ON THE ALGEBRAIC CLOSURE IN RINGS

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Abstract. The “algebraic closure” of a subset \( K \subseteq A \) of a ring is an algebraic analogue of topological closure.

Suppose \( A \) is a ring, or additive category, with identity 1 and invertible group \( A^{-1} \). Then we make the following

Definition 1. The algebraic closure of a subset \( K \subseteq A \) is the set

\[
\text{cl}_{\text{alg}}(K) = \{ a \in A : \forall b, c \in A \exists a' \in K , 1 - b(a - a')c \in A^{-1} \}. \tag{1.1}
\]

Equivalently

\[
a \in \text{cl}_{\text{alg}}(K) \iff \forall b \in A \exists a' \in K , 1 - b(a - a') \in A^{-1} \tag{1.2}
\]

for if this holds, then for arbitrary \( b, c \in A \) there is \( a' \in A \) with

\[
1 - cb(a - a') \in A^{-1} \iff 1 - b(a - a')c \in A^{-1} .
\]

For example if \( A \) is a Banach algebra, then the norm closure is a subset of the algebraic closure: using the geometric series \([10]\)

\[
\|z\|\|a-a'\|\|w\| < 1 \implies 1 - z(a - a')w \in A^{-1}. \tag{1.3}
\]

The reverse inclusion is not clear: for example

\[
\text{cl}_{\text{alg}}\{a\} = a + \text{Rad}(A), \tag{1.4}
\]

the algebraic closure of a singleton is the coset modulo the Jacobson radical.

The algebraic closure has almost all of the properties of topological closure, and is compatible with the algebraic operations:

Theorem 2. For arbitrary \( K, H \subseteq A \) there is inclusion

\[
K \subseteq \text{cl}_{\text{alg}}(K) , \tag{2.1}
\]

implication

\[
K \subseteq H \implies \text{cl}_{\text{alg}}(K) \subseteq \text{cl}_{\text{alg}}(H) , \tag{2.2}
\]

and inclusion (necessarily equality)

\[
\text{cl}_{\text{alg}}\text{cl}_{\text{alg}}(K) \subseteq \text{cl}_{\text{alg}}(K) . \tag{2.3}
\]
There is also inclusion

\[(2.4) \quad \text{cl}_{\text{alg}}(K) + \text{cl}_{\text{alg}}(H) \subseteq \text{cl}_{\text{alg}}(K + H) \]

and

\[(2.5) \quad \text{cl}_{\text{alg}}(K) \cdot \text{cl}_{\text{alg}}(H) \subseteq \text{cl}_{\text{alg}}(K \cdot H). \]

**Proof.** \((2.1)\) and \((2.2)\) are clear; towards \((2.3)\), if \(x \in \text{cl}_{\text{alg}}(K)\), then for arbitrary \(z, w \in A\) there is \(x' \in \text{cl}_{\text{alg}}(K)\) for which \(1 - z(x - x')w = c \in A^{-1}\), and then \(x'' \in K\) for which \(1 - c^{-1}z(x' - x'')w \in A^{-1}\). Now

\[1 - z(x - x'')w = 1 - z(x - x')w - z(x' - x'')w = c(1 - c^{-1}z(x' - x'')w) \in A^{-1}.\]

Towards \((2.4)\) and \((2.5)\), if \(x \in \text{cl}_{\text{alg}}(K)\) and \(y \in \text{cl}_{\text{alg}}(H)\) and if \(z, w \in A\), then, with \(1 - z(x - x')w = c \in A^{-1}\),

\[1 - z(x + y - x' - y')w = 1 - z(x - x')w - z(y - y')w \]

\[= (1 - z(x - x')w)(1 - c^{-1}z(y - y')w) \in A^{-1}A^{-1} = A^{-1} \]

provided \(1 - c^{-1}z(y - y')w \in A^{-1}\) and, with \(1 - z(x - x')yw = d \in A^{-1}\),

\[1 - z(xy - x'y')w = 1 - z(x - x')yw - z(x' - y')w \]

\[= (1 - z(x - x')yw)(1 - d^{-1}z(x' - y')w) \in A^{-1} \]

provided \(1 - d^{-1}z(x' - y')w \in A^{-1} \). □

One of the Kuratowski closure axioms however seems to fail:

**Example 3.** For \(K, H \subseteq A\), inclusion

\[(3.1) \quad \text{cl}_{\text{alg}}(K \cup H) \subseteq \text{cl}_{\text{alg}}(K) \cup \text{cl}_{\text{alg}}(H) \]

is liable to fail.

**Proof.** In the ring \(A = \mathbb{C}^2\) take

\[(3.2) \quad K = \{1\} \times \mathbb{C}, \quad H = \{-1\} \times \mathbb{C}\]

and claim

\[(3.3) \quad (2, 3) \notin \text{cl}_{\text{alg}}(K \cup H) \setminus (\text{cl}_{\text{alg}}K \cup \text{cl}_{\text{alg}}H). \]

To see that \((2, 3) \notin \text{cl}_{\text{alg}}K\) notice

\[(1, 1) - (1, 1)((2, 3) - (1, t)) = (0, t - 2) \notin A^{-1}. \]

To see that \((2, 3) \notin \text{cl}_{\text{alg}}H\) notice

\[(1, 1) - (1/3, 1)((2, 3) - (-1, t)) = (0, t - 2) \notin A^{-1}. \]

Finally to see that \((2, 3) \in \text{cl}_{\text{alg}}(K \cup H)\) suppose \((r, s) \in A\) is arbitrary. If \(r \neq 1\), then for arbitrary \(t \in \mathbb{C}\)

\[(1, 1) - (r, s)((2, 3) - (1, t)) = (1 - r, 1 - s(t - 3)) \in A^{-1}\] if \(1 - s(3 - t) \neq 0 \).

If \(r = 1\), then for arbitrary \(t \in \mathbb{C}\)

\[(1, 1) - (r, s)((2, 3) - (-1, t)) = (-2, 1 - s(3 - t)) \in A^{-1}\] if \(1 - s(3 - t) \neq 0 \). □

The algebraic closure of the invertibles participates in a curious minuet with the left and the right invertibles:
Theorem 4. There is equality
\begin{equation}
A_{\text{left}}^{-1} \cap \cl_{\text{alg}} A^{-1} = A_{\text{right}}^{-1} \cap \cl_{\text{alg}} A^{-1},
\end{equation}
and inclusion
\begin{equation}
\cl_{\text{alg}} A^{-1} \subseteq A^{-1} + A^{-1} + A^{-1}.
\end{equation}
Necessary and sufficient for
\begin{equation}
0 \in \cl_{\text{alg}} A^{-1}
\end{equation}
is inclusion
\begin{equation}
A \subseteq A^{-1} + A^{-1}.
\end{equation}

Proof. Towards (4.1) we prove more, and claim
\begin{equation}
A_{\text{left}}^{-1} \cap \cl_{\text{alg}} A_{\text{right}}^{-1} = A_{\text{right}}^{-1} \cap \cl_{\text{alg}} A_{\text{left}}^{-1}.
\end{equation}
Towards the first equality suppose \( a \in A_{\text{left}}^{-1} \cap \cl_{\text{alg}} A_{\text{right}}^{-1} \) so that there are \( a', b, b' \in A \) for which \( a' a = 1 = b b' \) with \( 1 - a'(a - b) \in A^{-1} \). But this means \( a' a = 1 \) with \( a' b \in A^{-1} \); hence, \( b \in A_{\text{left}}^{-1} \) and so \( b \in A_{\text{right}}^{-1} \). The argument for the second equality in (4.5) is the same.

Towards (4.2) suppose \( a \in \cl_{\text{alg}} A^{-1} \). Then for arbitrary \( b \in A^{-1} \) there is \( c \in A^{-1} \) for which
\begin{equation}
1 - b(a - c) = d \in A^{-1} \implies a = b^{-1}(1 - d) + c \in A^{-1} + A^{-1} + A^{-1}.
\end{equation}

Finally if (4.3) holds, then for arbitrary \( b \in A \) there is \( c \in A^{-1} \) for which \( 1 - b(-c) = d \in A^{-1} \), giving \( b = (d - 1)c^{-1} \in A^{-1} + A^{-1} \), as in (4.4). Conversely for arbitrary \( c, d \in A^{-1} \) we have
\begin{equation}
1 - (c + d)d^{-1} = -cd^{-1} \in A^{-1};
\end{equation}
in the presence of (4.3) this is (4.5).

The analogue of (4.1) for the topological closure is familiar in Banach algebras [8], [10]. From (4.1) it follows that everything in the algebraic closure of the invertibles is “consistent in regularity” [6], [7]:
\begin{equation}
\cl_{\text{alg}} A^{-1} \subseteq A^{-1} \cup \left( A \setminus (A_{\text{left}}^{-1} \cap A_{\text{right}}^{-1}) \right) = \{ a \in A : ax \in A^{-1} \iff xa \in A^{-1} \}.
\end{equation}
The last part of Theorem 4 generalizes to radical elements and to idempotents:

Theorem 5. If
\begin{equation}
1 + qA^{-1} \subseteq A^{-1},
\end{equation}
then the condition (4.6) is necessary and sufficient for
\begin{equation}
q \in \cl_{\text{alg}} A^{-1}.
\end{equation}
If instead \( p = p^2 \in A \) is idempotent, then
\begin{equation}
0 \in \cl_{\text{alg}}(pA)^{-1} \iff 0 \in \cl_{\text{alg}}(pA)^{-1} \implies 1 - p \in \cl_{\text{alg}} A^{-1}.
\end{equation}
Proof. If \( q \in A \) is in \( \text{cl}_{\text{alg}} A^{-1} \), then for arbitrary \( b \in A \) there is \( c \in A^{-1} \) for which 
\[ 1 - b(q - c) = d \in A^{-1} \] and hence 
\[ bc = dc^{-1} - c^{-1} \text{ with } e = 1 - qc^{-1}. \]

Now if \( (6.1) \) holds, then \( c \in A^{-1} \) giving 
\[ b = (d - 1)(ec)^{-1} \in A^{-1} + A^{-1}, \]
which is included in \( A^{-1} \) by \( (4.3) \). Conversely for arbitrary \( c, d \in A^{-1} \)
\[ 1 - (c + d)(q + c^{-1}) = -(dc^{-1} + cq + dq) \in A^{-1} + A^{-1}q + A^{-1}q \text{,} \]
which is included in \( A^{-1} \) by \( (4.3) \). In the presence of \( (4.3) \) this applies to arbitrary \( b = c + d \in A \). Towards \( (5.3) \), if \( 0 \) is the algebraic closure in \( pAp \) of the invertible group \( (pAp)^{-1} \), then for arbitrary \( b \in A \) there is \( c = pcp \in (pAp)^{-1} \) for which \( p(1 + bc) \in (pAp)^{-1} \), giving 
\[ 1 + pbc = 1 - p + p(1 + bc) \in A^{-1}, \]
which says \( 0 \in \text{cl}_{\text{alg}} (pAp)^{-1} \). Conversely if this holds, then for arbitrary \( b \in A \), in particular \( p\beta p \in pAp \), there is \( c = pcp \in (pAp)^{-1} \) and \( d \in A \) for which \( (1 + pbc)d = 1 = d(1 + pbc) \), giving 
\[ (p + pbc)pdp = p = pd(p + pbc) \]
and hence \( p(1 + bc) \in (pAp)^{-1} \). Also if for arbitrary \( b \in A \) there is \( c, d \in pAp \) with \( 1 - bc \in A^{-1} \) and \( cd = p = dc \), then \( a = 1 - p - c \in A^{-1} \) with \( a^{-1} = 1 - p - d \) giving \( 1 - ba = 1 - bc \in A^{-1} \).

The algebraic closure intervenes in generalized inverse theory:

Lemma 6. With

\[ (6.1) \quad \hat{A} = \{ a \in A : a \in Aa \} \quad \text{and} \quad \hat{A} = \{ a \in A : a \in aA^{-1}a \} \]

there is inclusion

\[ (6.2) \quad \hat{A} \cap \text{cl}_{\text{alg}} A^{-1} \subseteq \hat{A}. \]

Necessary and sufficient for equality in \((6.2)\) is that

\[ (6.3) \quad A^* \equiv \{ p \in A : p = p^2 \} \subseteq \text{cl}_{\text{alg}} A^{-1}. \]

Proof. If \( a = aa'a \in \hat{A} \), so that \( a'a = p = p^2 \) is idempotent, and if 
\( b \in A^{-1} \) with \( 1 + (b - a)a' = c^{-1} \in A^{-1}, \)
then
\[ a = (cb)p \in \hat{A}. \]

For equality in \((6.2)\), observe \[ 9, \quad 10 \]
\[ \hat{A} = A^{-1}A^* = A^*A^{-1}. \]

In Banach algebras \((6.3)\) is clear \[ 9, \quad 10 \]; generally
\[ (6.4) \quad 0 \notin \text{int} \sigma(a) \Rightarrow a \in \text{cl}_{\text{alg}} A^{-1}, \]
where \( \sigma(a) = \{ \lambda \in C : a - \lambda \notin A^{-1} \} \) is the usual spectrum.

We can extend the algebraic closure to tuples:

Definition 7. For arbitrary \( K \subseteq A^n \)

\[ (7.1) \quad \text{cl}_{\text{alg}} K = \{ x \in A^n : \forall z, w \in A^n \exists x' \in K, \ 1 - \sum_{j=1}^n z_j(x_j - x'_j)w_j \in A^{-1} \}. \]
We can also define left and right invertible tuples:

\[(7.2)\quad A^{-n}_{\text{left}} = \{ a \in A^n : 1 \in \sum_{j=1}^{n} Aa_j \} \quad \text{and} \quad A^{-n}_{\text{right}} = \{ a \in A^n : 1 \in \sum_{j=1}^{n} a_jA \} .\]

If we interpret \( A \) as an additive category, then in a sense we have already dealt with \( n \)-tuples, and more generally \( n \times m \) matrices, over \( A \), which just form another additive category \( B = \text{Matrix}(A) \); thus most of Theorem 2 extends to \( n \)-tuples. For example the analogues of (2.1) and (2.2) for \( K,H \subseteq A^n \) are immediate, while for

the analogue of (2.3) we argue that if \( x \in \text{cl}_{\text{alg}}\text{cl}_{\text{alg}}(K) \), then for arbitrary \( z,w \in A^n \) there is \( x' \in \text{cl}_{\text{alg}}(K) \) for which \( 1 - \sum_j z_j(x_j - x_j')w_j = c \in A^{-1} \), and then \( x'' \in K \) for which \( 1 - c^{-1} \sum_j z_j(x_j' - x_j'')w_j \in A^{-1} \). Now

\[
1 - \sum_j z_j(x_j - x_j'')w_j = 1 - \sum_j (z_j(x_j - x_j')w_j - z_j(x_j' - x_j'')w_j) = c(1 - c^{-1} \sum_j z_j(x_j' - x_j'')w_j) \in A^{-1} .
\]

Towards the extension of (2.4), if \( x \in \text{cl}_{\text{alg}}(K) \) and \( y \in \text{cl}_{\text{alg}}(H) \) and if \( z,w \in A^n \) then, with \( 1 - \sum_j z_j(x_j - x_j')w_j = c \in A^{-1} \),

\[
1 - \sum_j z_j(x_j + y_j - x_j' - y_j')w_j = (\sum_j (1 - z_j(x_j - x_j')w_j)(1 - c^{-1} \sum_j z_j(y_j - y_j')w_j) \in A^{-1}A^{-1} = A^{-1} \quad \text{provided} \quad 1 - c^{-1} \sum_j z_j(y_j - y_j')w_j \in A^{-1} .
\]

Similarly, for an extended version of (2.5), with \( 1 - \sum_j z_j(x_j - x_j')y_jw_j = d \in A^{-1} \),

\[
1 - \sum_j z_j(x_jy_j - x_j'y_j')w_j = (1 - \sum_j z_j(x_j - x_j')y_jw_j)(1 - d^{-1} \sum_j z_jx_j'(y_j - y_j')w_j) \quad \text{\in} \quad A^{-1} \quad \text{provided} \quad 1 - d^{-1} \sum_j z_jx_j'(y_j - y_j')w_j \in A^{-1} .
\]

(2.4) would also extend, in a category \( A \), to \( K,H \) and a more general bilinear image \( K \ast H \).

Declare that a ring \( A \) has left stable range \( \leq n \) provided

\[(7.3)\quad \forall (a,b) \in A^n \times A , \quad (a,b) \in A^{-n-1}_{\text{left}} \iff \exists c \in A^n , \quad a - cb \in A^{-n-1}_{\text{left}} .\]

Corach and Suarez [4, 5], and Blackadar [3], have considered this kind of situation when \( A \) is commutative, or a C*-algebra.

Notice how the definition prefers the final element of an \( n+1 \) tuple; an alternative would say that if \( d \in A^{n+1} \) was “left invertible”, then there would exist an index \( 1 \leq j \leq n + 1 \) for which an analogue of (7.3) held. In commutative Banach algebras the condition that \( A \) has stable range \( \leq 1 \) reduces to the topological closure of the invertible group being the whole of \( A \). We offer here a curious hybrid result:
Theorem 8. If
\begin{equation}
A \subseteq \text{cl}_{\text{alg}} A_{\leftarrow}^{-1},
\end{equation}
then there is implication
\begin{equation}
(a, b) \in A_{\leftarrow}^{-2} \implies (a - Ab) \cap A_{\rightarrow}^{-1} \neq \emptyset.
\end{equation}

Proof. Suppose
\begin{equation*}
a' a + b' b = 1 \text{ with } a' \in \text{cl}_{\text{alg}} A_{\leftarrow}^{-1},
\end{equation*}
so that there are \(a''\), \(a'''\) in \(A\) for which
\begin{equation*}
b' b = 1 - a'a = d - a''a \text{ with } d \in A^{-1} \text{ and } a'''a'' = 1,
\end{equation*}
giving
\begin{equation*}
a - cb = a'''d \in A_{\rightarrow}^{-1} \text{ with } c = -a'''b'.
\end{equation*}
\[\square\]

When \(b = 0\) this gives back (4.1). The same argument gives back a weaker version of (7.3): if \((a, b) \in A \times A^n\), then if (8.1) holds there is implication
\begin{equation}
(a, b) \in A_{\leftarrow}^{-n-1} \implies (a - \sum_j Ab_j) \cap A_{\rightarrow}^{-1} \neq \emptyset.
\end{equation}

We remark that the condition (8.2) says that the element \(a \in A\) is in another kind \([1, 2]\) of “closure” of the semigroup \(A_{\rightarrow}^{-1}\): for the moment we find the precise relationship between this condition and ours elusive.

References


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