A SHARP RESULT ON $m$-COVERS

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Abstract. Let $A = \{a_s + n_s \mathbb{Z}\}_{s=1}^k$ be a finite system of residue classes which forms an $m$-cover of $\mathbb{Z}$ (i.e., every integer belongs to at least $m$ members of $A$). In this paper we show the following sharp result: For any positive integers $m_1, \ldots, m_k$ and $\theta \in [0, 1)$, if there is $I \subseteq \{1, \ldots, k\}$ such that the fractional part of $\sum_{s \in I} m_s/n_s$ is $\theta$, then there are at least $2^m$ such subsets of $\{1, \ldots, k\}$. This extends an earlier result of M. Z. Zhang and an extension by Z. W. Sun. Also, we generalize the above result to $m$-covers of the integral ring of any algebraic number field with a power integral basis.

1. Introduction

For an integer $a$ and a positive integer $n$, we simply let $a(n)$ represent the set $a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. Following Sun \cite{S95, S96} we call a finite system

\begin{equation}
A = \{a_s(n_s)\}_{s=1}^k
\end{equation}

of such sets an $m$-cover of $\mathbb{Z}$ (where $m \in \{1, 2, 3, \ldots\}$) if every integer lies in at least $m$ members of (1.1). We use the term cover (or covering system) instead of 1-cover. For problems and results in this area, the reader may consult \cite{G04} pp. 383–390, \cite{PS} and \cite{S05}. P. Erdős \cite{E97} once said: “Perhaps my favorite problem of all concerns covering systems.”

Example 1.1. For each integer $m \geq 1$, there is an $m$-cover of $\mathbb{Z}$ that is not the union of two covers of $\mathbb{Z}$. To wit, let $p_1, \ldots, p_r$ be distinct primes with $r \geq 2m-1$, and set $N = p_1 \cdots p_r$. Clearly $A_1 = \{\prod_{i \in I} p_s(N) \mid I \subseteq \{1, \ldots, r\}, |I| \geq m\}$ does not cover any integer relatively prime to $N$. Let $a_1, \ldots, a_n$ be the list of those integers in \{0, 1, \ldots, N-1\} not covered by $A_1$, with each occurring exactly $m$ times. If $x \in \mathbb{Z}$ is covered by $A_1$, then $x \in \bigcap_{s \in I} 0(p_s)$ for some $I \subseteq \{1, \ldots, r\}$ with $|I| \geq m$. Therefore

\[ A_0 = \{0(p_1), \ldots, 0(p_r), a_1(N), \ldots, a_n(N)\} \]

forms an $m$-cover of $\mathbb{Z}$. Suppose that $I_1 \cup I_2 = \{1, \ldots, r\}$, $J_1 \cup J_2 = \{1, \ldots, n\}$ and $I_1 \cap I_2 = J_1 \cap J_2 = \emptyset$. For $i = 1, 2$ let $A_i$ be the system consisting of those $0(p_s)$ with $s \in I_i$ and those $a_t(N)$ with $t \in J_i$. We claim that $A_1$ or $A_2$ is not a cover of $\mathbb{Z}$. Without loss of generality, let us assume that $|I_1| \leq |I_2|$. Since $2|I_2| \geq |I_1| + |I_2| > 2(m-1)$, we have $|I_2| \geq m$ and hence $\prod_{s \in I_2} p_s$ is covered.
Corollary 1.2. Suppose that consecutive integers at least \( m = J \) are relatively prime to \( a_s(n_s) \) for all \( s \), then for any \( 0 \leq \theta < 1 \) the set
\[
I_A(\theta) = \left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{m_s}{n_s} = \theta \right\}
\]
has at least \( 2^m \) elements if it is nonempty.

Remark 1.1. Clearly \( m \) copies of \( 0(1) \) form an \( m \)-cover of \( \mathbb{Z} \). This shows that the lower bound in Theorem 1.1 is best possible.

Corollary 1.1. Let (1.1) be an \( m \)-cover of \( \mathbb{Z} \), and let \( m_1, \ldots, m_k \) be any integers. Then \(|S(A)| \leq 2^{k-m}\), where
\[
S(A) = \left\{ \sum_{s \in I} \frac{m_s}{n_s} : I \subseteq \{1, \ldots, k\} \right\}.
\]

Proof. As \(|I_A(\theta)| \geq 2^m\) for all \( \theta \in S(A) \), we have
\[
|S(A)|2^m \leq |\{ I : I \subseteq \{1, \ldots, k\} \}| = 2^k
\]
and hence \(|S(A)| \leq 2^{k-m}\).

Remark 1.2. Sun [S95, S96] showed that if \( m_1, \ldots, m_k \) are relatively prime to \( n_1, \ldots, n_k \), respectively, then (1.1) forms an \( m \)-cover of \( \mathbb{Z} \) whenever it covers \(|S(A)| \) consecutive integers at least \( m \) times.

Corollary 1.2. Suppose that (1.1) forms an \( m \)-cover of \( \mathbb{Z} \) but \( \{a_s(n_s)\}_{s=1}^{k-1} \) does not. If the covering function \( w_A(x) = |\{1 \leq s \leq k : x \equiv a_s(n_s) \mod n_k\}| \) is periodic modulo \( n_k \), then for any \( r = 0, \ldots, n_k - 1 \) we have
\[
\left| \left\{ I \subseteq \{1, \ldots, k-1\} : \sum_{s \in I} \frac{1}{n_s} = \frac{r}{n_k} \right\} \right| \geq 2^{m-1}.
\]

Proof. By Theorem 1 of Sun [S07],
\[
\left| \left\{ \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \ldots, k-1\} \text{ and } \sum_{s \in I} \frac{1}{n_s} = \frac{r}{n_k} \right\} \right| \geq m.
\]
In particular, \(\{\sum_{s \in I} 1/n_s\} = r/n_k\) for some \(I \subseteq \{1, \ldots, k-1\}\), and hence (1.4) holds in the case \(m = 1\). For \(A_k = \{a_s(n_s)\}_{s=1}^{k-1}\), clearly \(w_{A_k}(x) \geq m - 1\) for all \(x \in \mathbb{Z}\). In the case \(m > 1\), we obtain (1.4) by applying Theorem 1.1 to \(A_k\) with \(m_1 = \cdots = m_{k-1} = 1\) and \(\theta = r/n_k\). \(\square\)

**Remark 1.3.** When \(n_k\) is divisible by all the moduli \(n_1, \ldots, n_k\), Corollary 1.2 was stated by the second author in [S03, Theorem 2.5]. When \(w_{A_k}(x) = m\) for all \(x \in \mathbb{Z}\), the following result stronger than (1.4) (with \(r \in \{0, \ldots, n_k - 1\}\)) was proved in [S97]:

\[
\left\{ I \subseteq \{1, \ldots, k-1\} : \sum_{s \in I} \frac{1}{n_s} = n + \frac{r}{n_k} \right\} \supseteq \left( \begin{array}{c} m - 1 \\ n \end{array} \right)
\]

for every \(n = 0, \ldots, m - 1\).

For an algebraic number field \(K\), let \(O_K\) be the ring of algebraic integers in \(K\). For \(\alpha, \beta \in O_K\), we set

\[\alpha + \beta O_K = \{\alpha + \beta \omega : \omega \in O_K\}\]

and call it a residue class in \(O_K\). For a finite system

(1.5)

of such residue classes, if \(\{|1 \leq s \leq k : x \in \alpha_s + \beta_s O_K\}| \geq m\) for all \(x \in O_K\) (where \(m \in \{1, 2, 3, \ldots\}\)), then we call \(A\) an \(m\)-cover of \(O_K\). Covers of the ring \(\mathbb{Z}[\sqrt{-2}] = O_{\mathbb{Q}(\sqrt{-2})}\) were investigated by J. H. Jordan [63].

An algebraic number field \(K\) of degree \(n\) is said to have a **power integral basis** if there is \(\gamma \in O_K\) such that \(1, \gamma, \ldots, \gamma^{n-1}\) form a basis of \(O_K\) over \(\mathbb{Z}\). It is well known that all quadratic fields and cyclotomic fields have power integral bases.

Here is a generalization of Theorem 1.1.

**Theorem 1.2.** Let \(K\) be an algebraic number field with a power integral basis. Suppose that (1.5) forms an \(m\)-cover of \(O_K\), and let \(\omega_1, \ldots, \omega_k \in O_K\). Then, for any \(\mu \in K\), the set

\[
\left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{\omega_s}{\beta_s} \in \mu + O_K \right\}
\]

is empty or it has at least \(2^n\) elements.

**Remark 1.4.** We conjecture that the requirement in Theorem 1.2 that \(K\) has a power integral basis can be cancelled.

2. **Proof of Theorem 1.1**

**Lemma 2.1.** Let (1.1) be an \(m\)-cover of \(\mathbb{Z}\). Let \(m_1, \ldots, m_k \in \mathbb{Z}\) and define \(S(A)\) as in (1.3). Then, for any given \(\theta \in S(A)\), there exists \(t \in \{1, \ldots, k\}\) such that both \(\theta\) and \(\{\theta - m_t/n_t\}\) lie in \(S(A_t)\), where \(A_t = \{a_s(n_s)\}_{1 \leq s \leq k, s \neq t}\).

**Proof.** Choose a maximal \(J \subseteq \{1, \ldots, k\}\) such that \(\{\sum_{s \in J} m_s/n_s\} = \theta\). As (1.1) is a cover of \(\mathbb{Z}\), by [S95, Theorem 2] or [S99, Theorem 1(i)] there exists \(I \subseteq \{1, \ldots, k\}\) for which \(I \neq J\) and \(\{\sum_{s \in I} m_s/n_s\} = \theta\). Note that \(J \not\subseteq I\) and hence \(t \in J \setminus I\) for some \(1 \leq t \leq k\). Clearly \(\theta = \{\sum_{s \in I} m_s/n_s\} \in S(A_t)\) and also \(\{\theta - m_t/n_t\} = \{\sum_{s \in J \setminus \{t\}} m_s/n_s\} \in S(A_t)\). This concludes the proof. \(\square\)
Proof of Theorem 1.1. We use induction on \( m \).

The \( m = 1 \) case, as mentioned above, has been handled in [S95, S99].

Now let \( m > 1 \) and assume that Theorem 1.1 holds for smaller positive integers. Let \( \theta \in S(A) \). In light of Lemma 2.1, there is \( t \in \{1, \ldots, k\} \) such that both \( \theta \) and \( \theta' = (\theta - mt/nt) \) lie in \( S(A_t) \). As \( A_t \) forms an \( (m-1) \)-cover of \( \mathbb{Z} \), by the induction hypothesis we have \( |I_{A_t}(\theta)| \geq 2^{m-1} \) and \( |I_{A_t}(\theta')| \geq 2^{m-1} \). Observe that

\[
I_{A}(\theta) = I_{A_t}(\theta) \cup \{ I \cup \{ t \} : I \in I_{A_t}(\theta') \}.
\]

Therefore

\[
|I_{A}(\theta)| = |I_{A_t}(\theta)| + |I_{A_t}(\theta')| \geq 2^{m-1} + 2^{m-1} = 2^{m}.
\]

We are done.

3. Proof of Theorem 1.2

At first we give a lemma on algebraic number fields with power integral bases.

**Lemma 3.1.** Let \( K \) be an algebraic number field with a power integral basis \( 1, \gamma, \ldots, \gamma^{n-1} \). For any \( \mu = \sum_{r=0}^{n-1} \mu_r \gamma^r \in K \) with \( \mu_0, \ldots, \mu_{n-1} \in \mathbb{Q} \), we have

\[
\mu \in O_K \iff \psi(\mu), \psi(\mu\gamma), \ldots, \psi(\mu\gamma^{n-1}) \in \mathbb{Z},
\]

where \( \psi(\mu) \) denotes the last coordinate \( \mu_{n-1} \) of \( \mu \).

**Proof.** If \( \mu \in O_K \), then \( \mu_0, \mu_1, \ldots, \mu_{n-1} \in \mathbb{Z} \) and hence \( \psi(\mu\gamma^j) \in \mathbb{Z} \) for every \( j = 0, \ldots, n - 1 \).

Now assume that \( \psi(\mu\gamma^j) \in \mathbb{Z} \) for all \( j = 0, \ldots, n - 1 \). We want to show that \( \mu \in O_K \) (i.e., \( \mu_0, \ldots, \mu_{n-1} \in \mathbb{Z} \)). Clearly \( \mu_{n-1} = \psi(\mu\gamma^0) \in \mathbb{Z} \). If \( 0 < r < n - 1 \) and \( \mu_{r+1}, \ldots, \mu_{n-1} \in \mathbb{Z} \), then

\[
\mu_r = \psi(\mu_0 \gamma^{n-1-r} + \mu_1 \gamma^{n-r} + \cdots + \mu_r \gamma^0) - \psi(\mu_{r+1} \gamma^n + \cdots + \mu_{n-1} \gamma^{2n-2-r})
\]

and hence \( \mu_r \in \mathbb{Z} \) since \( \mu_{r+1} \gamma^n + \cdots + \mu_{n-1} \gamma^{2n-2-r} \in O_K \). So, by induction, \( \mu_r \in \mathbb{Z} \) for all \( r = 0, \ldots, n - 1 \). We are done. 

**Proof of Theorem 1.2.** In the spirit of the proof of Theorem 1.1, it suffices to handle the case \( m = 1 \). That is, we should prove that for any \( \theta \in O_K \) with \( \theta \neq \theta' = (\theta - mt/nt) \) such that \( \sum_{s \in I} \omega_s/\beta_s - \sum_{s \in J} \omega_s/\beta_s \in O_K \).

Let \( \{1, \gamma, \ldots, \gamma^{n-1}\} \) be a power integral basis of \( K \), and define \( \psi \) as in Lemma 3.1.

Let \( x_0, \ldots, x_{n-1} \in \mathbb{Z} \) and \( x = \sum_{r=0}^{n-1} x_r \gamma^r \). Since \( -x \in O_K \) is covered by \( A = \{ \alpha_s + \beta_s O_K \}_{s=1}^k \), we have

\[
0 = \prod_{s=1}^k \left( 1 - e^{2\pi i \psi(\omega_s(x+\alpha_s)/\beta_s)} \right) = \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} \prod_{s \in I} e^{2\pi i \psi(\omega_s(x+\alpha_s)/\beta_s)}
\]

\[
= \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} \prod_{s \in I} e^{2\pi i \psi(\sum_{r=0}^{n-1} x_r \gamma^r / \beta_s)}
\]

\[
= \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} e^{2\pi i \psi(\omega_s(x+\alpha_s)/\beta_s)} \prod_{r=0}^{n-1} e^{2\pi i x_r \psi(\omega_s \gamma^r / \beta_s)}
\]

\[
= \sum_{\theta_0 \in S_0} e^{2\pi i x_0 \theta_0} \sum_{\theta_1 \in S_1} e^{2\pi i x_1 \theta_1} \cdots \sum_{\theta_{n-1} \in S_{n-1}} e^{2\pi i x_{n-1} \theta_{n-1}} f(\theta_0, \ldots, \theta_{n-1}).
\]
where
\[ S_r = \left\{ \left\{ \psi \left( \sum_{s \in I} \frac{\omega_s \gamma^r}{\beta_s} \right) \right\} : I \subseteq \{1, \ldots, k\} \right\} \]
and
\[ f(\theta_0, \ldots, \theta_{n-1}) = \sum_{\substack{I \subseteq \{1, \ldots, k\} \\ \{\psi(\sum_{s \in I} \omega_s \alpha_s / \beta_s)\} = \theta_r}} (-1)^{|I|} e^{2\pi i \psi(\sum_{s \in I} \omega_s \alpha_s / \beta_s)}. \]

For each \( r = 0, \ldots, n-1 \), if \( \sum_{r \in S_r} e^{2\pi i x_r \theta_r} F(\theta_r) = 0 \) for all \( x_r = 0, \ldots, |S_r| - 1 \), then \( F(\theta_r) = 0 \) for every \( \theta_r \in S_r \), because the Vandermonde determinant \( \det(e^{2\pi i x_r \theta_r})_{0 \leq x_r < |S_r|, \theta_r \in S_r} \) does not vanish. So, by the above, we have \( f(\theta_0, \ldots, \theta_{n-1}) = 0 \) for all \( \theta_0 \in S_0, \ldots, \theta_{n-1} \in S_{n-1} \).

Now suppose that \( \mu \in K \) and \( \sum_{s \in J} \omega_s / \beta_s = \mu + O_K \) for a unique subset \( J \) of \( \{1, \ldots, k\} \). We want to deduce a contradiction.

Set \( \theta_r = \{\psi(\mu \gamma^r)\} \) for \( r = 0, \ldots, n-1 \). For any \( I \subseteq \{1, \ldots, k\} \) we have
\[
\left\{ \psi \left( \sum_{s \in I} \frac{\omega_s \gamma^r}{\beta_s} \right) \right\} = \theta_r \quad \text{for all } r = 0, \ldots, n-1
\]
\[ \iff \psi \left( \left( \sum_{s \in I} \frac{\omega_s}{\beta_s} - \mu \right) \gamma^r \right) \in \mathbb{Z} \quad \text{for all } r = 0, \ldots, n-1
\]
\[ \iff \sum_{s \in I} \frac{\omega_s}{\beta_s} \in \mu + O_K \quad \text{(by Lemma 3.1)}
\]
\[ \iff I = J.
\]
Thus the expression of \( f(\theta_0, \ldots, \theta_{n-1}) \) only contains one summand, and therefore
\[ 0 = f(\theta_0, \ldots, \theta_{n-1}) = (-1)^{|J|} e^{2\pi i \psi(\sum_{s \in J} \omega_s \alpha_s / \beta_s)} \neq 0,
\]
which is a contradiction.

The proof of Theorem 1.2 is now complete. \( \square \)

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**References**


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