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ON THE SUM FORMULA FOR THE q-ANALOGUE OF NON-STRICT MULTIPLE ZETA VALUES

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Abstract. In this article, the q-analogues of the linear relations of non-strict multiple zeta values called "the sum formula" and "the cyclic sum formula" are established.

1. Introduction

For any multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_r)$ ($k_i \in \mathbb{Z}$, $k_i > 0$), the *weight* $\mathrm{wt}(\mathbf{k})$ and *depth* $\mathrm{dep}(\mathbf{k})$ of $\mathbf{k}$ are by definition the integers $k = k_1 + k_2 + \cdots + k_r$ and r respectively. We denote by $I(k, r)$ the set of multi-indices $\mathbf{k}$ of weight k, and depth r, and by $I_0(k, r)$ the subset of *admissible* indices, i.e., indices with the extra requirement that $k_1 \geq 2$.

For an admissible index $(k_1, \ldots, k_r)$, the multiple zeta value [23, 11] (MZV, for short) and the non-strict multiple zeta value are defined as follows:

$$\zeta(k_1, \ldots, k_r) := \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \quad \zeta^\star(k_1, \ldots, k_r) := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

The latter is also called multiple zeta-star value (MZSV, for short) in [1, 19]. Both values can be written as a $\mathbb{Z}$-linear combination of each other.

These values are known to be related to many objects of mathematics and quantum physics, for example, connection formulae for hypergeometric functions [20], knot invariants [16], Feynman diagrams [15] and so on. They also appear in the coefficients of the Drinfel′d's KZ-associator [7]. The properties of the KZ-associator are related to the representations of the fundamental group of a configuration space.

Study of MZSVs has been initiated by Leonhard Euler [8], and he got many results including the well-known formula:

$$\zeta^\star(k-1, 1) = \frac{k+1}{2}\zeta(k) - \frac{1}{2}\sum_{r=2}^{k-2} \zeta(r)\zeta(k-r).$$

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It seems that Euler wanted to give an answer to the question, "When are MZSVs in the algebra generated by Riemann zeta values $\zeta(k)$?" It is a basic and an important question even now. The next two equivalent formulae were conjectured in [11], and proved by Andrew Granville [10] and Don Zagier independently.

Sum Formula. *For positive integers $0 < r < k$, there holds*

$$\sum_{\mathbf{k}\in I_0(k,r)} \zeta(\mathbf{k}) = \zeta(k), \qquad \sum_{\mathbf{k}\in I_0(k,r)} \zeta^\star(\mathbf{k}) = \binom{k-1}{r-1}\zeta(k). \tag{1}$$

These formulae are so fundamental that they are re-proved again and again [2, 12, 13, 17, 18, 20, 22]. For the MZSV case, there are two more proofs. One is the conclusion of the cyclic sum formula for MZSVs [19]. The other is using the special value of the generating function of multiple polylogarithms [14].

Cyclic Sum Formula. *For $(k_1, \dots, k_r) \in I_0(k, r)$,*

$$\sum_{i=1}^{r}\sum_{j=0}^{k_i-2} \zeta^\star(k_i - j, k_{i+1}, \dots, k_r, k_1, \dots, k_{i-1}, j+1) = k\,\zeta(k+1), \tag{2}$$

where the empty sum means zero.

Generating Function. *For the multiple polylogarithms with equality defined by*

$$\mathrm{Li}^\star_{k_1,\dots,k_r}(t) := \sum_{n_1\ge\cdots\ge n_r\ge 1} \frac{t^{n_1}}{n_1^{k_1}\cdots n_r^{k_r}},$$

the generating function and its special value at $t = 1$ are expressed as follows:

$$\begin{aligned}\sum_{k>r>0}\left\{\sum_{\mathbf{k}\in I_0(k,r)} \mathrm{Li}^\star_{\mathbf{k}}(t)\right\} x^{k-r-1}y^{r-1} &= \frac{1}{1-x-y}\int_0^t (1-s)^{-y}\,{}_2F_1(1-x-y, 1-y, 2-x-y; s)\,ds,\\ \sum_{k>r>0}\left\{\sum_{\mathbf{k}\in I_0(k,r)} \zeta^\star(\mathbf{k})\right\} x^{k-r-1}y^{r-1} &= \sum_{n=1}^{\infty}\frac{1}{n-x-y}\frac{1}{n-y},\end{aligned} \tag{3}$$

where ${}_2F_1$ is Gauß's hypergeometric series.

In this article, we define the q-analogues of MZSVs and construct the q-analogues of the above formulae.

For $0 < q < 1$ and $\alpha \in \mathbb{C}$, $[\alpha]$ is defined by $[\alpha] := (1-q^\alpha)/(1-q) \overset{q\to 1}{\longrightarrow} \alpha$. The q-Pochhammer symbol is defined by $(\alpha; q)_\infty := \prod_{j=0}^{\infty}(1-\alpha q^j)$ and $(\alpha; q)_n := (\alpha; q)_\infty/(\alpha q^n; q)_\infty$ for any integer n. Then the q-analogues of MZV are defined by

$$\zeta_q[k_1, \dots, k_r] := \sum_{n_1>\cdots>n_r>0} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1}\cdots[n_r]^{k_r}}.$$

A q-analogue of the relations for MZVs are considered in [3, 5, 6, 21]. A finite q-analogue of $\zeta^\star(\mathbf{k})$ is studied in [4].

Definition 1. For any admissible index $(k_1, \dots, k_r)$, the q-analogue of MZSV and the multiple polylogarithm with equality are as follows:

$$\zeta_q^\star[k_1, \dots, k_r] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1} \cdots [n_r]^{k_r}},$$

$$\mathrm{Li}^\star_{k_1,\dots,k_r}[t] := \sum_{n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_1}}{[n_1]^{k_1} \cdots [n_r]^{k_r}}.$$

As the q-analogue of (2), we have the next formula:

Theorem 1 (Cyclic Sum Formula)**.** *For $(k_1, \dots, k_r) \in I_0(k, r)$,*

$$\sum_{i=1}^{r} \sum_{j=0}^{k_i-2} \zeta_q^\star[k_i - j, k_{i+1}, \dots, k_r, k_1, \dots, k_{i-1}, j+1] = \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q[k-l+1], \tag{4}$$

where the empty sum means zero.

Moreover, there also holds the q-analogue of (3):

Theorem 2 (Generating Function of Multiple Polylogarithms)**.**

$$\sum_{k>r>0} \left\{ \sum_{\mathbf{k} \in I_0(k,r)} \mathrm{Li}^\star_{\mathbf{k}}[t] \right\} u^{k-r-1} v^{r-1} = \frac{1}{1-u-v} \int_0^t \frac{(s;q)_\infty}{(bs;q)_\infty} {}_2\phi_1\left(a, b, aq; s, q\right) \, d_q s,$$

where ${}_2\phi_1$ is Heine's q-hypergeometric series [9]*, $q^{-a-1} = \frac{1}{1-(1-q)(u+v)}$ and $b = \frac{1-(1-q)u}{1-(1-q)(u+v)}$, and the integral is the Jackson q-integral* [9]

$$\int_0^t f(s) \, d_q s := (1-q)t \sum_{n=0}^{\infty} f(q^n t) q^n.$$

As the corollary of these theorems, we obtain the q-analogue of (1):

Corollary 3 (Sum Formula)**.** *For integers $0 < r < k$,*

$$\sum_{\mathbf{k} \in I_0(k,r)} \zeta_q^\star[\mathbf{k}] = \frac{1}{k-1} \binom{k-1}{r-1} \sum_{l=0}^{r-1} \binom{r-1}{l} (k-1-l)(1-q)^l \zeta_q[k-l]. \tag{5}$$

2. Proof of Theorem 1

For index $(k_1, \dots, k_r) \in I(k, r)$ with $k_i \geq 2$ for some i, we set the convergent series

$$T(k_1, \dots, k_r) := \sum_{\substack{n_1 \geq \cdots \geq n_{r+1} \geq 1 \\ n_1 \neq n_{r+1}}} \frac{q^{n_1(k_1-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1} \cdots [n_r]^{k_r}} \frac{q^{n_1-n_{r+1}}}{[n_1 - n_{r+1}]}.$$

This series satisfies the equation

$$\begin{aligned} &T(k_1, k_2, \dots, k_r) - T(k_2, \dots, k_r, k_1) \\ &= \sum_{l=0}^{r} \binom{r}{l} \left(k_1 - \frac{l}{r}\right) (1-q)^l \zeta_q[k-l+1] - \sum_{j=0}^{k_1-2} \zeta_q^\star[k_1 - j, k_2, \dots, k_r, j+1]. \end{aligned} \tag{6}$$

Summing up these equations by rotating the indices, and we have the theorem.

To prove (6), by using the equation

$$\frac{1}{[n_1]}\frac{q^{n_1-n_{r+1}}}{[n_1-n_{r+1}]}=\left(\frac{1}{[n_1-n_{r+1}]}-\frac{1}{[n_1]}\right)\frac{1}{[n_{r+1}]},$$

we have

$$\begin{aligned}
&T(k_1,k_2,\dots,k_r)\\
&=\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}}\frac{q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1-1}[n_2]^{k_2}\cdots[n_r]^{k_r}}\left(\frac{1}{[n_1-n_{r+1}]}-\frac{1}{[n_1]}\right)\frac{1}{[n_{r+1}]}\\
&=\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}}\frac{q^{n_1(k_1-2)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1-1}[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{1}}\frac{q^{n_1}}{[n_1-n_{r+1}]}\\
&\qquad\qquad-\zeta_q^\star[k_1,k_2,\dots,k_r,1]+\sum_{n=1}^{\infty}\frac{q^{n(k-r)}}{[n]^{k+1}}\\
&=\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}}\frac{q^{n_1(k_1-2)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1-2}[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{1}}\left(\frac{1}{[n_1-n_{r+1}]}-\frac{1}{[n_1]}\right)\frac{q^{n_{r+1}}}{[n_{r+1}]}\\
&\qquad\qquad-\zeta_q^\star[k_1,k_2,\dots,k_r,1]+\sum_{n=1}^{\infty}\frac{q^{n(k-r)}}{[n]^{k+1}}\\
&=\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}}\frac{q^{n_1(k_1-3)+n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(2-1)}}{[n_1]^{k_1-2}[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{2}}\frac{q^{n_1}}{[n_1-n_{r+1}]}\\
&\qquad\qquad-\zeta_q^\star[k_1,k_2,\dots,k_r,1]-\zeta_q^\star[k_1-1,k_2,\dots,k_r,2]+2\sum_{n=1}^{\infty}\frac{q^{n(k-r)}}{[n]^{k+1}}\\
&=\cdots\cdots\cdots\cdots\cdots\\
&=\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}}\frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_1-2)}}{[n_1]^{1}[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{k_1-1}}\frac{q^{n_1}}{[n_1-n_{r+1}]}\\
&\qquad\qquad-\sum_{j=0}^{k_1-2}\zeta_q^\star[k_1-j,k_2,\dots,k_r,j+1]+(k_1-1)\sum_{n=1}^{\infty}\frac{q^{n(k-r)}}{[n]^{k+1}}.
\end{aligned}$$

By using the equation

$$\frac{1}{[n_1]}\frac{q^{n_1}}{[n_1-n_{r+1}]}=\left(\frac{q^{n_1-n_{r+1}}}{[n_1-n_{r+1}]}-\frac{q^{n_1}}{[n_1]}\right)\frac{q^{n_{r+1}}}{[n_{r+1}]},$$

the first term is

$$\begin{aligned}
&\sum_{\substack{n_1\geq\cdots\geq n_{r+1}\geq 1\\ n_1\neq n_{r+1}}} \frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_1-2)}}{[n_1]^1[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{k_1-1}}\frac{q^{n_1}}{[n_1-n_{r+1}]}\\
&=\sum_{\substack{n_2\geq\cdots\geq n_{r+1}\geq 1\\ n_2\neq n_{r+1}}} \frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_1-2)}}{[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{k_1-1}}\\
&\qquad\times\left(\sum_{n_1=n_2}^{\infty}\frac{q^{n_1-n_{r+1}}}{[n_1-n_{r+1}]}-\frac{q^{n_1}}{[n_1]}\right)\frac{q^{n_{r+1}}}{[n_{r+1}]}\\
&\qquad+\sum_{n_2=1}^{\infty}\frac{q^{n_2(k_2+\cdots+k_r+k_1-r)}}{[n_2]^{k_2+\cdots+k_r+k_1}}\left(\sum_{n_1=n_2+1}^{\infty}\frac{q^{n_1-n_2}}{[n_1-n_2]}-\frac{q^{n_1}}{[n_1]}\right)\\
&=\sum_{\substack{n_2\geq\cdots\geq n_{r+2}\geq 1\\ n_2\neq n_{r+2}}} \frac{q^{n_2(k_2-1)+\cdots+n_r(k_r-1)+n_{r+1}(k_1-1)}}{[n_2]^{k_2}\cdots[n_r]^{k_r}[n_{r+1}]^{k_1}}\frac{q^{n_2-n_{r+2}}}{[n_2-n_{r+2}]}+\sum_{n=1}^{\infty}\frac{q^{n(k-r+1)}}{[n]^{k+1}}.
\end{aligned}$$

Moreover, by substituting the equation

$$\sum_{n=1}^{\infty}\frac{q^{n(k-m)}}{[n]^{k+1}}=\sum_{l=0}^{m}\binom{m}{l}(1-q)^l\zeta_q[k-l+1],$$

we obtain (6).

Furthermore, for $\mathbf{k}=(k_1,\ldots,k_r)\in I_0(k,r)$, we set

$$J_0(\mathbf{k}):=\bigcup_{i=1}^{r}\bigcup_{j=0}^{k_i-2}\{(k_i-j,k_{i+1},\ldots,k_r,k_1,\ldots,k_{i-1},j+1)\}\subset I_0(k+1,r+1).$$

Then

$$I_0(k+1,r+1)=\bigcup_{\mathbf{k}\in I_0(k,r)}J_0(\mathbf{k}),\qquad\text{and}\qquad J_0(\mathbf{k})\cap J_0(\mathbf{k}')=\emptyset\quad\text{if }\mathbf{k}\neq\mathbf{k}'.$$

From (4) we have

$$\sum_{\mathbf{k}'\in J_0(\mathbf{k})}\zeta_q^{\star}[\mathbf{k}']=\frac{\#J_0(\mathbf{k})}{k-r}\sum_{l=0}^{r}(k-l)\binom{r}{l}(1-q)^l\zeta_q[k-l+1],$$

and summing up about $\mathbf{k}$ we obtain

$$\begin{aligned}
\sum_{\mathbf{k}''\in I_0(k+1,r+1)}\zeta_q^{\star}[\mathbf{k}'']&=\sum_{\mathbf{k}\in I_0(k,r)}\sum_{\mathbf{k}'\in J_0(\mathbf{k})}\zeta_q^{\star}[\mathbf{k}']\\
&=\frac{\#I_0(k+1,r+1)}{k-r}\sum_{l=0}^{r}(k-l)\binom{r}{l}(1-q)^l\zeta_q[k-l+1]\\
&=\frac{1}{k-r}\binom{k-1}{r}\sum_{l=0}^{r}(k-l)\binom{r}{l}(1-q)^l\zeta_q[k-l+1].
\end{aligned}$$

Thus we have Corollary 3.

3. Proof of Theorem 2

We denote the generating functions of $\mathrm{Li}^\star_{\mathbf{k}}$ as follows:

$$\begin{cases} \Psi^\star(u,v;t,q) := \displaystyle\sum_{k\geq r>0}\left(\sum_{\mathbf{k}\in I(k,r)} \mathrm{Li}^\star_{\mathbf{k}}[t]\right) u^{k-r}v^{r-1}, \\ \Psi^\star_0(u,v;t,q) := \displaystyle\sum_{k>r>0}\left(\sum_{\mathbf{k}\in I_0(k,r)} \mathrm{Li}^\star_{\mathbf{k}}[t]\right) u^{k-r-1}v^{r-1}. \end{cases}$$

To investigate the above generating functions we use the q-differential equation, where the q-differential operator D_q is defined by

$$(D_q f)(t) := \frac{f(t)-f(qt)}{t-qt}.$$

From the q-differential equation for $\mathrm{Li}^\star_{\mathbf{k}}$

$$D_q \mathrm{Li}^\star_{k_1,k_2,\dots,k_r}[t] = \begin{cases} \dfrac{1}{t}\ \mathrm{Li}^\star_{k_1-1,k_2,\dots,k_r}[t] & (k_1\geq 2), \\ \dfrac{1}{t}\dfrac{1}{1-t}\ \mathrm{Li}^\star_{k_2,\dots,k_r}[t] & (k_1=1 \text{ and } r\geq 2), \\ \dfrac{1}{1-t} & (k_1=1 \text{ and } r=1), \end{cases}$$

$\Psi^\star$ and $\Psi^\star_0$ satisfy the following q-differential equations:

$$\begin{cases} D_q\Psi^\star_0(u,v;t,q) = \dfrac{1}{t}\Psi^\star(u,v;t,q), \\ D_q(\Psi^\star - u\Psi^\star_0)(u,v;t,q) = \dfrac{1}{1-t} + \dfrac{1}{t}\dfrac{1}{1-t} v\Psi^\star(u,v;t,q). \end{cases}$$

By eliminating $\Psi^\star$ from the above equations, we have that $\Psi^\star_0$ satisfies the inhomogeneous linear q-differential equation of second order:

$$qt(1-t)D_q^2 f + \{(1-t)(1-u)-v\}D_q f = 1. \tag{7}$$

$\Psi^\star_0$ is characterized as the regular solution of (7) around the origin and the value at the origin is 0.

We must find such a solution of (7) in another way. At first we put $g := D_q f$ and solve the equation

$$qt(1-t)D_q g + \{(1-t)(1-u)-v\}g = 1, \tag{8}$$

by variation of parameter. We choose $C_0\, t^a (t;q)_\infty/(bt;q)_\infty$ for the solution of the homogeneous equation

$$qt(1-t)D_q h + \{(1-t)(1-u)-v\}h = 0,$$

where $q^{-a-1} = \frac{1}{1-(1-q)(u+v)}$, $b = \frac{1-(1-q)u}{1-(1-q)(u+v)}$ and $C_0\in\mathbb{C}$. We assume that

$$g(t) = C(t)\ t^a \frac{(t;q)_\infty}{(bt;q)_\infty},$$

and substitute this into (8); then we have

$$C'(t) = q^{-a-1}t^{-a-1}\frac{(bqt;q)_\infty}{(t;q)_\infty}.$$

The Jackson integral of $C'(t)$ is as follows:

$$\begin{aligned}\int_0^t q^{-a-1}s^{-a-1}\frac{(bqs;q)_\infty}{(s;q)_\infty}\,d_qs &= q^{-a-1}\int_0^t s^{-a-1}\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}s^n\,d_qs\\ &= q^{-a-1}\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}\frac{t^{n-a}}{[n-a]}\\ &= \frac{t^{-a}}{1-u-v}{}_2\phi_1(q^{-a},bq,q^{-a+1};t,q),\end{aligned}$$

where the first equality is by virtue of the q-binomial theorem [9]. So we obtain the solution of (8) which is regular at the origin:

$$g(t)=\frac{1}{1-u-v}\frac{(t;q)_\infty}{(bt;q)_\infty}{}_2\phi_1(q^{-a},bq,q^{-a+1};t,q).$$

We consider the Jackson integral again and get the solution of (7):

$$f(t)=\frac{1}{1-u-v}\int_0^t\frac{(s;q)_\infty}{(bs;q)_\infty}{}_2\phi_1(q^{-a},bq,q^{-a+1};s,q)\,d_qs.$$

By executing the Jackson integral, we have

$$\begin{aligned}f(t)&=\frac{1}{1-u-v}\sum_{n=0}^{\infty}\frac{(1-q^{-a})(bq;q)_n}{(1-q^{n-a})(q;q)_n}\int_0^t s^n\frac{(s;q)_\infty}{(bs;q)_\infty}\,d_qs\\ &=\frac{1}{1-u-v}\sum_{n=0}^{\infty}\frac{(1-q^{-a})(bq;q)_n}{(1-q^{n-a})(q;q)_n}(1-q)t^{n+1}\frac{(t)_\infty}{(bt)_\infty}\sum_{j=0}^{\infty}q^{j(n+1)}\frac{(bt)_j}{(t)_j},\end{aligned}\tag{9}$$

which is zero at $t=0$. Thus we obtain the theorem.

In the same way as [21], the special value of $\mathrm{Li}^\star$ and the generating function are expressed by the combination of the q-analogue of MZSVs : Substitute $t=q$ and the value is

$$\begin{aligned}\mathrm{Li}^\star_{k_1,k_2,\dots,k_r}[q]=\sum_{a_1=0}^{k_1-2}\sum_{a_2=0}^{k_2-1}\cdots\sum_{a_r=0}^{k_r-1}&\binom{k_1-2}{a_1}\binom{k_2-1}{a_2}\cdots\binom{k_r-1}{a_r}\\ &\times(1-q)^{k_1+\cdots+k_r-a_1-\dots-a_r}\zeta_q^\star[a_1,a_2,\dots,a_r],\end{aligned}$$

and the generating function is

$$\Psi^\star(u,v;q,q)=\frac{1}{1+(1-q)x}\sum_{k>r>0}\left\{\sum_{\mathbf{k}\in I_0(k,r)}\zeta_q^\star[\mathbf{k}]\right\}x^{k-r-1}y^{r-1},$$

where $x=\frac{u}{1-(1-q)u}$ and $y=\frac{v}{1-(1-q)u}$.

On the other hand, by substituting $t = q$ to (9),

$$\begin{aligned}
\Psi^\star(u,v;q,q) &= \frac{1}{1-u-v}\sum_{n=0}^{\infty}\frac{(1-q^{-a})(bq;q)_n}{(1-q^{n-a})(q;q)_n}(1-q)q^{n+1}\frac{(q;q)_\infty}{(bq;q)_\infty} \\
&\quad\times\sum_{j=0}^{\infty}q^{j(n+1)}\frac{(bq;q)_j}{(q;q)_j} \\
&= \frac{1-q}{1-u-v}\sum_{n=0}^{\infty}\frac{(1-q^{-a})(bq;q)_n}{(1-q^{n-a})(q;q)_n}q^{n+1}\frac{(q;q)_\infty}{(bq;q)_\infty}\frac{(bq^{n+2};q)}{(q^{n+1};q)} \\
&= \frac{1-q}{1-u-v}\sum_{n=0}^{\infty}\frac{(1-q^{-a})}{(1-q^{n-a})}\frac{q^{n+1}}{(1-bq^{n+1})} \\
&= \sum_{n=1}^{\infty}\frac{q^n}{[n]-(u+v)}\frac{1-(1-q)(u+v)}{[n]-(1-q^n)u-v} \\
&= \frac{1}{1+(1-q)x}\sum_{n=1}^{\infty}\frac{q^n(1-(1-q)y)}{([n]-y)([n]-xq^n-y)}.
\end{aligned}$$

Hence we have

$$\sum_{k>r>0}\left\{\sum_{\mathbf{k}\in I_0(k,r)}\zeta_q^\star[\mathbf{k}]\right\}x^{k-r-1}y^{r-1} = \sum_{n=1}^{\infty}\frac{q^n(1-(1-q)y)}{([n]-y)([n]-xq^n-y)},$$

and expanding the right hand by geometric series, we obtain Corollary 3.

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