A SHARP VANISHING THEOREM FOR LINE BUNDLES
ON K3 OR ENRIQUES SURFACES

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Abstract. Let $L$ be a line bundle on a K3 or Enriques surface. We give a vanishing theorem for $H^1(L)$ that, unlike most vanishing theorems, gives necessary and sufficient geometrical conditions for the vanishing. This result is essential in our study of Brill-Noether theory of curves on Enriques surfaces (2006) and of Enriques-Fano threefolds (2006 preprint).

1. Introduction

Since Grothendieck’s introduction of basic tools such as the cohomology of sheaves and the Grothendieck-Riemann-Roch theorem, vanishing theorems have proved to be essential in many studies in algebraic geometry.

Perhaps the most influential one, at least for line bundles, is the well-known Kawamata-Viehweg vanishing theorem ([K, V]) which, in its simplest form, asserts that $H^i(K_X + L) = 0$ for $i > 0$ and any big and nef line bundle $L$ on a smooth variety $X$. On the other hand, as most vanishing theorems (even for special surfaces [CD, Thm.1.5.1]), it gives only sufficient conditions for the vanishing. Practice shows though that, in many situations, it would be very useful to know that a certain vanishing is equivalent to some geometrical/numerical properties of $L$.

In this short note we accomplish the above goal for line bundles on a K3 or Enriques surface, by proving that, when $L^2 > 0$, the vanishing of $H^1(L)$ is equivalent to the fact that the intersection of $L$ with all effective divisors of self-intersection $-2$ is at least $-1$.

In the statement of the theorem we will employ the following

Definition 1.1. Let $X$ be a smooth surface. We will denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on $X$. We will say that a line bundle $L$ is primitive if $L \equiv kL'$ for some line bundle $L'$ and some integer $k$ implies $k = \pm 1$.

Theorem. Let $X$ be a K3 or an Enriques surface and let $L$ be a line bundle on $X$ such that $L > 0$ and $L^2 \geq 0$. Then $H^1(L) \neq 0$ if and only if one of the three following occurs:

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(i) \( L \sim nE \) for \( E > 0 \) nef and primitive with \( E^2 = 0 \), \( n \geq 2 \) and \( h^1(L) = n - 1 \)
if \( X \) is a K3 surface, \( h^1(L) = \left\lfloor \frac{n}{2} \right\rfloor \) if \( X \) is an Enriques surface;
(ii) \( L \sim nE + K_X \) for \( E > 0 \) nef and primitive with \( E^2 = 0 \), \( X \) is an Enriques
surface, \( n \geq 3 \) and \( h^1(L) = \left\lfloor \frac{n-1}{2} \right\rfloor \);
(iii) there is a divisor \( \Delta > 0 \) such that \( \Delta^2 = -2 \) and \( \Delta \cdot L \leq -2 \).

Note that the hypothesis \( L > 0 \) is not restrictive since, if \( L \) is nontrivial, from
\( L^2 \geq 0 \) we get by Riemann-Roch that either \( L > 0 \) or \( K_X - L > 0 \), and \( h^1(L) = h^1(K_X - L) \) by Serre duality.

The theorem has of course many possible applications. For example, if \( L \) is
base-point free and \( |P| \) is an elliptic pencil on \( X \), the knowledge of \( h^0(L - nP) \) for
\( n \geq 1 \) (which follows by Riemann-Roch if we know that \( h^1(L - nP) = 0 \)) determines
the type of scroll spanned by the divisors of \( |P| \) in \( \mathbb{P}H^0(L) \) and containing \( \varphi_L(X) \)
([SD, KJ, Co]). Most importantly for us, this result proves crucial in our study
of the Brill-Noether theory [KL1, KL2] and Gaussian maps [KL3] of curves lying
on an Enriques surface, and especially in our proof of a genus bound for threefolds
having an Enriques surface as a hyperplane section given in [KLM].

2. Proof of the Theorem

We first record the following simple but useful fact.

**Lemma 2.1.** Let \( X \) be a smooth surface and let \( A > 0 \) and \( B > 0 \) be divisors on
\( X \) such that \( A^2 \geq 0 \) and \( B^2 \geq 0 \). Then \( A.B \geq 0 \) with equality if and only if there
exists a primitive divisor \( F > 0 \) and integers \( a \geq 1, b \geq 1 \) such that \( F^2 = 0 \) and
\( A \equiv aF, B \equiv bF \).

**Proof.** The first assertion follows from the signature theorem [BPV, VIII.1]. If
\( A.B = 0 \), then we cannot have \( A^2 > 0 \), otherwise the Hodge index theorem implies
the contradiction \( B \equiv 0 \). Therefore \( A^2 = B^2 = 0 \). Now let \( H \) be an ample
line bundle on \( X \) and set \( \alpha = A.H, \beta = B.H \). We have \((\beta A - \alpha B)^2 = 0 \) and
\((\beta A - \alpha B).H = 0 \), therefore \( \beta A \equiv \alpha B \) by the Hodge index theorem. As there is no
torsion in \( \text{Num}(X) \) we can find a divisor \( F \) as claimed. \( \square \)

We now proceed with the theorem.

**Proof.** One immediately sees that \( h^1(L) \) has the given values in (i) and (ii). In the
case (iii) we first observe that \( h^2(L - \Delta) = 0 \). In fact \((K_X - L + \Delta)^2 > 0 \), whence if
\( K_X - L + \Delta \geq 0 \) the signature theorem [BPV, VIII.1] implies \( 0 \leq L.(K_X - L + \Delta) =
-2L^2 + L \cdot \Delta \leq -2 \), a contradiction. Therefore by Riemann-Roch we get
\[
\frac{1}{2}L^2 + \chi(O_X) < \frac{1}{2}L^2 - \Delta \cdot L + 1 + \chi(O_X) \leq h^0(L - \Delta) \leq h^0(L) = \frac{1}{2}L^2 + \chi(O_X) + h^1(L)
\]
whence \( h^1(L) > 0 \).

Now assume that \( h^1(L) > 0 \).

First we suppose that \( L \) is nef. By Riemann-Roch we have that \( L + K_X > 0 \).
Since \( h^1(-L + K_X)) = h^1(L) > 0 \), by [BPV, Lemma12.2], we deduce that \( L + K_X \)
is not 1-connected, whence that there exist \( L' > 0 \) and \( L'' > 0 \) such that \( L + K_X \sim
L' + L'' \) and \( L'.L'' \leq 0 \). Now \((L')^2 \geq (L')^2 + L'.L'' = L'.L \geq 0 \) and similarly
\((L'')^2 \geq 0 \), whence Lemma 2.1 implies that \( L' \equiv aE, L'' \equiv bE \) for some \( a, b \geq 1 \)
and for \( E > 0 \) nef and primitive with \( E^2 = 0 \). This gives us the two cases (i) and
(ii).
Now assume that $L$ is not nef, so that the set
$$\mathcal{A}_1(L) := \{ \Delta > 0 : \Delta^2 = -2, \Delta.L \leq -1 \}$$
is not empty. Similarly define the set
$$\mathcal{A}_2(L) = \{ \Delta > 0 : \Delta^2 = -2, \Delta.L \leq -2 \}.$$

If $\mathcal{A}_2(L) \neq \emptyset$ we are done. Assume therefore that $\mathcal{A}_2(L) = \emptyset$ and pick $\Gamma \in \mathcal{A}_1(L)$. Then $\Gamma.L = -1$, and we can clearly assume that $\Gamma$ is irreducible. Hence if we set $L_1 = L - \Gamma$ we have that $L_1 > 0$, $L_1^2 = L^2$ and, since $h^0(L_1) = h^0(L)$, also that $h^1(L_1) = h^1(L) > 0$.

If $L_1$ is nef, by what we have just seen, we have $L_1 \equiv nE$, for $n \geq 2$, whence $L \equiv nE + \Gamma$ and $-1 = \Gamma.L = nE.\Gamma = -2$, a contradiction.

Therefore $L_1$ is not nef and $\mathcal{A}_1(L_1) \neq \emptyset$.

If $\mathcal{A}_2(L_1) \neq \emptyset$ we pick a $\Delta \in \mathcal{A}_2(L_1)$. We have $-2 \geq \Delta.L_1 = \Delta.(L - \Gamma) \geq -1 - \Delta.\Gamma$, whence $\Delta.\Gamma \geq 1$, $(\Delta + \Gamma)^2 \geq -2$ and $(\Delta + \Gamma).L_1 \leq -1$. Now Lemma 2.3 yields $(\Delta + \Gamma)^2 = -2$, so that $\Delta.\Gamma = 1$. Also $-1 \leq \Delta.L = \Delta.(L_1 + \Gamma) \leq -1$, whence $\Delta.L = -1$ and $(\Delta + \Gamma).L = -2$, contradicting $\mathcal{A}_2(L) = \emptyset$.

We have therefore shown that $\mathcal{A}_2(L_1) = \emptyset$.

This means that we can continue the process. But the process must eventually stop, since we always remove base components. This gives the desired contradiction.

\[ \square \]

Remark 2.2. A naive guess, to insure the vanishing of $H^1(L)$ for a line bundle $L > 0$ with $L^2 \geq 0$, could be that it is enough to add the hypothesis $L.R \geq -1$ for every irreducible rational curve $R$. However this is not true. Take, for example, a nef divisor $B$ with $B^2 \geq 4$ and two irreducible rational curves $R_1, R_2$ such that $B.R_i = 0$, $R_1.R_2 = 1$. Then $L := B + R_1 + R_2$ satisfies the above requirements, but $L.(R_1 + R_2) = -2$, whence $H^1(L) \neq 0$ by the theorem.

Remark 2.3. It would be of interest to know if, in the statement of the theorem, it is possible to replace divisors $\Delta > 0$ such that $\Delta^2 = -2$ with chains of irreducible rational curves.

Definition 2.4. An effective line bundle $L$ on a K3 or Enriques surface is said to be quasi-nef if $L^2 \geq 0$ and $L.\Delta \geq -1$ for every $\Delta$ such that $\Delta > 0$ and $\Delta^2 = -2$.

An immediate consequence of the theorem is

Corollary 2.5. An effective line bundle $L$ on a K3 or Enriques surface is quasi-nef if and only if $L^2 \geq 0$ and either $h^1(L) = 0$ or $L \equiv nE$ for some $n \geq 2$ and some primitive and nef divisor $E$ with $E^2 = 0$.

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