose \( K \) is a Galois extension over \( F \) with Galois group \( S_4 \). There must exist intermediate fields \( F^{tr} \) and \( F^{ur} \) such that \( \text{Gal}(K/F^{tr}) \) is a \( p \)-group, and \( \text{Gal}(F^{tr}/F) \) and \( \text{Gal}(F^{ur}/F) \) are cyclic groups. By Galois theory, there is a \( p \)-group \( S' \) which is a normal subgroup of \( S_4 \). Since \( p \geq 3 \), \( S' \) must be \((1)\). Since \( S_4 \) does not have a cyclic normal subgroup \( S \) such that \( S_4/S \) is also cyclic, this is a contradiction.

Similarly we get \( \nu(F, A_4) = 0 \). \( \Box \)

**Proposition 4.2.** Let \( p = 2 \). Then
\[
\nu(F, A_4) = \begin{cases} 
4(2^m - 1)/3 & \text{if } \mu_3 \subset F, \\
(2^m - 1)/3 & \text{if } \mu_3 \not\subset F.
\end{cases}
\]

**Proof.** Let \( K \) be an \( A_4 \)-extension over \( F \). Since \( K_4 \) is a normal subgroup of \( A_4 \), there exists a (unique) Galois subfield \( F' \) of degree 3 over \( F \), where \( K_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

By [2],
\[
|F'^*/(F'^*)^2| = 4q^{3e},
|F^*/(F^*)^2| = 4q^e.
\]

It is clear that the natural map of \( F'^*/(F'^*)^2 \to F^*/(F^*)^2 \) is an injection since \( [F': F] = 3 \). We consider the action on \( F'^*/(F'^*)^2 \) of the Galois group \( \text{Gal}(F'/F) \).

(1) Denote \( G' = \text{Gal}(F'/F) \). Then the following result holds:
\[
(F^*/(F^*)^2)^G' \cong F^*/(F^*)^2.
\]

As we’ve already noted injectivity, it remains to show that the natural map is surjective. Let \( a \in F'^*/(F'^*)^2 \) be a fixed point of \( \text{Gal}(F'/F) \) and \( a \neq 1 \) in \( F'^*/(F'^*)^2 \). Then \( F'(\sqrt{a'}) \) is a Galois extension over \( F \), where \( a' \) represents a lifting of \( a \) in \( F'^* \).

There isn’t an order-2 normal subgroup in \( S_3 \), so
\[
\text{Gal}(F'(\sqrt{a'})/F) \cong \mathbb{Z}/6\mathbb{Z}.
\]

Let \( F'' \) be the fixed field of the normal subgroup \( \mathbb{Z}/3\mathbb{Z} \). There exists an element \( b \in F^*/(F^*)^2 \) such that
\[
F'' = F(\sqrt{b}).
\]

Then
\[
F'(\sqrt{a'}) = F'(\sqrt{b}).
\]

So \( a = b \) in \( F'^*/(F'^*)^2 \).

(2) Let \( \sigma \) be a generator of \( G' \). Assume \( x \in F'^*/(F'^*)^2 \) such that \( F^*/(F^*)^2 \); then there are the following two cases:

(i) \( N_{F'/F}(x) = 1 \) in \( F^*/(F^*)^2 \),
(ii) \( N_{F'/F}(x) \neq 1 \) in \( F^*/(F^*)^2 \).
In (i), the field $F'((\sqrt{x}, \sqrt{\sigma x})$ is an $A_4$-extension over $F$ since the Galois group of $F'((\sqrt{x}, \sqrt{\sigma x})/F$ isomorphic to $K_4 \times \mathbb{Z}/3\mathbb{Z}$, where $K_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In (ii),

$$F'((\sqrt{x}, \sqrt{\sigma x}, \sqrt{\sigma^2 x}) = F'((\sqrt{x/\sigma(x)}, \sqrt{\sigma x/\sigma^2 x}, \sqrt{N_{F'/F}(x)}),$$

So $F'((\sqrt{x}, \sqrt{\sigma x}, \sqrt{\sigma^2 x})$ is an $(A_4 \times \mathbb{Z}/2\mathbb{Z})$-extension over $F$. Any $(A_4 \times \mathbb{Z}/2\mathbb{Z})$-extension $K$ is generated by an $A_4$-extension over $F$ and an extension of degree 2 over $F$. Denote the unique extension of degree 3 by $F'$; then there exist $x \in F^*/(F^*)^2$ and $a \in F^*/(F^*)^2$ satisfying $x \neq F^*/(F^*)^2$, $N_{F'/F}(x) \in (F^*)^2$ and $a \notin (F^*)^2$, such that $K = F'((\sqrt{x}, \sqrt{\sigma x}, \sqrt{a})$. It is easy to see that

$$K = F'((\sqrt{ax/\sigma x}, \sqrt{a\sigma x/\sigma^2 x}, \sqrt{a\sigma^2 x/x})$$

since $N_{F'/F}(x) = x\sigma x\sigma^2 x = 1$ in $F^*/(F^*)^2$. Then $\sigma x/\sigma^2 x = x\sigma^2 x/\sigma^2 x = x$ in $F^*/(F^*)^2$. Consider $K$ as an extension over $F'$; there exist 7 subfields with order 2 over $F'$ which are one-to-one correspondents to $\{y, \sigma y, \sigma^2 y, y\sigma y, y\sigma^2 y, y^2, y, N_{F'/F}(y)\}$, where $y = ax/\sigma x$. And $N_{F'/F}(y) = a \neq 1$ in $F^*/(F^*)^2$. The $\text{Gal}(F'/F)$-orbits are $\{y, \sigma y, \sigma^2 y\}$, $\{y\sigma y, y\sigma^2 y, y^2\}$, and $\{N_{F'/F}(y)\}$. So any $(A_4 \times \mathbb{Z}/2\mathbb{Z})$-extension over $F$ is in form of (ii) and

$$\nu(F, A_4 \times \mathbb{Z}/2\mathbb{Z}) = \nu(F, A_4)\nu(F, \mathbb{Z}/2\mathbb{Z})$$

(i) Suppose $\mu_3 \not\subset F$, $F'$ is the unique unramified extension of degree 3 over $F$, so

$$3\nu(F, A_4) + 3\nu(F, A_4 \times \mathbb{Z}/2\mathbb{Z}) = 4q^{3e} - 4q^e,$$

$$\nu(F, A_4 \times \mathbb{Z}/2\mathbb{Z}) = \nu(F, A_4)(4q^e - 1).$$

Then

$$\nu(F, A_4) = (q^{2e} - 1)/3.$$

(ii) Suppose $\mu_3 \subset F$, $F'$ is the unique unramified extension and 3 totally ramified extensions of degree 3 over $F$, so

$$3\nu(F, A_4) + 3\nu(F, A_4 \times \mathbb{Z}/2\mathbb{Z}) = 4(4q^{3e} - 4q^e),$$

$$\nu(F, A_4 \times \mathbb{Z}/2\mathbb{Z}) = \nu(F, A_4)(4q^e - 1).$$

Then

$$\nu(F, A_4) = 4(q^{2e} - 1)/3.$$

\[\square\]

**Proposition 4.3.** Let $p = 2$. Then

$$\nu(F, S_4) = \begin{cases} 
0 & \text{if } \mu_3 \subset F, \\
2m+1 - 1 & \text{if } \mu_3 \not\subset F \text{ and } m \text{ is even and } f = 1, \\
2m - 1 & \text{otherwise.}
\end{cases}$$

*Proof.* By Krasner’s theorem [3],

$$|M(4)| = 16q^{3e} - 4q^{2e} - 5.$$

By the local class field theory and the dual theory of the finite abelian group, the following equation holds:

$$|Ab(4)| = |\{S : S \text{ is the subgroup of order } 4 \text{ of } F^*/(F^*)^4\}|.$$
Let $T_1$ be the set consisting of the elements of order $\leq 2$ in $F^*/(F^*)^4$, and let $T_2$ be the set consisting of the elements of order 4 in $F^*/(F^*)^4$. The sequence
\[
0 \to T_1 \to F^*/(F^*)^4 \to (F^*)^2/(F^*)^4 \to 0
\]
is exact. The third map is $a \mapsto a^2$. So
\[
|T_1| = |F^*/(F^*)^4|/(F^*)^2/(F^*)^4|.
\]
Suppose $\mu_4 \not\subseteq F$; then
\[
|T_1| = 4\mu^e, \\
|T_2| = 8\mu^{2e} - 4\mu^e.
\]
Then
\[
|\text{Ab}(4)| = |T_2|/2 + (|T_1| - 1)(|T_1| - 2)/6 = 20\mu^{2e}/3 - 4\mu^e + 1/3.
\]
Suppose $\mu_4 \subseteq F$; then
\[
|T_1| = 4\mu^e, \\
|T_2| = 16\mu^{2e} - 4\mu^e.
\]
Then
\[
|\text{Ab}(4)| = |T_2|/2 + (|T_1| - 1)(|T_1| - 2)/6 = 32\mu^{2e}/3 - 4\mu^e + 1/3.
\]
Since $D_8$ is a 2-group, by Theorem 2.2 of [7],
\[
\nu(F, D_8) = \begin{cases} 
q^e(q^e - 1)(4q^e - 1) & \text{if } \mu_4 \subseteq F \text{ or } \mu_4 \not\subseteq F \text{ and } m \text{ is even and } f = 1, \\
q^e(2q^e - 1)^2 & \text{otherwise.}
\end{cases}
\]
By Lemma 2,
\[
(i) \text{ If } \mu_4 \subseteq F, \text{ then } \\
\nu(F, S_4) = (|M(4)| - |\text{Ab}(4)|)/4 - \nu(F, A_4) - \nu(F, D_8) = 4(q^{2e} - 1)/3 - \nu(F, A_4).
\]
\[
(ii) \text{ If } \mu_4 \not\subseteq F, \text{ and } m \text{ is odd or } m \text{ is even and } f \geq 2, \text{ then } \\
\nu(F, S_4) = (|M(4)| - |\text{Ab}(4)|)/4 - \nu(F, A_4) - \nu(F, D_8) = 4(q^{2e} - 1)/3 - \nu(F, A_4).
\]
\[
(iii) \text{ If } \mu_4 \not\subseteq F \text{ and } m \text{ is even and } f = 1, \text{ then } \\
\nu(F, S_4) = (|M(4)| - |\text{Ab}(4)|)/4 - \nu(F, A_4) - \nu(F, D_8) = (7q^{2e} - 4)/3 - \nu(F, A_4).
\]
By Proposition 4.2, we have
\[
\nu(F, S_4) = \begin{cases} 
0 & \text{if } \mu_3 \subseteq F, \\
2q^{2e} - 1 & \text{if } \mu_4 \not\subseteq F \text{ and } n \text{ is even and } f = 1, \\
q^{2e} - 1 & \text{otherwise.}
\end{cases}
\]

Remark. Since $K_4$ is a normal subgroup of $S_4$ and $S_4/K_4 \cong S_3$, there exists an $S_3$-subextension in an $S_4$-extension of $F$ by Galois theory. If $\mu_3 \subseteq F$ and $p \neq 3$, then $\nu(F, S_3) = 0$. So $\nu(F, S_4) = 0$. This gives another proof for a case of $\nu(F, S_4)$.

Using these propositions, the proof of Theorem 1.2 is obtained. This completes the proof of Theorem 1.2.
5. Examples

Example 5.1. Let $F = \mathbb{Q}_p$. Then

(1) \[
\nu(\mathbb{Q}_p, S_3) = \begin{cases} 
6 & \text{if } p = 3, \\
0 & \text{if } p \equiv 1 \pmod{3}, \\
3 & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]

(2) \[
\nu(\mathbb{Q}_p, A_4) = \begin{cases} 
1 & \text{if } p = 2, \\
0 & \text{if } p > 2.
\end{cases}
\]

(3) \[
\nu(\mathbb{Q}_p, S_4) = \begin{cases} 
3 & \text{if } p = 2, \\
0 & \text{if } p > 2.
\end{cases}
\]

(4) \[
\nu(\mathbb{Q}_p, S_n) = \nu(\mathbb{Q}_p, A_n) = 0 \quad (n \geq 5).
\]

References


DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, PEOPLE’S REPUBLIC OF CHINA 230026
E-mail address: dshwei@ustc.edu

DEPARTMENT OF MATHEMATICS, NANJING NORMAL UNIVERSITY, NANJING, PEOPLE’S REPUBLIC OF CHINA 210097
E-mail address: cgji@njnu.edu.cn