PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 10, October 2007, Pages 3367–3373
S 0002-9939(07)08997-6
Article electronically published on June 20, 2007

CROSS i-SECTIONS OF STAR BODIES AND DUAL MIXED VOLUMES

SONGJUN LV AND GANGSONG LENG

(Communicated by Jonathan M. Borwein)

Abstract. In this paper, we establish an extension of Funk's section theorem. Our result has the following corollary: If K is a star body in $\mathbb{R}^n$ whose central i-slices have the same volume (with appropriate dimension) as the central i-slices of a centered body M, then the dual quermassintegrals satisfy $\widetilde{W}_j(M) \leq \widetilde{W}_j(K)$, for any $0 \leq j < n-i$, with equality if and only if $K = M$. The case that K is a centered body implies Funk's section theorem.

1. Introduction

Let $G(n,i)$ denote the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$, and let $\mathrm{vol}_i(\cdot)$ be the i-dimensional Lebesgue measure. Let B denote the Euclidean unit ball, and let S^{n-1} be the Euclidean sphere in $\mathbb{R}^n$. A compact, convex set in $\mathbb{R}^n$ is said to be a *convex body* if it has a non-empty interior. Let $\mathcal{K}^n$, $\mathcal{K}_e^n$ be the classes of convex bodies and of origin-symmetric convex bodies, respectively. For $K \in \mathcal{K}^n$ and $\xi \in G(n,i)$, let $K|\xi$ denote the image of the orthogonal projection of K onto ξ. We will also work with general *star bodies* L, which are star-shaped bodies, meaning that $tL \subset L$ for all $t \in [0,1]$, with the additional requirement that their radial functions $\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}$ for $u \in S^{n-1}$ are continuous functions on S^{n-1}. Denote by $\mathcal{S}^n$, $\mathcal{S}_e^n$ the classes of star bodies and of origin-symmetric star bodies, respectively. For $M \in \mathcal{S}^n$ and $\xi \in G(n,i)$, let $M \cap \xi$ denote the intersection of M and ξ.

The well-known Aleksandrov's projection theorem (see [1] and [4]) states that if $K, L \in \mathcal{K}_e^n$ and if for some i such that $0 < i < n$,

$$\mathrm{vol}_i(K|\xi) = \mathrm{vol}_i(L|\xi) \ \forall \xi \in G(n,i),$$

then $K = L$.

In [2], Chakerian and Lutwak gave the following extension of Aleksandrov's projection theorem (in fact, they established a more general result):

Theorem 1.1 (Chakerian and Lutwak [2]). *If $K \in \mathcal{K}_e^n$, $L \in \mathcal{K}^n$ and if for some i such that $0 < i < n$,*

$$\mathrm{vol}_i(K|\xi) = \mathrm{vol}_i(L|\xi) \ \forall \xi \in G(n,i),$$

Received by the editors June 19, 2006.
2000 *Mathematics Subject Classification.* Primary 52A30, 52A40.
Key words and phrases. Star body, cross i-section, dual mixed volume, Radon transform.
This research was supported, in part, by NSFC Grant 10671117.

then

$$\mathrm{vol}_n(K) \geq \mathrm{vol}_n(L),$$

with equality if and only if L is a translate of K.

A dual form of Aleksandrov's projection theorem is the well-known Funk's section theorem (see [3] and [4]), which states that if $K, L \in \mathcal{S}_e^n$ and if for some i such that $0 < i < n$,

$$\mathrm{vol}_i(K \cap \xi) = \mathrm{vol}_i(L \cap \xi) \ \forall \xi \in G(n,i),$$

then $K = L$.

In this paper, we will give an extension of Funk's section theorem, which is a dual form of Chakerian and Lutwak's result (Theorem 1.1). Our result is the following theorem, which is a special case of Corollary 3.5 of this paper:

Theorem 1.2. *If $K \in \mathcal{S}_e^n$, $L \in \mathcal{S}^n$ and if for some i such that $0 < i < n$,*

$$\mathrm{vol}_i(K \cap \xi) = \mathrm{vol}_i(L \cap \xi) \ \forall \xi \in G(n,i),$$

then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L),$$

with equality if and only if $K = L$.

Obviously, Funk's section theorem is an immediate consequence: If K, L are both in $\mathcal{S}_e^n$, and

$$\mathrm{vol}_i(K \cap \xi) = \mathrm{vol}_i(L \cap \xi) \ \forall \xi \in G(n,i),$$

then Theorem 1.2 gives both

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L) \text{ and } \mathrm{vol}_n(L) \leq \mathrm{vol}_n(K).$$

But the equality conditions of the inequality of Theorem 1.2 now show that $K = L$.

The key idea used by Chakerian and Lutwak in [2] to prove Theorem 1.1 is from Weil [14, 15, 16] and Schneider and Weil [13]. However, it seems that there is no "dual" method to arrive at Theorem 1.2, and we use an entirely "non-dual" approach to prove it. Our main tools are Radon transforms and the dual mixed volume theory which was developed by Lutwak in [9] and [10].

2. Notation and background material

Let $C(S^{n-1})$ be the space of continuous functions on the unit sphere S^{n-1}, and let $C_e(S^{n-1})$ be the subspace of $C(S^{n-1})$ that contains the even continuous functions on S^{n-1}. The subset of $C_e(S^{n-1})$ that contains the infinitely differentiable functions will be denoted by $C_e^\infty(S^{n-1})$. Denote by $C(G(n,i))$ the space of continuous functions on $G(n,i)$. For $f \in C(S^{n-1})$, $g \in C(G(n,i))$, $1 \leq i \leq n-1$, the i-dimensional spherical Radon transform $R_i f$ and its dual transform $R_i^t g$ are defined by

$$(R_i f)(\xi) = \int_{S^{n-1}\cap\xi} f(u)d\sigma_i(u), \ (R_i^t g)(u) = \int_{\xi\in G(n,i)} g(\xi)d\upsilon_i(\xi), \tag{2.1}$$

where σ_i is the Haar probability measure on S^{i-1} (and we have identified S^{i-1} with $S^{n-1} \cap \xi$), and υ_i is the Haar probability measure on the homogeneous space $\{\xi \in G(n,i) : u \in \xi\}$.

The corresponding duality relation reads (see [7, 8] or [12])

$$\int_{G(n,i)} (R_i f)(\xi) g(\xi) d\xi = \int_{S^{n-1}} f(u)(R_i^t g)(u)du. \tag{2.2}$$

This allows us to define $R_i\mu$ and $R_i^t\nu$ for arbitrary finite Borel measures μ on S^{n-1} and ν on $G(n,i)$ as follows:

$$\int_{G(n,i)}(R_i\mu)(\xi)g(\xi)d\xi=\int_{S^{n-1}}(R_i^tg)(u)d\mu(u),\ g\in C(G(n,i)); \tag{2.3}$$

$$\int_{S^{n-1}}(R_i^t\nu)(u)f(u)du=\int_{G(n,i)}(R_if)(\xi)d\nu(\xi),\ f\in C(S^{n-1}). \tag{2.4}$$

We will also write (2.2), (2.3), and (2.4) briefly as

$$(R_if,g)=(f,R_i^tg),\ (R_i\mu,g)=(\mu,R_i^tg),\ (R_i^t\nu,f)=(\nu,R_if).$$

We shall say that an origin-symmetric star body K in $\mathbb{R}^n$ is an i-intersection body (see [5] and [18]) if there is a non-negative Borel measure μ on $G(n,i)$ such that $\rho_K^{n-i}=R_i^t\mu$. Denote by $\mathcal{I}_i^n$ the class of i-intersection bodies. The case $i=n-1$ is associated with the notion of intersection body due to Lutwak [9] (we refer to a slightly more general notion than the original notion by Lutwak). We shall simply denote the class of intersection bodies by $\mathcal{I}^n$ (rather than $\mathcal{I}_{n-1}^n$).

The k-*radial sum* of two star bodies L_1,L_2 is defined as the star body L satisfying $\rho_L^k=\rho_{L_1}^k+\rho_{L_2}^k$. When $k=1$ this operation will simply be referred to as the *radial sum* and denoted by $\widetilde{+}$. It is well known (e.g. [5], [11]) that the class of i-intersection bodies is closed under taking k-radial sums.

We shall assume that the space of star bodies in $\mathbb{R}^n$ is always endowed with the natural *radial metric* $\widetilde{\delta}$, defined by $\widetilde{\delta}(L_1,L_2)=\max\limits_{u\in S^{n-1}}|\rho_{L_1}(u)-\rho_{L_2}(u)|$. For i star bodies $K_1,\cdots,K_i$ and $\xi\in G(n,i)$, *the dual mixed volume*, $\widetilde{v}_\xi(K_1\cap\xi,\cdots,K_i\cap\xi)$, of $K_1\cap\xi,\cdots,K_i\cap\xi$ is defined by

$$\widetilde{v}_\xi(K_1\cap\xi,\cdots,K_i\cap\xi)=\frac{1}{i}\int_{S^{n-1}\cap\xi}\rho_{K_1}(u)\cdots\rho_{K_i}(u)du. \tag{2.5}$$

If $K_1=\cdots=K_i$, then we get the ith section volume function of K:

$$\mathrm{vol}_i(K\cap\xi)=\frac{1}{i}\int_{S^{n-1}\cap\xi}\rho_K^i(u)du. \tag{2.6}$$

Thus, by (2.1), the Radon transform R_i has the following close connection with the central sections of star bodies:

$$(R_i\rho_K^i)(\xi)=\frac{1}{\omega_i}\mathrm{vol}_i(K\cap\xi),\ \xi\in G(n,i). \tag{2.7}$$

When $i=n$ in (2.5), the dual mixed volume of the star bodies $K_1,\cdots,K_n$ is denoted by $\widetilde{V}(K_1,\cdots,K_n)$, and the dual mixed volume $\widetilde{V}(K,n-i;B,i)$ is called the *dual quermassintegral* of K, denoted by $\widetilde{W}_i(K)$, where there are $n-i$ copies of K and i copies of the unit ball B in $\mathbb{R}^n$.

3. Main result and its proof

In this section, we shall establish the following theorem, which has Theorem 1.2 as a direct consequence.

Theorem 3.1. *If $K_1,\cdots,K_i\in\mathcal{S}^n, M\in\mathcal{S}_e^n$, and*

$$\widetilde{v}_\xi(K_1\cap\xi,\cdots,K_i\cap\xi)=\mathrm{vol}_i(M\cap\xi)\ \forall\xi\in G(n,i),$$

then

$$\widetilde{W}_{n-i}(M) = \widetilde{V}(K_1, \cdots, K_i, \underbrace{B, \cdots, B}_{n-i}),$$

and for all j such that $0 \le j < n-i$,

$$\widetilde{W}_j(M)^i \le \widetilde{W}_j(K_1) \cdots \widetilde{W}_j(K_i),$$

with equality, for any j, implying that $K_1, \cdots, K_i$ all are dilations of M.

To prove Theorem 3.1, the class of *generalized intersection bodies*, introduced by Zhang in [17], is needed. Therefore, we give the notion and some properties of generalized intersection bodies in advance.

For the case $i = n-1$, the i-dimensional spherical Radon transform, R_{n-1}, will be simply denoted by R. When restricted to $C_e^\infty(S^{n-1})$, the spherical Radon transform $R : C_e^\infty(S^{n-1}) \to C_e^\infty(S^{n-1})$ is a continuous bijection (see [8] and [17], Lemma 5.7). For $K \in \mathcal{S}_e^n$, the distribution $R^{-1}\rho_K$ is called the dual generating distribution of K, denoted by $\widetilde{\mu}_K$. If $\widetilde{\mu}_K$ is a measure, then $\rho_K = R\widetilde{\mu}_K$ determines an intersection body K. A body $K \in \mathcal{S}_e^n$ is called a generalized intersection body if the dual generating distribution $\widetilde{\mu}_K$ is a signed measure. Denote by $\mathcal{I}_g^n$ the set of generalized intersection bodies. If $Q \in \mathcal{I}_g^n$, then $\widetilde{\mu}_Q = R^{-1}\rho_Q$ is a signed measure on S^{n-1}. Thus by the Jordon decomposition of $\widetilde{\mu}_Q$, there exist measures μ_1, μ_2 on S^{n-1} such that $\widetilde{\mu}_Q = \mu_1 - \mu_2$. Now from the fact that $\widetilde{\mu}_Q$ is finite, it is easily shown that μ_1 and μ_2 are both finite (see [6]). For $i = 1, 2$, define

$$\widetilde{\mu}_i(E) = \frac{1}{2}[\mu_i(E) + \mu_i(-E)],$$

where E is a Borel set on S^{n-1}. Then $\widetilde{\mu}_1, \widetilde{\mu}_2$ are even finite measures on S^{n-1}. Since $\widetilde{\mu}_Q$ also is even, we have $\widetilde{\mu}_Q = \widetilde{\mu}_1 - \widetilde{\mu}_2$. Let Q_1, Q_2 be intersection bodies determined by $\widetilde{\mu}_1, \widetilde{\mu}_2$, respectively. Then $\rho_Q = \rho_{Q_1} - \rho_{Q_2}$. But this implies that $Q_1 = Q \mathbin{\widetilde{+}} Q_2$. That is to say, we can define a body in $\mathcal{I}_g^n$ in the following different way: A body $K \in \mathcal{S}_e^n$ is said to be a generalized intersection body if there exists an $M \in \mathcal{I}^n$ such that $K \mathbin{\widetilde{+}} M \in \mathcal{I}^n$.

As main ingredients in the proof of Theorem 3.1, the following lemmas are also required.

Lemma 3.2 (Milman [11])**.** *Let $K_1 \in \mathcal{I}_{n-i_1}^n$, and $K_2 \in \mathcal{I}_{n-i_2}^n$ for $i_1, i_2 < n$ such that $i_3 = i_1 + i_2 < n$. Then the origin-symmetric star body K_3 defined by $\rho_{K_3}^{i_3} = \rho_{K_1}^{i_1}\rho_{K_2}^{i_2}$ satisfies $K_3 \in \mathcal{I}_{n-i_3}^n$.*

Lemma 3.3 (Zhang [17])**.** *The generalized intersection bodies are dense in the class of origin-symmetric star bodies.*

Lemma 3.4. *If $K_1, \cdots, K_i \in \mathcal{S}^n$ $(1 \le i \le n-1), L \in \mathcal{S}_e^n$, and*

$$\widetilde{v}_\xi(K_1 \cap \xi, \cdots, K_i \cap \xi) = \mathrm{vol}_i(L \cap \xi) \ \forall \xi \in G(n,i), \tag{3.1}$$

then

$$\widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i}) = \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i}), \tag{3.2}$$

for all $Q_j \in \mathcal{S}_e^n$.

Proof. Define a star body K by $\rho_K^i = \rho_{K_1} \cdots \rho_{K_i}$. Then

$$\widetilde{v}_\xi(K_1 \cap \xi, \cdots, K_i \cap \xi) = \mathrm{vol}_i(K \cap \xi) \ \forall \xi \in G(n,i). \tag{3.3}$$

Let $Q_j \in \mathcal{I}^n$ $(j = 1, \cdots, n-i)$. Applying Lemma 3.2 to $\{Q_j\}$ successively shows that the origin-symmetric star body Q defined by $\rho_Q^{n-i} = \rho_{Q_1} \cdots \rho_{Q_{n-i}}$ is an i-intersection body. Thus there exists a non-negative measure μ on $G(n,i)$ so that $\rho_Q^{n-i} = R_i^t \mu$.

From (3.1), (3.3), (2.4) and (2.7), we have

$$\begin{aligned}
&\widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i}) - \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i}) \\
&= \frac{1}{n} \int_{S^{n-1}} \rho_K^i \rho_Q^{n-i} du - \frac{1}{n} \int_{S^{n-1}} \rho_L^i \rho_Q^{n-i} du \\
&= \frac{1}{n} \left[(\rho_K^i, \rho_Q^{n-i}) - (\rho_L^i, \rho_Q^{n-i}) \right] \\
&= \frac{1}{n} \left[(\rho_K^i, R_i^t \mu) - (\rho_L^i, R_i^t \mu) \right] \\
&= \frac{1}{n} \left[(R_i \rho_K^i, \mu) - (R_i \rho_L^i, \mu) \right] \\
&= \frac{1}{n\omega_i} \int_{G(n,i)} \left[\mathrm{vol}_i(K \cap \xi) - \mathrm{vol}_i(L \cap \xi) \right] d\mu(\xi) \\
&= \frac{1}{n\omega_i} \int_{G(n,i)} \left[\widetilde{v}_\xi(K_1 \cap \xi, \cdots, K_i \cap \xi) - \mathrm{vol}_i(L \cap \xi) \right] d\mu(\xi).
\end{aligned}$$

This proves that (3.2) must hold for all $Q_j \in \mathcal{I}^n$.

Next we show that (3.2) holds for all $Q_j \in \mathcal{I}_g^n$. Since the dual mixed volume $\widetilde{V}$ is symmetric in its arguments, we need only to show that (3.2) holds for one of $\{Q_j\} \subset \mathcal{I}_g^n$. Without loss of generality, suppose $Q \in \mathcal{I}_g^n$ and that Q is the last argument of $\widetilde{V}$. By the alternative definition of generalized intersection body, there exists an $M \in \mathcal{I}^n$ such that $Q \mathbin{\widetilde{+}} M \in \mathcal{I}^n$. Since (3.2) holds for any body in $\{Q_j\} \subset \mathcal{I}^n$, we have

$$\widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q \mathbin{\widetilde{+}} M) = \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i-1}, Q \mathbin{\widetilde{+}} M). \tag{3.4}$$

But from the definition of the radial sum and (2.5),

$$\begin{aligned}
\widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q \mathbin{\widetilde{+}} M) &= \widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, Q) \\
&\quad + \widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, M),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i-1}, Q \mathbin{\widetilde{+}} M) &= \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i-1}, Q) \\
&\quad + \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i-1}, M).
\end{aligned} \tag{3.6}$$

Again by the fact that (3.2) holds for any body in $\{Q_j\} \subset \mathcal{I}^n$, from $M \in \mathcal{I}^n$ it follows that

$$\widetilde{V}(K_1, \cdots, K_i, Q_1, \cdots, Q_{n-i-1}, M) = \widetilde{V}(\underbrace{L, \cdots, L}_{i}, Q_1, \cdots, Q_{n-i-1}, M).$$

Then from (3.4), (3.5), and (3.6) we get

$$\widetilde{V}(K_1,\cdots,K_i,Q_1,\cdots,Q_{n-i-1},Q)=\widetilde{V}(\underbrace{L,\cdots,L}_{i},Q_1,\cdots,Q_{n-i-1},Q),$$

where $Q\in\mathcal{I}_g^n$. This proves that (3.2) holds for all $Q_j\in\mathcal{I}_g^n$.

By Lemma 3.3, every member of $\mathcal{S}_e^n$ is the limit of generalized intersection bodies in the radial metric $\widetilde{\delta}$, and it follows from the continuity of dual mixed volumes that (3.2) must hold for all $Q_j\in\mathcal{S}_e^n$ $(j=1,\cdots,n-i)$. □

Proof of Theorem 3.1. Taking $Q_1=\cdots=Q_{n-i}=B$ in Lemma 3.4 immediately gives

$$\widetilde{W}_{n-i}(L)=\widetilde{V}(K_1,\cdots,K_i,\underbrace{B,\cdots,B}_{n-i}).$$

Suppose $0\le j<n-i$. Take $Q_1=\cdots=Q_j=B$ and $Q_{j+1}=\cdots=Q_{n-i}=L$ in Lemma 3.4, and get

$$\widetilde{W}_j(L)=\widetilde{V}(K_1,\cdots,K_i,\underbrace{L,\cdots,L}_{n-i-j},\underbrace{B,\cdots,B}_{j}). \tag{3.7}$$

Applying Hölder's inequality to the right-hand side of (3.7), we get

$$\widetilde{W}_j(L)^{n-j-1}\le\widetilde{W}_j(L)^{n-j-1-i}\prod_{l=1}^{i}\widetilde{V}(\underbrace{K_l,\cdots,K_l}_{n-j-1},L,\underbrace{B,\cdots,B}_{j}). \tag{3.8}$$

Again applying the Hölder inequality to the dual mixed volumes in the product on the right-hand side of (3.8), we have

$$\widetilde{W}_j(L)^i\le\widetilde{W}_j(K_1)\cdots\widetilde{W}_j(K_i);$$

the equality condition now follows from that of the Hölder inequality, that is, $K_1,\cdots,K_i$ are dilations of L. This proves the theorem. □

Of interest is the special case $K_1=\cdots=K_i=K$ of Theorem 3.1:

Corollary 3.5. *Let $L\in\mathcal{S}_e^n, K\in\mathcal{S}^n$, and*

$$\mathrm{vol}_i(K\cap\xi)=\mathrm{vol}_i(L\cap\xi)\ \forall\xi\in G(n,i).$$

Then $\widetilde{W}_{n-i}(L)=\widetilde{W}_{n-i}(K)$, and for all j such that $0\le j<n-i$,

$$\widetilde{W}_j(L)\le\widetilde{W}_j(K),$$

with equality, for any j, if and only if $K=L$.

Obviously, taking $j=0$ in Corollary 3.5, we immediately get Theorem 1.2 presented in Section 1.

Acknowledgments

The authors would like to thank Professor Gaoyong Zhang for many valuable suggestions for improving the original manuscript.

References

1. A. D. Aleksandrov, *On the theory of mixed volumes. II. New inequalities beteeen mixed volumes and their application*, Mat. Sbornik N. S. 2 (1937), 1205-1238, Russian.
2. G. D. Chakerian, E. Lutwak, *Bodies with similar projections*, Trans. Amer. Math. Soc. 349 (1997), 1811-1820. MR1390034 (98a:52011)
3. P. Funk, *Über Flächen mit lauter geschlossenen geodätischen Linien*, Math. Ann. 74 (1913), 278-300. MR1511763
4. R. J. Gardner, *Geometric Tomography*, New York, Cambridge Univ. Press, 1995. MR1356221 (96j:52006)
5. E. Grinberg, G. Zhang, *Convolutions, transforms and convex bodies*, Proc. London Math. Soc. 78 (1999), 77-115. MR1658156 (99m:52009)
6. P. R. Halmos, *Measure Theory*, Princeton, Van Nostrand, 1950. MR0033869 (11:504d)
7. S. Helgason, *Groups and geometric analysis*, Academic Press, 1984. MR754767 (86c:22017)
8. S. Helgason, *The Radon transform*, Birkhäuser, Boston, Second edition, 1999. MR1723736 (2000m:44003)
9. E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. 71 (1988), 232-261. MR963487 (90a:52023)
10. E. Lutwak, *Dual mixed volumes*, Pacific J. Math. 58 (1975), 531-538. MR0380631 (52:1528)
11. E. Milman, *Generalized intersection bodies*, J. Funct. Anal. 240 (2006), 530-567. MR2261694
12. B. Rubin, *Inversion formulas for the spherical Radon transform and the generalized cosine transform*, Adv. Appl. Math. 29 (2002), 471-497. MR1942635 (2004c:44006)
13. R. Schneider and W. Weil, *Zonoids and related topics, Convexity and its Applications* (P.M. Gruber and J.M. Wills, Eds.), Birkhäuser, Basel, 1983, pp. 296-317. MR731116 (85c:52010)
14. W. Weil, *Über den Vektorraum der Differenzen von Stützfunktionen konvexer Körper*, Math. Nachr. 59 (1974), 353-369. MR0341283 (49:6033)
15. W. Weil, *Kontinuierliche Linearkombination von Strecken*, Math. Z. 148 (1976), 71-84. MR0400052 (53:3887)
16. W. Weil, *Centrally symmetric convex bodies and distributions II*, Israel J. Math. 32 (1979), 143-182. MR531260 (80g:52003)
17. G. Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. 345 (1994), 777-801. MR1254193 (95d:52008)
18. G. Zhang, *Sections of convex bodies*, Amer. J. Math. 118 (1996), 319-340. MR1385280 (97f:52015)

Department of Mathematics, Shanghai University, Shanghai, People's Republic of China 200444

E-mail address: lvsongjun@126.com

Department of Mathematics, Shanghai University, Shanghai, People's Republic of China 200444

E-mail address: gleng@staff.shu.edu.cn