THE ROGERS-RAMANUJAN CONTINUED FRACTION
AND A QUINTIC ITERATION FOR 1/π

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(Communicated by Jonathan M. Borwein)

ABSTRACT. Properties of the Rogers-Ramanujan continued fraction are used to obtain a formula for calculating 1/π with quintic convergence.

1. INTRODUCTION

Let q be a complex number satisfying |q| < 1. The Rogers-Ramanujan continued fraction is

\[ R(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}. \]

The purpose of this article is to use properties of the Rogers-Ramanujan continued fraction to derive the following iteration for 1/π.

Theorem 1.1. Let \( g = (1 + \sqrt{5})/2 \). Define sequences by

\[
\begin{align*}
    s_0 &= \left( \sqrt{g^{10} + 1} - g^5 \right)^{1/5}, \\
    k_0 &= 0, \\
    r_{n+1} &= \left( \frac{1 - g^5 s_n^5}{g^5 + s_n^5} \right)^{1/5}, \\
    s_{n+1} &= \frac{1 - g r_{n+1}}{g + r_{n+1}}, \\
    k_{n+1} &= \frac{(s_{n+1} + g)^4 (g^2 s_{n+1}^2 + g^2 s_{n+1} + 1)}{g^2 (s_{n+1}^2 - g^2 s_{n+1} + g^2)} k_n \\
    &\quad + \frac{2 \times 5^{-1/2} g^2 s_{n+1} (1 - g s_{n+1}) (g^2 s_{n+1}^2 - s_{n+1} + 1)}{(s_{n+1} + g) (s_{n+1}^2 - g^2 s_{n+1} + g^2)} f(s_{n+1}),
\end{align*}
\]

where

\[ f(s) = 4s^4 - (2 + 5g)s^3 + (5 - 3g)s^2 + (6 + 7g)s + (5 + 3g). \]

Then \( k_n \) converges to 1/π, and the rate of convergence is order 5.
A cubic iteration for $1/\pi$, based on Ramanujan’s cubic continued fraction

$$G(q) = \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^6}{1 + \cdots}}}},$$

has been given by H. H. Chan and K. P. Loo [11]. Our Theorem 1.1 is the analogue of [11, Theorem 2.1], for which the Rogers-Ramanujan continued fraction takes the role that Ramanujan’s cubic continued fraction played in [11]. The method, in both the present work and in [11], is based on ideas developed in [10].

Theorem 1.1 is different from the quintic iterations of J. M. and P. B. Borwein in [6, p. 175], [7] and [8, p. 202], which were obtained using quintic modular equations. Other iterations for $1/\pi$ based on Dedekind’s $\eta$-function and modular functions were given by J. M. Borwein and F. G. Garvan [9], and iterations based on elliptic functions were given by Chan [10].

2. SOME PRELIMINARY RESULTS

In this section, we collect some important results concerning the Rogers-Ramanujan continued fraction and some allied functions. Two good sources of information about the Rogers-Ramanujan continued fraction are the last chapter of the introductory book by B. C. Berndt [4] and the expository article by W. Duke [13].

The first significant fact about the Rogers-Ramanujan continued fraction is its expression in terms of an infinite product:

$$R(q) = q^{1/5} \prod_{j=1}^{\infty} \frac{(1 - q^{5j-4})(1 - q^{5j-1})}{(1 - q^{5j-3})(1 - q^{5j-2})}.$$

An outline of a proof of this result, together with references, can be found in Berndt’s book [4].

Let

$$X(q) = R^5(q),$$

$$y(q) = R(q^5),$$

and

$$Z(q) = \prod_{j=1}^{\infty} \frac{(1 - q^j)^5}{(1 - q^{5j})}.$$  

When it is not necessary to emphasize the parameter $q$, we will simply write $R$, $X$, $y$ and $Z$ for $R(q)$, $X(q)$, $y(q)$ and $Z(q)$, respectively.

We will use the golden ratio, which we denote by

$$g = \frac{1 + \sqrt{5}}{2}.$$

We will require the formulas

$$\frac{1}{R} - 1 - R = \frac{1}{q^{1/5}} \prod_{j=1}^{\infty} \frac{(1 - q^{j/5})}{(1 - q^{5j})},$$

and

$$\frac{1}{X} - 11 - X = \frac{1}{q} \prod_{j=1}^{\infty} \frac{(1 - q^j)^6}{(1 - q^{5j})^6}.$$
Simple proofs of these results, using only the Jacobi triple product identity, have been given by M. Hirschhorn [13]. More information about the identities (2.5) and (2.6), and references to other proofs, can be found in the book by G. E. Andrews and B. C. Berndt [1, pp. 11–12].

The function $Z$ has a simple Lambert series expansion:

$$Z = 1 - 5 \sum_{j=1}^{\infty} \left( \frac{j}{5} \right) \frac{jq^j}{1 - q^j},$$

where $\left( \frac{4}{5} \right)$ is the Legendre symbol. This formula was given by Ramanujan [16, Chapter 19, Entry 9 (v)]. For proofs, see Berndt’s book [2] pp. 257–261 or the papers by J. M. Dobbie [12] and Hirschhorn [14]. References to other proofs are given in [2] and [14].

The functions $R$ and $R^5$ satisfy the modular properties [13, eqs. (3.2) and (7.3)]

$$R \left( e^{-2\pi/\alpha} \right) = 1 - gR(e^{-2\pi\alpha}/g + R(e^{-2\pi\alpha})),$$

$$R^5 \left( e^{-2\pi/5\alpha} \right) = 1 - g^5R^5(e^{-2\pi\alpha}/g^5 + R^5(e^{-2\pi\alpha})),$$

where $\alpha$ is any complex number satisfying $\text{Re}(\alpha) > 0$. If we let $\alpha = \sqrt{t/5}$ and rearrange, then (2.9) may be rewritten as

$$\left( g^5 + X \left( e^{-2\pi \sqrt{t/5}} \right) \right) \left( g^5 + X \left( e^{-2\pi / \sqrt{5t}} \right) \right) = 1 + g^{10}.$$  

This result appears in Ramanujan’s lost notebook [1, p. 91], [17, p. 364]. If we replace $\alpha$ with $5\alpha$ in (2.8) and combine the result with (2.9), we obtain a relation between $u = R(q)$ and $v = R(q^5)$ given by

$$\left( \frac{1 - gv}{g + v} \right)^5 = 1 - g^5u^5,$$

where $g = 1 - 2v + 4v^2 - 3v^3 + v^4$.

If we solve for $u^5$, we obtain

$$u^5 = \frac{1 - 2v + 4v^2 - 3v^3 + v^4}{1 + 3v + 4v^2 + 2v^3 + v^4}.$$  

On the other hand, if we solve (2.11) for $v$, we obtain

$$v = \frac{1 - g \left( \frac{1-u^5}{g+v} \right)^{1/5}}{g + \left( \frac{1-u^5}{g+v} \right)^{1/5}}.$$  

Equation (2.12) was given by Ramanujan in his first letter to Hardy [5, p. 29]. Equation (2.13) will be used in our iteration for $1/\pi$.

3. A FORMULA FOR $1/\pi$

3.1. The functions $A(q)$ and $\kappa(t)$. Let

$$q = \exp \left( \frac{-2\pi \sqrt{7}}{\sqrt{5}} \right), \quad p = \exp \left( \frac{-2\pi}{\sqrt{5t}} \right), \quad t > 0.$$
If we logarithmically differentiate (2.2) and use (2.7), we obtain

\[
\frac{dX}{dq} = X \left( 1 - 5 \sum_{j=1}^{\infty} \frac{j}{5} \frac{jq^j}{1-q^j} \right) = ZX.
\]

(3.1)

Differentiating (2.10) and using (3.1), we get

\[
t \frac{Z(q)X(q)}{g^5 + X(q)} = \frac{Z(p)X(p)}{g^5 + X(p)}.
\]

(3.2)

We may rewrite (2.10) as

\[
X(q) = \frac{(g^5 + X(q))(1 - g^5X(p))}{g^{10} + 1},
\]

and replacing \(t\) with \(1/t\), we obtain

\[
X(p) = \frac{(g^5 + X(p))(1 - g^5X(q))}{g^{10} + 1}.
\]

(3.3)

Substituting (3.3) and (3.4) into (3.2), we deduce that

\[
t \frac{Z(q)}{1-g^5X(q)} = \frac{Z(p)}{1-g^5X(p)}.
\]

(3.5)

If we define

\[
A(q) = \frac{Z(q)}{1-g^5X(q)},
\]

then (3.5) reduces to

\[
tA(q) = A(p).
\]

(3.6)

Differentiating (3.7) with respect to \(t\), we find that

\[
A(q) - \frac{\pi \sqrt{t}}{\sqrt{5}} \widetilde{A}(q) = \frac{\pi}{\sqrt{5t^3}} \widetilde{A}(p),
\]

where

\[
\widetilde{f}(z) = z \frac{df}{dz}.
\]

Multiplying both sides by \(2/\pi A(q)\), we deduce that

\[
\left( \frac{1}{\pi} - \frac{2\sqrt{t}}{\sqrt{5}} \widetilde{A}(q) \right) + \left( \frac{1}{\pi} - \frac{2\sqrt{t}}{\sqrt{5t}} \widetilde{A}(p) \right) = 0.
\]

(3.8)

If we define

\[
\kappa(t) = \frac{1}{\pi A(q)} - \frac{2\sqrt{t}}{\sqrt{5}} \frac{\widetilde{A}(q)}{A^2(q)},
\]

then (3.5) becomes, after dividing by \(A(q)\), simply

\[
\kappa(t) + t\kappa \left( \frac{1}{t} \right) = 0.
\]

(3.9)
3.2. The multiplier. Let

\begin{equation}
M_N(q) = \frac{A(q)}{A(q^N)}.
\end{equation}

We will be particularly interested in \(M_5(q)\). Observe that by (3.6),

\begin{equation}
M_5(q) = \frac{A(q)}{A(q^5)} = \frac{Z(q)}{(1 - g^5 X(q))} \frac{(1 - g^5 X(q^5))}{Z(q^5)} = \frac{Z(q)}{Z(q^5)} \frac{(1 - g^5 y^5)}{(1 - g^5 X)}.
\end{equation}

By (2.4), we have

\begin{equation}
\frac{\partial}{\partial y} \left( \prod_{j=1}^{\infty} \frac{(1 - q^j)^5}{(1 - q^{5j})^5} \frac{(1 - q^{25j})}{(1 - q^{125j})} \right) = \frac{1}{q} \prod_{j=1}^{\infty} \frac{(1 - q^j)^3}{(1 - q^{5j})^3} = \prod_{j=1}^{\infty} \frac{1 - q^{25j}}{1 - q^{125j}}.
\end{equation}

By (2.5) and (2.6), we obtain

\begin{equation}
M_5(q) = \frac{\prod_{j=1}^{\infty} (1 - q^j)^5}{\prod_{j=1}^{\infty} (1 - q^{5j})^5} \frac{(1 - g^5) y^5}{(1 - g^5) X}.
\end{equation}

Now using (2.12) and the relations \(u^5 = X\) and \(v = y\), we may express \(X\) in terms of \(y\). The final result is

\begin{equation}
M_5(q) = \frac{(y + g)^4 (g^2 y^2 + g^2 y + 1)}{g^2 (y^2 - g^2 y + g^2)}.
\end{equation}

Differentiating (3.12) gives

\begin{equation}
\frac{dM_5}{dy} = \frac{(y + g)^3}{(y^2 - g^2 y + g^2)^2} f(y),
\end{equation}

where

\begin{equation}
f(y) = 4y^4 - (2 + 5g) y^3 + (5 - 3g) y^2 + (6 + 7g) y + (5 + 3g).
\end{equation}

By the chain rule, together with (2.20), (2.23) and (3.11), we obtain

\begin{equation}
q \frac{dM_5}{dq} = Z(q^5) y \frac{dM_5}{dy}.
\end{equation}

Therefore, using (3.6), (3.12) and (3.13), we obtain

\begin{equation}
\frac{\bar{M}_5(q)}{M_5(q) A(q^5)} = \frac{Z(q^5) y}{M_5(q) A(q^5)} \frac{dM_5}{dy} = \frac{y(1 - g^5 y^5)}{M_5(q) A(q^5)} \frac{dM_5}{dy} = \frac{g^2 y (1 - g y)(g^2 y^2 - y + 1)}{(y + g)(y^2 - g^2 y + g^2)} f(y).
\end{equation}
3.3. A functional equation for $\kappa$. In this section, we obtain a formula that expresses $\kappa(tN^2)$ in terms of $\kappa(t)$. The iteration for $1/\pi$ is based on this formula.

Logarithmically differentiating (3.11), we get
\[
\frac{\widetilde{M}_N(q)}{M_N(q)} = \frac{\widetilde{A}(q)}{A(q)} - N\frac{\widetilde{A}(q^N)}{A(q^N)}.
\]

Divide by $A(q^N)$ and use (3.11) again to get
\[
\frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)} = \frac{\widetilde{A}(q)}{A(q)A(q^N)} - N\frac{\widetilde{A}(q^N)}{A^2(q^N)}
\]
\[= M_N(q)\frac{\tilde{A}(q)}{A^2(q)} - N\frac{\widetilde{A}(q^N)}{A^2(q^N)}.
\]

Now multiply by $2\sqrt{t/5}$ and use (3.10) to get
\[
2\sqrt{\frac{t}{5}} \frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)} = M_N(q)2\sqrt{\frac{t}{5}} \frac{\tilde{A}(q)}{A^2(q)} - 2\sqrt{\frac{tN^2}{5}} \frac{\tilde{A}(q^N)}{A^2(q^N)}
\]
\[= M_N(q)\left(1 - \frac{\pi A(q)}{A(q^N)} - \kappa(t)\right) - \left(\frac{\pi A(q^N)}{A^2(q^N)} - \kappa(tN^2)\right)
\]
\[= \kappa(tN^2) - M_N(q)\kappa(t).
\]

Therefore,
\[\kappa(tN^2) = M_N(q)\kappa(t) + 2\sqrt{\frac{t}{5}} \frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)}.
\]

3.4. An iteration for $1/\pi$. If we let $\alpha = 1/\sqrt{5}$ in (2.20) and solve the resulting quadratic equation in $R^5$, we obtain
\[R(e^{-2\pi\sqrt{5}}) = \left(\sqrt{g^{10} + 1} - g^5\right)^{1/5}.
\]

Now let $t = 1$ in (3.10) to get
\[\kappa(1) = 0.
\]

Define two sequences by
\[k_n = \kappa(5^{2n}),
\]
\[s_n = R(e^{-2\pi\sqrt{5} - 1})
\]

where $n$ is a non-negative integer. By the calculations just done, we have
\[k_0 = 0,\quad s_0 = \left(\sqrt{g^{10} + 1} - g^5\right)^{1/5}.
\]

Furthermore, expanding (3.9) in a series gives
\[\kappa(t) = \frac{1}{\pi} - (1 + 5\sqrt{5}) \left(\frac{1}{2\pi} + \frac{t}{5}\right) q + O(q^2), \quad \text{as } t \to \infty.
\]

Therefore
\[k_n - \frac{1}{\pi} \sim -(1 + 5\sqrt{5}) \left(\frac{1}{2\pi} + \frac{5^n}{\sqrt{5}}\right) \exp\left(-\frac{2\pi}{\sqrt{5}} 5^n\right), \quad \text{as } n \to \infty.
\]
It follows that \( k_n \) converges to \( 1/\pi \) and the rate of convergence is order 5. The identity (3.13) with \( q = e^{-2\pi\sqrt{5n-1}} \) gives

\[
(3.18) \quad s_{n+1} = \frac{1 - g \left( \frac{1-g^5 s_n}{g^5 + s_n} \right)^{1/5}}{g + \left( \frac{1-g^5 s_n}{g^5 + s_n} \right)^{1/5}}.
\]

Let \( N = 5 \) and \( t = 5^{2n} \) in (3.16). We find that

\[
k_{n+1} = M_5 \left( e^{-2\pi\sqrt{5n-1}} \right) k_n + 2 \times 5^{n-1/2} \frac{M_5 \left( e^{-2\pi\sqrt{5n-1}} \right) A \left( e^{-2\pi\sqrt{5n-1}} \right)}{M_5 \left( e^{-2\pi\sqrt{5n-1}} \right)}.
\]

Using (3.12) and (3.15), we have

\[
(3.19) \quad k_{n+1} = \frac{(s_{n+1} + g)^4 (g^2 s_{n+1}^2 + g^2 s_{n+1} + 1)}{g^2 (s_{n+1}^2 - g^2 s_{n+1} + g^2)} k_n
\]

\[
+ 2 \times 5^{n-1/2} \frac{g^2 s_{n+1} (1 - g s_{n+1}) (g^2 s_{n+1}^2 - s_{n+1} + 1)}{(s_{n+1} + g)^2 (s_{n+1}^2 - g^2 s_{n+1} + g^2)} f(s_{n+1}).
\]

Identities (3.17), (3.18) and (3.19) imply Theorem 1.1.

Remark 3.1. The values of \( 1/k_1, 1/k_2, 1/k_3, 1/k_4 \) and \( 1/k_5 \) give \( \pi \) correct to 3, 27, 148, 758 and 3808 decimal places, respectively.

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