ON THE SUM FORMULA FOR THE $q$-ANALOGUE OF NON-STRICT MULTIPLE ZETA VALUES

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Abstract. In this article, the $q$-analogues of the linear relations of non-strict multiple zeta values called “the sum formula” and “the cyclic sum formula” are established.

1. Introduction

For any multi-index $k = (k_1, k_2, \ldots, k_r)$ ($k_i \in \mathbb{Z}$, $k_i > 0$), the weight $\text{wt}(k)$ and depth $\text{dep}(k)$ of $k$ are by definition the integers $k = k_1 + k_2 + \cdots + k_r$ and $r$ respectively. We denote by $I(k, r)$ the set of multi-indices $k$ of weight $k$, and depth $r$, and by $I_0(k, r)$ the subset of admissible indices, i.e., indices with the extra requirement that $k_1 \geq 2$.

For an admissible index $(k_1, \ldots, k_r)$, the multiple zeta value $\zeta((k_1, \ldots, k_r)) : = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$, $\zeta^*((k_1, \ldots, k_r)) : = \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$. The latter is also called multiple zeta-star value (MZSV, for short) in $[1, 19]$. Both values can be written as a $\mathbb{Z}$-linear combination of each other.

These values are known to be related to many objects of mathematics and quantum physics, for example, connection formulae for hypergeometric functions $[20]$, knot invariants $[16]$, Feynman diagrams $[15]$ and so on. They also appear in the coefficients of the Drinfel’d’s KZ-associator $[7]$. The properties of the KZ-associator are related to the representations of the fundamental group of a configuration space.

Study of MZSVs has been initiated by Leonhard Euler $[8]$, and he got many results including the well-known formula:

$$\zeta^*(k - 1, 1) = \frac{k + 1}{2} \zeta(k) - \frac{1}{2} \sum_{r=2}^{k-2} \zeta(r) \zeta(k - r).$$
It seems that Euler wanted to give an answer to the question, “When are MZSVs in the algebra generated by Riemann zeta values \( \zeta(k) \)?” It is a basic and an important question even now. The next two equivalent formulae were conjectured in [11], and proved by Andrew Granville [10] and Don Zagier independently.

**Sum Formula.** For positive integers \( 0 < r < k \), there holds

\[
(1) \quad \sum_{k \in I_0(k,r)} \zeta(k) = \zeta(k), \quad \sum_{k \in I_0(k,r)} \zeta^*(k) = \binom{k-1}{r-1} \zeta(k).
\]

These formulae are so fundamental that they are re-proved again and again [2, 12, 13, 17, 18, 20, 22]. For the MZV case, there are two more proofs. One is the conclusion of the cyclic sum formula for MZSVs [19]. The other is using the special value of the generating function of multiple polylogarithms [14].

**Cyclic Sum Formula.** For \((k_1, \ldots, k_r) \in I_0(k,r)\),

\[
(2) \quad \sum_{i=1}^{k_r-2} \sum_{j=0}^{k_i-j} \zeta^*(k_i-j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j+1) = k \zeta(k+1),
\]

where the empty sum means zero.

**Generating Function.** For the multiple polylogarithms with equality defined by

\[
\text{Li}^*_{k_1, \ldots, k_r}(t) := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}},
\]

the generating function and its special value at \( t = 1 \) are expressed as follows:

\[
(3) \quad \sum_{k \geq r > 0} \left\{ \sum_{k \in I_0(k,r)} \text{Li}^*_k(t) \right\} x^{k-r-1} y^{r-1} = \frac{1}{1-x-y} \int_0^t (1-s)^{-y} F_2(1-x-y, 1-y, 2-x-y; s) \, ds,
\]

\[
\sum_{k \geq r > 0} \left\{ \sum_{k \in I_0(k,r)} \zeta^*(k) \right\} x^{k-r-1} y^{r-1} = \sum_{n=1}^\infty \frac{1}{n-x-y} \frac{1}{n-y}.
\]

where \( F_2 \) is Gauß's hypergeometric series.

In this article, we define the \( q \)-analogues of MZSVs and construct the \( q \)-analogues of the above formulae.

For \( 0 < q < 1 \) and \( \alpha \in \mathbb{C} \), \( [\alpha] \) is defined by \( [\alpha] := (1-q^\alpha)/(1-q) \). The \( q \)-Pochhammer symbol is defined by \( (\alpha; q)_{\infty} := \prod_{n=0}^{\infty} (1-\alpha q^n) \) and \( (\alpha; q)_n := (\alpha; q)_{\infty}/(\alpha q^n; q)_{\infty} \) for any integer \( n \). Then the \( q \)-analogues of MZV are defined by

\[
\zeta_q[k_1, \ldots, k_r] := \sum_{n_1 > \cdots > n_r > 0} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1} \cdots [n_r]^{k_r}}.
\]

A \( q \)-analogue of \( \zeta^*(k) \) is studied in [4].
Definition 1. For any admissible index \((k_1, \ldots, k_r)\), the \(q\)-analogue of MZSV and the multiple polylogarithm with equality are as follows:

\[
\zeta_q^*[k_1, \ldots, k_r] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]_{k_1} \cdots [n_r]_{k_r}},
\]

\[
\text{Li}_{k_1, \ldots, k_r}^*[t] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_1}}{[n_1]_{k_1} \cdots [n_r]_{k_r}}.
\]

As the \(q\)-analogue of (2), we have the next formula:

Theorem 1 (Cyclic Sum Formula). For \((k_1, \ldots, k_r) \in I_0(k, r),\)

\[
\sum_{i=1}^{r} \sum_{j=0}^{k-i-2} \zeta_q^*[k_i - j, k_i+1, \ldots, k_r, k_1, \ldots, k_i-1, j+1] = \sum_{l=0}^{r} (k-l-1-r) \left(\begin{array}{c} r+1 \\ l \end{array}\right) (1-q)^l \zeta_q[k - l + 1],
\]

where the empty sum means zero.

Moreover, there also holds the \(q\)-analogue of (3):

Theorem 2 (Generating Function of Multiple Polylogarithms).

\[
\sum_{k > r > 0} \left\{ \sum_{k \in I_0(k, r)} \text{Li}_k^*[t] \right\} u^{k-r-1} v^{r-1} = \frac{1}{1-u-v} \int_0^t \frac{(s; q)_n q_{\infty}}{(bs; q)_n q_{\infty}} 2\phi_1 (a, b, aq; s, q) \, dq s,
\]

where \(2\phi_1\) is Heine’s \(q\)-hypergeometric series [9], \(q^{-a-1} = \frac{1}{1-(1-q)(a+v)}\) and \(b = \frac{1-(1-q)u}{1-(1-q)(a+v)}\), and the integral is the Jackson \(q\)-integral [9].

As the corollary of these theorems, we obtain the \(q\)-analogue of (1):

Corollary 3 (Sum Formula). For integers \(0 < r < k,\)

\[
\sum_{k \in I_0(k, r)} \zeta_q^*[k] = \frac{1}{k-1} \binom{k-1}{r-1} \sum_{l=0}^{r-1} \binom{r-1}{l} (k-1-l) (1-q)^l \zeta_q[k-l].
\]

2. Proof of Theorem 1

For index \((k_1, \ldots, k_r) \in I(k, r)\) with \(k_i \geq 2\) for some \(i\), we set the convergent series

\[
T(k_1, \ldots, k_r) := \sum_{n_1 \geq \cdots \geq n_r+1 \geq 1} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]_{k_1} \cdots [n_r]_{k_r}} \frac{q^{n_1-n_r+1}}{[n_1-n_r+1]_{k_r}}.
\]

This series satisfies the equation

\[
T(k_1, k_2, \ldots, k_r) - T(k_2, \ldots, k_r, k_1) = \sum_{l=0}^{r} \left(\begin{array}{c} r \\ l \end{array}\right) (k_1 - l) \left(1-q\right)^l \zeta_q[k-l+1] - \sum_{j=0}^{k_1-2} \zeta_q^*[k_1-j, k_2, \ldots, k_r, j+1].
\]
Summing up these equations by rotating the indices, and we have the theorem.

To prove (5), by using the equation

\[ \frac{1}{n_1} q^{n_1-n_{r+1}} = \left( \frac{1}{n_1-n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}. \]

we have

\[
T(k_1, k_2, \ldots, k_r) = \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1 \atop n_1 \neq n_{r+1}} q^{n_1(k_1-1) + n_2(k_2-1) + \cdots + n_r(k_r-1)} \left( \frac{1}{n_1-n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}
\]

\[
= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1 \atop n_1 \neq n_{r+1}} q^{n_1(k_1-2) + n_2(k_2-1) + \cdots + n_r(k_r-1)} q^{n_1} \left( \frac{1}{n_1-n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}
\]

\[
- \zeta_q^* [k_1, k_2, \ldots, k_r, 1] + \sum_{n=1}^{\infty} \frac{q^{n(k-r)}}{[n]^{k+1}}
\]

\[
= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1 \atop n_1 \neq n_{r+1}} q^{n_1(k_1-2) + n_2(k_2-1) + \cdots + n_r(k_r-1) + n_{r+1}(2-1)} q^{n_1} \left( \frac{1}{n_1-n_{r+1}} - \frac{1}{n_1} \right) \frac{1}{n_{r+1}}
\]

\[
- \zeta_q^* [k_1, k_2, \ldots, k_r, 1] + \sum_{n=1}^{\infty} \frac{q^{n(k-r)}}{[n]^{k+1}}
\]

\[
= \sum_{n_1 \geq \cdots \geq n_{r+1} \geq 1 \atop n_1 \neq n_{r+1}} q^{n_2(k_2-1) + \cdots + n_r(k_r-1) + n_{r+1}(k_1-2)} q^{n_1} \left( \frac{1}{n_1} \right) \frac{1}{n_{r+1}}
\]

\[
- \sum_{j=0}^{k_1-2} \zeta_q^*[k_1-j, k_2, \ldots, k_r, j+1] + (k_1-1) \sum_{n=1}^{\infty} \frac{q^{n(k-r)}}{[n]^{k+1}}.
\]

By using the equation

\[ \frac{1}{n_1} q^{n_1} = \left( \frac{q^{n_1-n_{r+1}}}{n_1-n_{r+1}} - \frac{q^{n_1}}{n_1} \right) \frac{q^{n_{r+1}}}{n_{r+1}}. \]
Thus we have Corollary 3.

Moreover, by substituting the equation

\[\sum_{n_2 \geq \ldots \geq n_{r+2} \geq 1} q^{n_2(k_2-1)+\ldots+n_r(k_r-1)+n_{r+1}(k_{r+1}-2)} \left( \sum_{n_1=1}^{\infty} \frac{q^{n_1-n_{r+1}}}{[n_1][n_1-n_{r+1}]} - \frac{q^{n_{r+1}}}{[n_{r+1}]} \right) \]

we obtain (II).

Furthermore, for \( k = (k_1, \ldots, k_r) \in I_0(k, r) \), we set

\[ J_0(k) := \bigcup_{i=1}^{r} \bigcup_{j=0}^{k_i-2} \{ (k_i-j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j+1) \} \subset I_0(k+1, r+1). \]

Then

\[ I_0(k+1, r+1) = \bigcup_{k \in I_0(k, r)} J_0(k), \quad \text{and} \quad J_0(k) \cap J_0(k') = \emptyset \quad \text{if} \ k \neq k'. \]

From (II) we have

\[ \sum_{k' \in J_0(k)} \zeta^*_q[k'] = \frac{#J_0(k)}{k-r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1], \]

and summing up about \( k \) we obtain

\[ \sum_{k' \in I_0(k+1, r+1)} \zeta^*_q[k'] = \sum_{k \in I_0(k, r)} \sum_{k' \in J_0(k)} \zeta^*_q[k'] \]

\[ = \frac{#I_0(k+1, r+1)}{k-r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1] \]

\[ = \frac{1}{k-r} \binom{k-1}{r} \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1]. \]

Thus we have Corollary 3.
3. Proof of Theorem 2

We denote the generating functions of \( \text{Li}_k^* \) as follows:

\[
\Psi^*(u, v; t, q) := \sum_{k \geq r > 0} \left( \sum_{k \in l(k, r)} \text{Li}^*_k[t] \right) u^{k-r} v^{r-1},
\]

\[
\Psi_0^*(u, v; t, q) := \sum_{k \geq r > 0} \left( \sum_{k \in l_0(k, r)} \text{Li}^*_k[t] \right) u^{k-r-1} v^{r-1}.
\]

To investigate the above generating functions we use the \( q \)-differential equation, where the \( q \)-differential operator \( D_q \) is defined by

\[
(D_q f)(t) := \frac{f(t) - f(qt)}{t - qt}.
\]

From the \( q \)-differential equation for \( \text{Li}_k^* \)

\[
D_q \text{Li}^*_k[u, v; t, q] = \left\{ \begin{array}{ll}
\frac{1}{t} \text{Li}^*_{k-1, k_1, \ldots, k_r}[t] & (k_1 \geq 2), \\
\frac{1}{t} \frac{1}{1 - t} \text{Li}^*_{k_2, \ldots, k_r}[t] & (k_1 = 1 \text{ and } r \geq 2), \\
\frac{1}{1 - t} & (k_1 = 1 \text{ and } r = 1),
\end{array} \right.
\]

\( \Psi^* \) and \( \Psi_0^* \) satisfy the following \( q \)-differential equations:

\[
D_q \Psi^*(u, v; t, q) = \frac{1}{t} \Psi^*(u, v; t, q),
\]

\[
D_q (\Psi^* - u \Psi_0^*)(u, v; t, q) = \frac{1}{1 - t} + \frac{1}{t} - v \Psi^*(u, v; t, q).
\]

By eliminating \( \Psi^* \) from the above equations, we have that \( \Psi_0^* \) satisfies the inhomogeneous linear \( q \)-differential equation of second order:

\[
qt(1 - t)D_q^2 f + \{(1 - t)(1 - u) - v\}D_q f = 1.
\]

\( \Psi_0^* \) is characterized as the regular solution of (7) around the origin and the value at the origin is 0.

We must find such a solution of (7) in another way. At first we put \( g := D_q f \) and solve the equation

\[
qt(1 - t)D_q g + \{(1 - t)(1 - u) - v\}g = 1,
\]

by variation of parameter. We choose \( C_0 t^a(t; q)_\infty / (bt; q)_\infty \) for the solution of the homogeneous equation

\[
qt(1 - t)D_q h + \{(1 - t)(1 - u) - v\}h = 0,
\]

where \( q^{-a-1} = \frac{1}{1 - (1 - q)(u + v)} \), \( b = \frac{1 - (1 - q)v}{1 - (1 - q)(u + v)} \), and \( C_0 \in \mathbb{C} \). We assume that

\[
g(t) = C(t) t^a \frac{(t; q)_\infty}{(bt; q)_\infty},
\]

and substitute this into (8); then we have

\[
C'(t) = q^{-a-1} t^{-a-1} \frac{bqt; q)_\infty}{(t; q)_\infty}.
\]
The Jackson integral of $C'(t)$ is as follows:

\[
\int_0^t q^{-a-1} s^{-a-1} \frac{(bs; q)_\infty}{(s; q)_\infty} \, dq \, ds = q^{-a-1} \int_0^t s^{-a-1} \sum_{n=0}^{\infty} \frac{(bs; q)_n}{(q; q)_n} s^n \, dq \, ds
\]

\[
= q^{-a-1} \sum_{n=0}^{\infty} \frac{(bs; q)_n}{(q; q)_n} \frac{t^{n-a}}{n-a}
\]

\[
= \frac{t^{-a}}{1 - u - v} \phi_1(q^{-a}, bq, q^{-a+1}; t, q),
\]

where the first equality is by virtue of the $q$-binomial theorem [9]. So we obtain the solution of (8) which is regular at the origin:

\[
g(t) = \frac{1}{1 - u - v} \frac{(t; q)_\infty}{(bt; q)_\infty} 2\phi_1(q^{-a}, bq, q^{-a+1}; t, q).
\]

We consider the Jackson integral again and get the solution of (7):

\[
f(t) = \frac{1}{1 - u - v} \int_0^t \frac{(s; q)_\infty}{(bs; q)_\infty} 2\phi_1(q^{-a}, bq, q^{-a+1}; s, q) \, dq \, ds.
\]

By executing the Jackson integral, we have

\[
f(t) = \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bs; q)_n}{(1 - q^{n-a})(q; q)_n} \int_0^t s^n \frac{(s; q)_\infty}{(bs; q)_\infty} \, dq \, ds
\]

\[
= \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \frac{(1 - q^{-a})(bs; q)_n}{(1 - q^{n-a})(q; q)_n} (1 - q)^{t_{n+1}} \frac{(t; q)_\infty}{(bt; q)_\infty} \sum_{j=0}^{\infty} q^{j(n+1)} \frac{(bt)_j}{(t)_j},
\]

which is zero at $t = 0$. Thus we obtain the theorem.

In the same way as [21], the special value of $\text{Li}^*_{k_1, k_2, \ldots, k_r}$ and the generating function are expressed by the combination of the $q$-anologue of MZSVs: Substitute $t = q$ and the value is

\[
\text{Li}^*_{k_1, k_2, \ldots, k_r} [q] = \sum_{a_1=0}^{k_1-1} \sum_{a_2=0}^{k_2-1} \cdots \sum_{a_r=0}^{k_r-1} \binom{k_1-2}{a_1} \binom{k_2-1}{a_2} \cdots \binom{k_r-1}{a_r} \times (1-q)^{k_1+\cdots+k_r-a_1-\cdots-a_r} \zeta_q^*[a_1, a_2, \ldots, a_r],
\]

and the generating function is

\[
\Psi^*(u, v; q, q) = \frac{1}{1 + (1-q)x} \sum_{k > r > 0} \left\{ \sum_{k \in I_0(k, r)} \zeta_q^*[k] \right\} x^{k-r-1} y^{r-1},
\]

where $x = \frac{u}{1-(1-q)u}$ and $y = \frac{v}{1-(1-q)u}$.
On the other hand, by substituting $t = q$ to \[9\],

\[
\Psi(u, v; q, q) = \frac{1}{1 - u - v} \sum_{n=0}^{\infty} \left(1 - q^{-a}\right) \left(\frac{bq}{q}\right)^n (1 - q) q^{n+1} \frac{(q; q)_{\infty}}{(1 - bq^n; q)_{\infty}} \\
\times \sum_{j=0}^{\infty} q^{j(n+1)} \frac{(bq; q)_j}{(q; q)_j}.
\]

\[
= \frac{1 - q}{1 - u - v} \sum_{n=0}^{\infty} \left(1 - q^{-a}\right) (1 - q) q^n \frac{(q; q)_{\infty}}{(1 - bq^n; q)_{\infty}} \frac{(bq^{n+2}; q)_{\infty}}{(q^{n+1}; q)_{\infty}}
\]

\[
= \frac{1 - q}{1 - u - v} \sum_{n=0}^{\infty} \left(1 - q^{-a}\right) q^{n+1} (1 - bq^{n+1})
\]

\[
= \sum_{n=1}^{\infty} q^n \frac{1 - (1 - q)(u + v)}{[n] - (u + v) [n] - (1 - q^n)u - v}
\]

\[
= \frac{1}{1 + (1 - q)x} \sum_{n=1}^{\infty} \frac{q^n (1 - (1 - q)y) [n] - y \cdot [n] - xq^n - y}{[n] - y \cdot [n] - xq^n - y}.
\]

Hence we have

\[
\sum_{k > r > 0} \left\{ \sum_{k \in I_0(k, r)} \zeta^r[k]\right\} x^{k-r-1} y^{r-1} = \sum_{n=1}^{\infty} q^n \frac{(1 - (1 - q)y) [n] - y \cdot [n] - xq^n - y}{[n] - y \cdot [n] - xq^n - y},
\]

and expanding the right hand by geometric series, we obtain Corollary 3.

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