ON LITTLEWOOD-PALEY FUNCTIONS

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Abstract. We prove that, for a compactly supported $L^q$ function $\Phi$ with vanishing integral on $\mathbb{R}^n$, the corresponding square function operator $S_\Phi$ is bounded on $L^p$ for $|1/p - 1/2| < \min\{(q - 1)/2, 1/2\}$.

1. Introduction

Let $n \geq 1$ and $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. For a function $\Phi \in L^1(\mathbb{R}^n)$ which satisfies

\begin{equation}
\int_{\mathbb{R}^n} \Phi(x)dx = 0,
\end{equation}

we define the square function operator $S_\Phi$ by

\begin{equation}
(S_\Phi f)(x) = \left( \int_0^\infty \left| \Phi_t \ast f(x) \right|^2 \frac{dt}{t} \right)^{1/2},
\end{equation}

where $\Phi_t(x) = t^{-n}\Phi(x/t)$. The operator $S_\Phi$ is often called a square function or a Littlewood-Paley function. Such operators have long played important roles in harmonic analysis. The main problem under investigation concerns the boundedness of these operators on various $L^p$ spaces.

It has been well known that, if the function $\Phi$ is sufficiently nice (in terms of decaying and smoothness properties), the corresponding operator $S_\Phi$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The following result is due to Benedek, Calderón and Panzone (\cite{2}).

Theorem A. Suppose that $\Phi$ satisfies (1.1) and for some positive $\alpha$,

\begin{equation}
|\Phi(x)| \leq C(1 + |x|)^{-n-\alpha}, \int_{\mathbb{R}^n} |\Phi(x - y) - \Phi(x)|dx \leq C|y|^\alpha.
\end{equation}

Then $S_\Phi$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Examples of functions satisfying (1.3) include the Schwartz functions, as well as the following which arise from the Poisson kernel on $\mathbb{R}^n$:

$$\Phi_0(x) = \frac{\partial}{\partial t} \left( \frac{t}{(|x|^2 + t^2)^{(n+1)/2}} \right) \bigg|_{t=1}$$

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and
\[ \Phi_j(x) = \frac{\partial}{\partial x_j} \left( \frac{1}{(|x|^2 + 1)^{(n+1)/2}} \right). \]

When \( \Phi \) is given by
\[ \Phi(x) = |x|^{-n+1} \chi_{[0,1]}(|x|), \]
where \( \Omega \) is homogeneous of degree 0 and has mean value zero on \( S^{n-1} \), then \( S_\Phi \) becomes the Marcinkiewicz integral operator (13). See also [1], [13], [16] and the extensive list of references given in the survey [6].

In [12] S. Sato proved that, among other things, the conclusion of Theorem A is still true if the smoothness condition (1.3) is eliminated (for \( L^2 \) this had been known earlier; see [5], [10]).

In this paper we shall study the \( L^p \) boundedness of \( S_\Phi \) without imposing conditions (1.2) or (1.3) on \( \Phi \), or the assumption that \( \Phi \) be given as in (1.4). The following is a known result:

**Theorem B.** Suppose that \( \Phi \) satisfies (1.1) and is compactly supported.

(i) If \( \Phi \in L^q(R^n) \) for some \( q \geq 2 \), then \( S_\Phi \) is a bounded operator on \( L^p(R^n, w) \) for \( p > q' \) and \( w \in A_{p/q'} \).

(ii) If \( \Phi \in L^2(R^n) \), then \( S_\Phi \) is a bounded operator on \( L^p(R^n) \) for \( 1 < p < \infty \).

In the above statement, \( L^p(R^n, w) \) represents the weighted \( L^p \) space with weight \( w \) (the definition of the weight class \( A_p \) can be found in [11], [1] or [13]). When \( w \equiv 1 \) we write \( L^p(R^n, w) \) as \( L^p(R^n) \).

Part (i) of Theorem B is due to S. Sato (Theorem 3 in [12]). Part (ii) follows from (i) by using duality and interpolation (with \( w \equiv 1 \)).

Theorem B (ii) covers the cases \( \Phi \in L^q(R^n) \), \( q > 2 \) as well because of the compact support assumption. However, the approach used in [12] does not appear to work when \( q < 2 \) (see also [9], page 241). The main purpose of the present paper is to establish the following theorem dealing with the case where \( \Phi \in L^q(R^n) \) for \( q < 2 \).

**Theorem C.** Suppose that \( \Phi \) is a compactly supported function satisfying (1.1). If \( \Phi \in L^q(R^n) \) for some \( q \in (1, 2] \), then \( S_\Phi \) is a bounded operator on \( L^p(R^n) \) for \( |1/p - 1/2| < (q - 1)/2 \).

Remarks 1. (i) The range of \( p \) given by \( |1/p - 1/2| < (q - 1)/2 \) is the same as \( 2/q < p < 2/(2 - q) \), which becomes \((1, \infty) \) when \( q = 2 \).

(ii) A result relevant to the theorems mentioned above is Theorem 1 in [7]. While in general an \( L^q \) function \( \Phi \) does not satisfy the pointwise decay condition imposed on its Fourier transform in Theorem 1 of [7], a modification of the proof given in [7] can yield the \( L^p \) boundedness of \( S_\Phi \) for \( |1/p - 1/2| < (q - 1)/(2q) \) under the conditions in Theorem C. Since \( q > 1 \), the range of \( p \) given by \( |1/p - 1/2| < (q - 1)/2 \) in our theorem is considerably better.

(iii) While the inequality \( |1/p - 1/2| < (q - 1)/2 \) gives the full range \( 1 < p < \infty \) when \( q = 2 \), it would be an interesting problem to determine whether it also represents the best possible range for \( p \) when \( q < 2 \).

The proof of Theorem C will be given in Section 2. Section 3 contains a result on the boundedness of \( S_\Phi \) when the compact support condition is replaced by some other conditions.
2. Proof of Theorem C

Lemma 2.1. Let \( \Psi \) be a compactly supported function in \( L^q(\mathbb{R}^n) \) for some \( q \in (1, 2) \). Then, for every \( p \) satisfying \( 1/p - 1/2 < (q - 1)/2 \), there exists a \( C_p > 0 \) such that

\[
(2.1) \quad \left\| \left( \int_0^\infty |(\Psi_t \ast F^t)(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C_p \left( \int_0^\infty |F^t(x)|^2 \frac{dt}{t} \right)^{1/2} \left\| \right\|_{L^p(\mathbb{R}^n)}
\]

holds for every measurable function \( F^t(x) = F(t, x) \) on \((0, \infty) \times \mathbb{R}^n\).

Proof. By duality and interpolation we may assume that \( p > 2 \). We shall also assume that \( \text{supp}(\Psi) \) is contained in \( B(0, 1) \), where \( B(x_0, t) = \{ x \in \mathbb{R}^n : |x - x_0| \leq t \} \). Let \( T \) be the operator acting on functions defined on \((0, \infty) \times \mathbb{R}^n \) given by

\[
(2.2) \quad T(F)(t, x) = (\Psi_t \ast F^t)(x),
\]

where \( F^t(y) = F(t, y) \) for \((t, y) \in (0, \infty) \times \mathbb{R}^n \). For \( 1 \leq p, q < \infty \) we shall use the following notation:

\[
(2.3) \quad \|F\|_{L^p(L^q(t^{-1}dt), dx)} = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty |F(t, x)|^{q/p} \frac{dt}{t} \right) dx \right)^{1/p}.
\]

Thus we have

\[
(2.4) \quad \|T(F)\|_{L^p(L^q(t^{-1}dt), dx)} = \int_{\mathbb{R}^n} \left( \int_0^\infty |(\Psi_t \ast F^t)(x)| t^{-1} dt \right) dx
\]

\[
= \int_0^\infty \|\Psi_t \ast F^t\|_{L^q(\mathbb{R}^n)} t^{-1} dt
\]

\[
\leq \|\Psi\|_{L^q(\mathbb{R}^n)} \int_0^\infty \|F^t\|_{L^q(t^{-1}dt), dx} t^{-1} dt
\]

\[
= \|\Psi\|_{L^q(\mathbb{R}^n)} \|F\|_{L^q(\mathbb{R}^n)}\|F^t\|_{L^q(t^{-1}dt), dx}.
\]

It follows from \( 1/2 - 1/p < (q - 1)/2 \) that

\[
p'q/2 > 1.
\]

Let \( r = (p'q/2)' \). By \( p > 2 \) we have \( r > q' \). Thus, for any \( F \) that satisfies

\[
\|T(F)\|_{L^r(L^{q'}(t^{-1}dt), dx)} < \infty,
\]

there exists a function \( h \in L^{(r/q')'}(\mathbb{R}^n) \) such that

\[
\|h\|_{L^{(r/q')'}(\mathbb{R}^n)} = 1
\]

and

\[
(2.5) \quad \|T(F)\|_{L^r(L^{q'}(t^{-1}dt), dx)} = \int_{\mathbb{R}^n} \left( \int_0^\infty |(\Psi_t \ast F^t)(x)|^{q't^{-1}} dt \right) h(x) dx.
\]

By Hölder’s inequality,

\[
|\Psi_t \ast F^t(x)|^{q'} = t^{-nq} \left| \int_{\mathbb{R}^n} \Psi \left( \frac{x - y}{t} \right) F(t, y) dy \right|^{q'}
\]

\[
\leq \|\Psi\|_{L^q(\mathbb{R}^n)} |B(0, t)|^{-1} \int_{\mathbb{R}^n} |F(t, y)|^{q'} \chi_{B(0, t)}(x - y) dy.
\]
Let $M$ denote the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Then by (2.5) and (2.6),
\[
\|T(F)\|_{L^p(L^2(t^{-1}dt),dx)} \leq \|\Psi\|_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left( \int_0^\infty |F(t,y)|q't^{-1}dt \right)(Mh)(y)dy \\
\leq \|\Psi\|_{L^q(\mathbb{R}^n)} \left( \int_0^\infty |F(t,y)|q't^{-1}dt \right)^{q'/(q'+1)} \|Mh\|_{L^q(\mathbb{R}^n)}.
\]
(2.7)

Since
\[
\frac{1}{2} = \frac{\theta}{q'} + \frac{(1-\theta)}{1}, \quad \frac{1}{p} = \frac{\theta}{r} + \frac{(1-\theta)}{1},
\]
hold with $\theta = q/2$, by interpolating between (2.4) and (2.7) (see, for example, [3]) we obtain
\[
\|T(F)\|_{L^p(L^2(t^{-1}dt),dx)} \leq C_p \|F\|_{L^p(L^2(t^{-1}dt),dx)},
\]
which proves (2.1) for $|1/p - 1/2| < (q-1)/2$.

For $s > 0$ we let $s^{\pm\alpha} = \min\{s^\alpha, s^{-\alpha}\}$.

**Lemma 2.2.** Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and $\Phi \in L^q(\mathbb{R}^n)$ for some $q \in (1,2]$. Suppose that $\Phi$ is compactly supported and $\text{supp}(\hat{\Psi}) \subset \{1/2 < |\xi| < 2\}$. Then for every $p$ satisfying $|1/p - 1/2| < (q-1)/2$, there exist $C_p, \alpha_p > 0$ such that
\[
\left( \int_0^\infty |\Phi_t * \Psi_{st} * f|^2 dt \right)^{1/2} \leq C_p(s^{\pm\alpha_p}) \|f\|_{L^p(\mathbb{R}^n)}
\]
(2.8)
for all $f \in L^p(\mathbb{R}^n)$ and $s > 0$.

**Proof.** Let
\[
T_sf(x) = \left( \int_0^\infty |\Phi_t * \Psi_{st} * f(x)|^2 dt \right)^{1/2}.
\]
Then, by Lemma 2.1 and Theorem A, for each $p$ satisfying $|1/p - 1/2| < (q-1)/2$,
\[
\|T_sf\|_{L^p(\mathbb{R}^n)} \leq C_p \|sf\|_{L^p(\mathbb{R}^n)} \\
\leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]
(2.9)

On the other hand, by Plancherel’s Theorem,
\[
\|T_sf\|_{L^2(\mathbb{R}^n)}^2 = \int_0^\infty \int_{\mathbb{R}^n} |\Phi_t * \Psi_{st} * f(x)|^2 dx dt^{-1} dt \\
= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left( \int_0^\infty \hat{\Phi}(t\xi)^2 |\hat{\Psi}(st\xi)|^2 dt \right) d\xi.
\]
(2.10)

It follows from Lemmas 2, 3 of [12] and (1.1) that
\[
\int_{1/2}^1 |\hat{\Phi}(t\xi)|^2 dt \leq C|\xi|^{\pm 1/(2q')},
\]
(2.11)
Let $\xi' = |\xi|^{-1}\xi$ for $\xi \neq 0$. By (2.10) and (2.11) we have
\[
\|T_s f\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} |f(\xi)|^2 \left( \int_{1/2}^{2} |\tilde{\Phi}(ts^{-1}\xi')|^2 \frac{dt}{t} \right) d\xi
\]
(2.12)

By (2.9), (2.12) and interpolation we conclude that (2.8) holds for $|1/p - 1/2| < (q - 1)/2$.

**Proof of Theorem C.** It suffices to establish
\[
\|Sf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}
\]
for $f \in S(\mathbb{R}^n)$ and $|1/p - 1/2| < (q - 1)/2$.

Let $\eta \in C^\infty(\mathbb{R})$ such that supp($\eta$) is contained in $(1/4, 1)$ and
\[
\int_0^\infty \eta(s) \frac{ds}{s} = 2.
\]
Define the Schwartz function $\Psi$ on $\mathbb{R}^n$ by
\[
\tilde{\Psi}(\xi) = \eta(|\xi|^2)
\]
for $\xi \in \mathbb{R}^n$. Then by (2.13) we have
\[
\int_0^\infty \tilde{\Psi}(s\xi) \frac{ds}{s} = 1
\]
and
\[
\tilde{\Phi}_t * f = \int_0^\infty (\tilde{\Phi}_t * \tilde{\Psi}_s * f) \frac{ds}{s}
\]
for all $f \in S(\mathbb{R}^n)$ and $t > 0$. By Minkowski’s inequality we have
\[
|S\Phi f(x)| \leq \int_0^\infty \left( \int_0^\infty |\tilde{\Phi}_t * \tilde{\Psi}_s * f(x)|^2 \frac{dt}{t} \right)^{1/2} \frac{ds}{s}
\]
By Lemma 2.2, for every $p$ satisfying $|1/p - 1/2| < (q - 1)/2$,
\[
\|S\Phi f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \int_0^\infty s^{\pm \alpha_p} \frac{ds}{s}
\]
(2.15)

The proof of Theorem C is now complete.

### 3. Further results

For $\Phi$’s which are not necessarily compactly supported, one can easily deduce the following result from Theorem 2 of [12] by using duality and interpolation:

**Theorem 3.1.** Let $\Phi \in L^1(\mathbb{R}^n)$ and satisfy (1.1). Suppose that
\[(i) \int_{|x|<1} |\Phi(x)|^{1+\varepsilon} dx + \int_{|x|>1} |\Phi(x)||x|^2 dx < \infty \text{ for some } \varepsilon > 0; \]
\[(ii) |\Phi(x)| \leq h(|x|)\Omega(x'), \text{ where} \]
\[(ii.a) h \text{ is non-negative, non-increasing and satisfies } \int_0^\infty h(r)r^{n-1} dr < \infty; \]
\[(ii.b) \Omega \in L^q(S^{n-1}) \text{ for some } q \geq 2. \]

Then $S\Phi$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. 

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Using the above theorem one can see that the condition (1.4) in Theorem A is redundant, as observed in [12].

Below we shall show that the requirement “\(q \geq 2\)” in (ii.b) of the above theorem can be lowered to \(q > 1\) without affecting the validity of the claim.

**Theorem 3.2.** If the condition (ii.b) in Theorem 3.1 is replaced by the weaker condition (ii.b'): \(\Omega \in L^q(S^{n-1})\) for some \(q > 1\), while all other conditions in Theorem 3.1 remain unchanged, then \(S_\Phi\) is bounded on \(L^p(\mathbb{R}^n)\) for \(1 < p < \infty\).

**Proof.** By Lemmas 1–3 in [12], we see that (2.11) still holds (note that \(q > 1\) is needed when applying Lemma 2 of [12]). Thus, it suffices to show that (2.1) holds for all \(p \in (1, \infty)\).

For each \(y' \in S^{n-1}\) and \(x \in \mathbb{R}^n\), let

\[
M_{y'}f(x) = \sup_{s > 0} \left( \frac{1}{s} \int_0^s |f(x - sy')|ds \right).
\]

For \(t > 0\) and \(x \in \mathbb{R}^n\),

\[
|\Phi_t * f(x)| \leq \int_{S^{n-1}} |\Omega(y')| \left( \int_0^{\infty} |f(x - try')|h(r)r^{n-1}dr \right)d\sigma(y')
\]

\[
\leq \int_{S^{n-1}} |\Omega(y')| \left( \sum_{j=-\infty}^{\infty} 2^{n(j+1)}h(2^j)M_{y'}f(x) \right)d\sigma(y')
\]

\[
\leq 2^n \left( \int_0^{\infty} h(r)r^{n-1}dr \right) \int_{S^{n-1}} |\Omega(y')|M_{y'}f(x)d\sigma(y').
\]

By (3.1) and the uniform boundedness of the operators \(M_{y'}\) on \(L^p(\mathbb{R}^n)\) for \(1 < p < \infty\), we have

\[
\| \sup_{t > 0} |\Phi_t * f| \|_{L^p(\mathbb{R}^n)} \leq C_p\|\Omega\|_1\|f\|_{L^p(\mathbb{R}^n)}
\]

for \(1 < p < \infty\). It follows from (3.2) and the proof of the lemma on p. 544 of [8] (after some trivial modifications) that (2.1) holds for \(1 < p < \infty\). \(\square\)

**References**


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