A CHARACTERIZATION OF MODULES
LOCALLY OF FINITE INJECTIVE DIMENSION

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Abstract. In this note, we characterize finite modules locally of finite injective dimension over commutative Noetherian rings in terms of vanishing of Ext modules.

1. Introduction

Let \( R \) be a commutative Noetherian ring. Goto [2] proved the following theorem.

**Theorem 1.1** (Goto). The following are equivalent:

1. \( R \) is Gorenstein;
2. For every finite \( R \)-module \( M \), there exists an integer \( n \) such that \( \text{Ext}^i_R(M,R) = 0 \) for all \( i > n \).

We should note that this theorem remains valid even in the case where the ring \( R \) has infinite Krull dimension.

The purpose of this note is to give a characterization of finite modules locally of finite injective dimension. Our theorem is the following.

**Theorem 1.2.** The following are equivalent for a finite \( R \)-module \( N \):

1. \( \text{id}_{R_p} N_p < \infty \) for every \( p \in \text{Spec} R \);
2. For every finite \( R \)-module \( M \), there exists an integer \( n \) such that \( \text{Ext}^i_R(M,N) = 0 \) for all \( i > n \).

This theorem is a generalization of Goto’s. In fact, applying our theorem to \( N = R \), we immediately obtain Goto’s theorem.

2. Proof of the Theorem

We denote by \( CM(R) \) the Cohen-Macaulay locus of \( R \), that is, the set of prime ideals \( p \) of \( R \) such that the local ring \( R_p \) is Cohen-Macaulay. The following lemma can be shown in a similar way to the proof of [3, Theorem 24.5].

**Lemma 2.1.** Let \( p \) be a prime ideal of \( R \) such that both \( R_p \) and \( R/p \) are Cohen-Macaulay rings. Then there exists an element \( f \in R - p \) such that \( D(f) \cap V(p) \subseteq CM(R) \).

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It is known that the Cohen-Macaulay locus of a homomorphic image of a Cohen-Macaulay ring is an open subset; see [3, Exercises 24.2]. To prove our theorem, we need to generalize this fact. For an ideal \( I \) of \( R \), let \( \text{CM}_R(R/I) \) denote the set of prime ideals \( p \in V(I) \) such that the local ring \((R/I)_p\) is Cohen-Macaulay.

**Lemma 2.2.** Let \( I \) and \( J \) be ideals of \( R \). Suppose that \( V(J) \) is contained in \( \text{CM}(R) \). Then the following hold:

1. For any prime ideal \( p \in \text{CM}_R(R/I) \cap V(J) \), there exists an element \( f \in R - p \) such that \( D(f) \cap V(p) \subseteq \text{CM}_R(R/I) \).
2. There exists an ideal \( K \) of \( R \) such that \( \text{CM}_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J) \).

In other words, \( \text{CM}_R(R/I \cap V(J)) \) is an open subset of \( V(I) \cap V(J) \) in the relative topology induced by the Zariski topology of \( \text{Spec} R \).

**Proof:**

1. Let \( p \in \text{CM}_R(R/I) \cap V(J) \). Then by the assumption that \( V(J) \) is contained in \( \text{CM}(R) \), the ring \( R_p \) is a Cohen-Macaulay local ring. Making a similar argument to the proof of [3, Theorem 24.5], we can assume without loss of generality that there is an \( R \)-regular sequence \( x = x_1, x_2, \ldots, x_n \) in \( p \) with \( p^r \subseteq xR \) for some \( r > 0 \) and that \( \overline{p}/\overline{p}^{r+1} \) is a free \( R/\overline{p} \)-module for all \( i > 0 \), where \( \overline{R} = R/\langle x \rangle \) and \( \overline{p} = p/\langle x \rangle R \).

We have only to prove that the residue ring \( R/p \) is Cohen-Macaulay. In fact, if \( R/p \) is Cohen-Macaulay, then so are \((R/I)/(p/I)) \) and \((R/I)_{p/I} \) since \( p \) is in \( \text{CM}_R(R/I) \). Hence Lemma 2.1 implies that there is an element \( f \in R - p \) such that \( D(f) \cap V(p/I) \) is contained in \( \text{CM}(R/I) \), where \( f \) denotes the residue class of \( f \) in \( R/I \). We easily see that \( D(f) \cap V(p) \) is contained in \( \text{CM}_R(R/I) \).

Let us show that \( R/p \) is a Cohen-Macaulay ring. It is easy to see from [3, Exercises 24.1] that \( R/p = \overline{R}/\overline{p} \) is Cohen-Macaulay if and only if so is \( \overline{R} \). Take a prime ideal \( q \in V(\overline{x}) = V(p) \). Then we have \( q \supseteq p \supseteq J \); hence \( q \in V(J) \subseteq \text{CM}(R) \).

Therefore \( R_q \) is a Cohen-Macaulay local ring, and so is \( \overline{R_q} \), because \( x \) is an \( R_q \)-regular sequence. This shows that \( \overline{R} \) is a Cohen-Macaulay ring. Thus we conclude that the residue ring \( R/p \) is Cohen-Macaulay, as desired.

2. Set \( U = \{ p/I + J \mid p \in \text{CM}_R(R/I) \cap V(J) \} \). This is a subset of \( \text{Spec} R/I + J \). Note that this subset is stable under generalization. Let \( P \in U \). Then there is a prime ideal \( p \in \text{CM}_R(R/I) \cap V(J) \) such that \( P = p/I + J \). By the assertion (1) of the lemma, the set \( D(f) \cap V(p) \) is contained in \( \text{CM}_R(R/I) \) for some \( f \in R - p \). Denote by \( \overline{f} \) the residue class of \( f \) in \( R/I \). It is easy to see that \( P \in D(\overline{f}) \cap V(P) \subseteq U \). Thus \( U \) contains a nonempty open subset of \( V(P) \). By virtue of the topological Nagata criterion [3, Theorem 24.2], \( U \) is an open subset of \( \text{Spec} R/I + J \); we have \( U = D(K/I + J) \) for some ideal \( K \) of \( R \) containing \( I + J \). Then it is easily checked that \( \text{CM}_R(R/I) \cap V(J) = D(K) \cap V(I) \cap V(J) \).

Now, we can prove our theorem.

**Proof of Theorem 1.2**

(2) \( \Rightarrow \) (1): Let \( p \) be a prime ideal of \( R \). Then there is an integer \( n \) such that \( \text{Ext}_R^i(R/p, N) = 0 \) for all \( i > n \). Hence we have \( \text{Ext}_R^i(p, \kappa(p), N_p) = 0 \) for all \( i > n \). Therefore by [1, Theorem 3.1.14] we obtain \( \text{id}_{R_p} \) \( N_p \leq n < \infty \).

(1) \( \Rightarrow \) (2): First of all, note that (2) is equivalent to the statement that for each ideal \( I \) of \( R \) there is an integer \( n \) such that \( \text{Ext}_R^i(R/I, N) = 0 \) for all \( i > n \). (This can easily be proved by induction on the number of generators of the \( R \)-module
Suppose that there exists an ideal $I$ of $R$ such that for any integer $n$ there is an integer $i > n$ such that $\text{Ext}^i_R(R/I, N) \neq 0$. We want to derive a contradiction. Since $R$ is Noetherian, one can choose $I$ to be a maximal one among such ideals. Making a similar argument to the proof of Theorem 1.1, we see that the ideal $I$ is prime and that for any element $f \in R - I$, the map

$$\text{Ext}^i_R(R/I, N) \to \text{Ext}^i_R(R/I, N)$$

is an isomorphism for $i \gg 0$.

**Claim 1.** One has $I \in \text{Supp}_R N \subseteq \text{CM}(R)$.

*Proof of Claim.* Our assumption (1) implies that for any $p \in \text{Supp}_R N$, the nonzero finite $R_p$-module $N_p$ has finite injective dimension. Hence $R_p$ is a Cohen-Macaulay local ring; see [1, Corollary 9.6.2 and Remarks 9.6.4]. Thus $\text{Supp}_R N$ is contained in $\text{CM}(R)$. On the other hand, assume that $I$ is not in $\text{Supp}_R N$. Then there exists an element $f \in \text{Ann}_R N - I$, and the map $\text{Ext}^i_R(R/I, N) \to \text{Ext}^i_R(R/I, N)$ is an isomorphism for $i \gg 0$. Since $fN = 0$, this map is the zero map, and we get $\text{Ext}^j_R(R/I, N) = 0$ for $i \gg 0$. It follows from this contradiction that $I$ belongs to $\text{Supp}_R N$.

Noting that $\text{Supp}_R N = V(\text{Ann}_R N)$, we see from Claim 1 and Lemma 2.2(2) that there is an ideal $K$ of $R$ such that $\text{CM}(R/I) \cap \text{Supp}_R N = D(K) \cap V(I) \cap \text{Supp}_R N$. The localization $(R/I)_f = \kappa(I)$ is a field, hence a Cohen-Macaulay ring. It is seen from Claim 1 again that $I \in \text{CM}(R/I) \cap \text{Supp}_R N \subseteq D(K)$. Thus there is an element $f \in K - I$.

**Claim 2.** For any prime ideal $p \in D(f)$ and any integer $i > \text{ht} I$, one has $\text{Ext}^i_{R_p}(R_p/IR_p, N_p) = 0$.

*Proof of Claim.* We may assume that $p$ belongs to both $V(I)$ and $\text{Supp}_R N$, because otherwise the module $\text{Ext}^i_{R_p}(R_p/IR_p, N_p)$ automatically vanishes. Hence Claim 1 implies that $p$ belongs to $\text{CM}(R)$, namely the local ring $R_p$ is Cohen-Macaulay. Added to it, since $D(f)$ is contained in $D(K)$, we have $p \in D(K) \cap V(I) \cap \text{Supp}_R N \subseteq \text{CM}(R/I)$, and therefore $R_p/IR_p$ is Cohen-Macaulay. Thus we get the following equalities:

$$\text{depth } R_p - \text{depth } R_p/IR_p = \dim R_p - \dim R_p/IR_p = \text{ht } IR_p = \text{ht } I.$$

Since $N_p$ is a finite $R_p$-module of finite injective dimension by assumption, it follows from the result of Ischebeck [1, Exercises 3.1.24] that $\text{Ext}^i_{R_p}(R_p/IR_p, N_p) = 0$ for every $i > \text{ht} I$.

Claim 2 means that $(\text{Ext}^i_{R_f}(R_f/IR_f, N_f))_p = 0$ for every $p \in \text{Spec } R_f$ and every $i > \text{ht} I$. Therefore, $\text{Ext}^i_{R_f}(R_f/IR_f, N_f) = 0$ for $i > \text{ht} I$. The $R$-module $\text{Ext}^i_R(R/I, N)$ is isomorphic to $\text{Ext}^i_{R_f}(R_f/IR_f, N_f)$ for $i \gg 0$, and thus $\text{Ext}^i_R(R/I, N) = 0$ for $i \gg 0$. This contradiction completes the proof of our theorem.

**References**


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