A CHARACTERIZATION OF MODULES
LOCALLY OFFINITE INJECTIVE DIMENSION

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ABSTRACT. In this note, we characterize finite modules locally of finite injective dimension over commutative Noetherian rings in terms of vanishing of Ext modules.

1. Introduction

Let $R$ be a commutative Noetherian ring. Goto \cite{2} proved the following theorem.

**Theorem 1.1** (Goto). The following are equivalent:

1. $R$ is Gorenstein;
2. For every finite $R$-module $M$, there exists an integer $n$ such that $\text{Ext}_R^i(M, R) = 0$ for all $i > n$.

We should note that this theorem remains valid even in the case where the ring $R$ has infinite Krull dimension.

The purpose of this note is to give a characterization of finite modules locally of finite injective dimension. Our theorem is the following.

**Theorem 1.2**. The following are equivalent for a finite $R$-module $N$:

1. $\text{id}_{R_p} N_p < \infty$ for every $p \in \text{Spec } R$;
2. For every finite $R$-module $M$, there exists an integer $n$ such that $\text{Ext}_R^i(M, N) = 0$ for all $i > n$.

This theorem is a generalization of Goto’s. In fact, applying our theorem to $N = R$, we immediately obtain Goto’s theorem.

2. Proof of the theorem

We denote by $\text{CM}(R)$ the Cohen-Macaulay locus of $R$, that is, the set of prime ideals $p$ of $R$ such that the local ring $R_p$ is Cohen-Macaulay. The following lemma can be shown in a similar way to the proof of \cite[Theorem 24.5]{3}.

**Lemma 2.1.** Let $p$ be a prime ideal of $R$ such that both $R_p$ and $R/p$ are Cohen-Macaulay rings. Then there exists an element $f \in R - p$ such that $D(f) \cap V(p) \subseteq \text{CM}(R)$.

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It is known that the Cohen-Macaulay locus of a homomorphic image of a Cohen-Macaulay ring is an open subset; see [3, Exercises 24.2]. To prove our theorem, we need to generalize this fact. For an ideal $I$ of $R$, let $\CM_R(R/I)$ denote the set of prime ideals $p \in V(I)$ such that the local ring $(R/I)_p$ is Cohen-Macaulay.

**Lemma 2.2.** Let $I$ and $J$ be ideals of $R$. Suppose that $V(J)$ is contained in $\CM(R)$. Then the following hold:

1. For any prime ideal $p \in \CM_R(R/I) \cap V(J)$, there exists an element $f \in R - p$ such that $D(f) \cap V(p) \subseteq \CM_R(R/I)$.
2. There exists an ideal $K$ of $R$ such that $\CM_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J)$.

In other words, $\CM_R(R/I) \cap V(J)$ is an open subset of $V(I) \cap V(J)$ in the relative topology induced by the Zariski topology of Spec $R$.

**Proof.** 1) Let $p \in \CM_R(R/I) \cap V(J)$. Then by the assumption that $V(J)$ is contained in $\CM(R)$, the ring $R_p$ is a Cohen-Macaulay local ring. Making a similar argument to the proof of [3, Theorem 24.5], we can assume without loss of generality that there is an $R$-regular sequence $x = x_1, x_2, \ldots, x_n$ in $p$ with $p^r \subseteq xR$ for some $r > 0$ and that $\overline{\mathcal{R}}/\overline{p}^{+1}$ is a free $R/\overline{p}$-module for all $i > 0$, where $\mathcal{R} = R/xR$ and $\overline{p} = p/\overline{xR}$.

We have only to prove that the residue ring $R/p$ is Cohen-Macaulay. In fact, if $R/p$ is Cohen-Macaulay, then so are $(R/I)/(p/I)$ and $(R/I)_{p/I}$ since $p$ is in $\CM_R(R/I)$. Hence Lemma 2.1 implies that there is an element $f \in R - p$ such that $D(f) \cap V(p/I)$ is contained in $\CM(R/I)$, where $\overline{f}$ denotes the residue class of $f$ in $R/I$. We easily see that $D(f) \cap V(p)$ is contained in $\CM_R(R/I)$.

Let us show that $R/p$ is a Cohen-Macaulay ring. It is easy to see from [3, Exercises 24.1] that $R/p = \overline{R}/\overline{p}$ is Cohen-Macaulay if and only if so is $\overline{R}$. Take a prime ideal $q \in V(\overline{x}) = V(p)$. Then we have $q \supseteq p \supseteq J$, hence $q \in V(J) \subseteq \CM(R)$. Therefore $R_q$ is a Cohen-Macaulay local ring, and so is $\overline{R}_q$, because $x$ is an $R_q$-regular sequence. This shows that $\overline{R}$ is a Cohen-Macaulay ring. Thus we conclude that the residue ring $R/p$ is Cohen-Macaulay, as desired.

2) Set $U = \{ p/I + J \mid p \in \CM_R(R/I) \cap V(J) \}$. This is a subset of Spec $R/I + J$. Note that this subset is stable under generalization. Let $P \in U$. Then there is a prime ideal $p \in \CM_R(R/I) \cap V(J)$ such that $P = p/I + J$. By the assertion (1) of the lemma, the set $D(f) \cap V(p)$ is contained in $\CM_R(R/I)$ for some $f \in R - p$. Denote by $\overline{f}$ the residue class of $f$ in $R/I$. It is easy to see that $P \in D(\overline{f}) \cap V(P) \subseteq U$. Thus $U$ contains a nonempty open subset of $V(P)$. By virtue of the topological Nagata criterion [3, Theorem 24.2], $U$ is an open subset of Spec $R/I + J$; we have $U = D(K/I + J)$ for some ideal $K$ of $R$ containing $I + J$. Then it is easily checked that $\CM_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J)$.

Now, we can prove our theorem.

**Proof of Theorem 2.2** (2) $\Rightarrow$ (1): Let $p$ be a prime ideal of $R$. Then there is an integer $n$ such that $\Ext_R^i(R/p, N) = 0$ for all $i > n$. Hence we have $\Ext_R^i(R/p, N_p) = 0$ for all $i > n$. Therefore by [1, Theorem 3.1.14] we obtain $\id_R N_p \leq n < \infty$.

(1) $\Rightarrow$ (2): First of all, note that (2) is equivalent to the statement that for each ideal $I$ of $R$ there is an integer $n$ such that $\Ext_R^i(R/I, N) = 0$ for all $i > n$. This can easily be proved by induction on the number of generators of the $R$-module
Suppose that there exists an ideal $I$ of $R$ such that for any integer $n$ there is an integer $i > n$ such that $	ext{Ext}^i_R(R/I, N) \neq 0$. We want to derive a contradiction. Since $R$ is Noetherian, one can choose $I$ to be a maximal one among such ideals. Making a similar argument to the proof of Theorem 1.1, we see that the ideal $I$ is prime and that for any element $f \in R - I$, the map
\[
\text{Ext}^i_R(R/I, N) \to \text{Ext}^i_R(R/I, N)
\]
is an isomorphism for $i \gg 0$.

**Claim 1.** One has $I \in \text{Supp}_R N \subseteq \text{CM}(R)$.

**Proof of Claim.** Our assumption (1) implies that for any $p \in \text{Supp}_R N$, the nonzero finite $R_p$-module $N_p$ has finite injective dimension. Hence $R_p$ is a Cohen-Macaulay local ring; see [1] Corollary 9.6.2 and Remarks 9.6.4. Thus $\text{Supp}_R N$ is contained in $\text{CM}(R)$. On the other hand, assume that $I$ is not in $\text{Supp}_R N$. Then there exists an element $f \in \text{Ann}_R N - I$, and the map $\text{Ext}^i_R(R/I, N) \to \text{Ext}^i_R(R/I, N)$ is an isomorphism for $i \gg 0$. Since $fN = 0$, this map is the zero map, and we get $\text{Ext}^i_R(R/I, N) = 0$ for $i \gg 0$. It follows from this contradiction that $I$ belongs to $\text{Supp}_R N$.

Noting that $\text{Supp}_R N = V(\text{Ann}_R N)$, we see from Claim 1 and Lemma 2.2 that there is an ideal $K$ of $R$ such that $\text{CM}_R(R/I) \cap \text{Supp}_R N = D(K) \cap V(I) \cap \text{Supp}_R N$. The localization $(R/I)_f = \kappa(I)$ is a field, hence a Cohen-Macaulay ring. It is seen from Claim 1 again that $I \in \text{CM}_R(R/I) \cap \text{Supp}_R N \subseteq D(K)$. Thus there is an element $f \in K - I$.

**Claim 2.** For any prime ideal $p \in D(f)$ and any integer $i > \text{ht} I$, one has $\text{Ext}^i_{R_p}(R_p/IR_p, N_p) = 0$.

**Proof of Claim.** We may assume that $p$ belongs to both $V(I)$ and $\text{Supp}_R N$, because otherwise the module $\text{Ext}^i_{R_p}(R_p/IR_p, N_p)$ automatically vanishes. Hence Claim 1 implies that $p$ belongs to $\text{CM}(R)$, namely the local ring $R_p$ is Cohen-Macaulay. Added to it, since $D(f)$ is contained in $D(K)$, we have $p \in D(K) \cap V(I) \cap \text{Supp}_R N \subseteq \text{CM}_R(R/I)$, and therefore $R_p/IR_p$ is Cohen-Macaulay. Thus we get the following equalities:
\[
\text{depth } R_p - \text{depth } R_p/IR_p = \dim R_p - \dim R_p/IR_p = \text{ht } IR_p = \text{ht } I.
\]
Since $N_p$ is a finite $R_p$-module of finite injective dimension by assumption, it follows from the result of Ischebeck [1] Exercises 3.1.24] that $\text{Ext}^i_{R_p}(R_p/IR_p, N_p) = 0$ for every $i > \text{ht} I$.

Claim 2 means that $(\text{Ext}^i_{R_f}(R_f/IR_f, N_f))_p = 0$ for every $p \in \text{Spec } R_f$ and every $i > \text{ht} I$. Therefore, $\text{Ext}^i_{R_f}(R_f/IR_f, N_f) = 0$ for $i > \text{ht} I$. The $R$-module $\text{Ext}^i_R(R/I, N)$ is isomorphic to $\text{Ext}^i_{R_f}(R_f/IR_f, N_f)$ for $i \gg 0$, and thus $\text{Ext}^i_R(R/I, N) = 0$ for $i \gg 0$. This contradiction completes the proof of our theorem.

**References**


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