MINIMAL SYSTEMS OF BINOMIAL GENERATORS AND THE INDISPENSABLE COMPLEX OF A TORIC IDEAL

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Abstract. Let \( A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n \) be a vector configuration and \( I_A \subset K[x_1, \ldots, x_m] \) its corresponding toric ideal. The paper consists of two parts. In the first part we completely determine the number of different minimal systems of binomial generators of \( I_A \). In the second part we associate to \( A \) a simplicial complex \( \Delta_{\text{ind}}(A) \). We show that the vertices of \( \Delta_{\text{ind}}(A) \) correspond to the indispensable monomials of the toric ideal \( I_A \), while one dimensional facets of \( \Delta_{\text{ind}}(A) \) with minimal binomial \( A \)-degree correspond to the indispensable binomials of \( I_A \).

1. Introduction

Let \( A = \{a_1, \ldots, a_m\} \) be a vector configuration in \( \mathbb{Z}^n \) such that the affine semi-group \( \mathbb{N}A := \{l_1a_1 + \cdots + l_ma_m \mid l_i \in \mathbb{N}\} \) is pointed. Recall that \( \mathbb{N}A \) is pointed if zero is the only invertible element. Let \( K \) be a field of any characteristic; we grade the polynomial ring \( K[x_1, \ldots, x_m] \) by setting \( \deg_A(x_i) = a_i \) for \( i = 1, \ldots, m \). For \( \mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m \), we define the \( A \)-degree of the monomial \( x^u := x_1^{u_1} \cdots x_m^{u_m} \) to be

\[
\deg_A(x^u) := u_1a_1 + \cdots + u_ma_m \in \mathbb{N}A.
\]

The toric ideal \( I_A \) associated to \( A \) is the prime ideal generated by all the binomials \( x^u - x^v \) such that \( \deg_A(x^u) = \deg_A(x^v) \) (see [13]). For such binomials, we define \( \deg_A(x^u - x^v) := \deg_A(x^v) \).

In general it is possible for a toric ideal \( I_A \) to have more than one minimal system of generators. We define \( \nu(I_A) \) to be the number of different minimal systems of binomial generators of the toric ideal \( I_A \), where the sign of a binomial does not count. Minimal systems of binomial generators of toric ideals have been studied in several papers; see [3] and its introduction. A recent problem arising from algebraic statistics (see [14]), is when a toric ideal possesses a unique minimal system of binomial generators, i.e. \( \nu(I_A) = 1 \). To study this problem, Ohsugi and Hibi introduced in [10] the notion of indispensable binomials while Aoki, Takemura and Yoshida introduced in [1] the notion of indispensable monomials. Moreover in [10]...
a necessary and sufficient condition is given for toric ideals associated with certain finite graphs to possess unique minimal systems of binomial generators. We recall that a binomial \( B = x^u - x^v \in I_A \) is indispensable if every system of binomial generators of \( I_A \) contains \( B \) or \(-B\), while a monomial \( x^u \) is indispensable if every system of binomial generators of \( I_A \) contains a binomial \( B \) such that the \( x^u \) is a monomial of \( B \).

In this article we use and extend ideas and techniques developed by Diaconis and Sturmfels (see [7]) and Takemura and Aoki (see [14]) to study minimal systems of binomial generators of the toric ideal \( I_A \) and also to investigate the notion of the indispensable complex of \( I_A \), denoted by \( \Delta_{\text{ind}(A)} \). In Section 2 we construct graphs \( G(b) \), for every \( b \in NA \), and use them to provide a formula for \( \nu(I_A) \). We give criteria for a toric ideal to be generated by indispensable binomials. In Section 3 we define \( \Delta_{\text{ind}(A)} \) and we show that this complex determines the indispensable monomials and binomials. As an application we characterize principal toric ideals in terms of \( \Delta_{\text{ind}(A)} \).

2. The Number of Minimal Generating Sets of a Toric Ideal

Let \( A \subset \mathbb{Z}^n \) be a vector configuration so that \( NA \) is pointed and let \( I_A \subset K[x_1,\ldots,x_m] \) be its corresponding toric ideal. A vector \( b \in NA \) is called a Betti \( A \)-degree if \( I_A \) has a minimal generating set containing an element of \( A \)-degree \( b \). The Betti \( A \)-degrees are independent of the choice of a minimal generating set of \( I_A \); see [4, 9, 13]. The \( A \)-graded Betti number \( \beta_{A,b} \) of \( I_A \) is the number of times \( b \) appears as the \( A \)-degree of a binomial in a given minimal generating set of \( I_A \) and is also an invariant of \( I_A \).

The semigroup \( NA \) is pointed, so we can partially order it with the relation

\[
\mathbf{c} \geq \mathbf{d} \iff \text{there is an } e \in NA \text{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e}.
\]

For \( I_A \neq \{0\} \) the minimal elements of the set \( \{ \deg_A(x^u) \mid x^u - x^v \in I_A \} \subset NA \) with respect to \( \geq \) are called minimal binomial \( A \)-degrees. Minimal binomial \( A \)-degrees are always Betti \( A \)-degrees but the converse is not true, as Example 2.3 demonstrates. For any \( b \in NA \) we define the ideal

\[
I_{A,b} := (x^u - x^v \mid \deg_A(x^u) = \deg_A(x^v) \leq b) \subset I_A.
\]

**Definition 2.1.** For a vector \( b \in NA \) we define \( G(b) \) to be the graph with vertices the elements of the fiber

\[
\deg^{-1}_A(b) = \{ x^u \mid \deg_A(x^u) = b \}
\]

and edges all the sets \( \{ x^u, x^v \} \) whenever \( x^u - x^v \in I_{A,b} \).

The fiber \( \deg^{-1}_A(b) \) has finitely many elements, since the affine semigroup \( NA \) is pointed. If \( x^u, x^v \) are vertices of \( G(b) \) such that \( \gcd(x^u, x^v) \neq 1 \), then \( \{ x^u, x^v \} \) is an edge of \( G(b) \). The next proposition follows easily from the definition.

**Proposition 2.2.** Let \( b \in NA \). Every connected component of \( G(b) \) is a complete subgraph. The graph \( G(b) \) is not connected if and only if \( b \) is a Betti \( A \)-degree.

**Example 2.3.** Let

\[
A = \{(2,2,2,0,0),(2,-2,-2,0,0),(2,2,-2,0,0),(2,-2,2,0,0), (3,0,0,3,3),(3,0,0,-3,-3),(3,0,0,3,-3),(3,0,0,-3,3)\}.
\]
Using CoCoA [6], we see that $I_A = \langle x_1 x_2 - x_3 x_4, x_5 x_6 - x_7 x_8, x_1^3 x_2^2 - x_5^2 x_6^2 \rangle$. The Betti $A$-degrees are $b_1 = (4, 0, 0, 0, 0), b_2 = (6, 0, 0, 0, 0)$ and $b_3 = (12, 0, 0, 0, 0)$. We note that $b_3 = 2b_2$, so $b_3$ is not a minimal binomial $A$-degree. The ideals $I_{A,b_1}$ and $I_{A,b_2}$ are zero, while $I_{A,b_3} = \langle x_1 x_2 - x_3 x_4, x_5 x_6 - x_7 x_8 \rangle$. The graphs $G(b)$ are connected for all $b \in NA$ except for the Betti $A$-degrees. In fact $G(b_1)$ and $G(b_2)$ consist of two connected components, $\{x_1 x_2\}$ and $\{x_3 x_4\}$ for $G(b_1)$, $\{x_3 x_6\}$ and $\{x_7 x_8\}$ for $G(b_2)$, while the connected components of $G(b_3)$ are $\{x_1^3 x_2^2, x_1^2 x_2^2 x_3 x_4, x_1 x_2 x_3^2 x_4^2, x_3^3 x_4^3\}$ and $\{x_5^2 x_6^2, x_5 x_6 x_7 x_8, x_7^2 x_8^2\}$.

Let $n_b$ denote the number of connected components of $G(b)$, this means that

$$G(b) = \bigcup_{i=1}^{n_b} G(b)_i,$$

and let $t_i(b)$ be the number of vertices of the $i$-component. The next proposition will be helpful in the sequel.

**Proposition 2.4.** An $A$-degree $b$ is a minimal binomial $A$-degree if and only if every connected component of $G(b)$ is a singleton.

**Proof.** If $b$ is a minimal binomial $A$-degree, then $I_{A,b} = \{0\}$ and every connected component of $G(b)$ is a singleton. Suppose now that $b$ is not minimal, i.e. $c \not\leq b$ for some minimal binomial $A$-degree $c$. Thus there is a binomial $B = x^u - x^v \in I_A$, with $\deg_A(B) = c$, and a monomial $x^a \not\equiv 1$ such that $b = c + \deg_A(x^a)$. Therefore $x^{a+u}, x^{a+v}$ are vertices of $G(b)$ and belong to the same component of $G(b)$ since $x^{a+u} - x^{a+v} = x^a B \in I_{A,b}$.

Let $G \subset I_A$ be a set of binomials. We recall the definition of the graph $\Gamma(b)_G$ (7) and a criterion for $G$ to be a generating set of $I_A$ (Theorem 2.5). Let $\Gamma(b)_G$ be the graph with vertices the elements of $\deg_A^{-1}(b)$ and edges the sets $\{x^u, x^v\}$ whenever the binomial

$$\frac{(x^u - x^v)}{\gcd(x^u, x^v)} \text{ or } \frac{(x^v - x^u)}{\gcd(x^u, x^v)}$$

belongs to $G$. In [7] the following theorem was proved.

**Theorem 2.5 (7).** $G$ is a generating set for $I_A$ if and only if $\Gamma(b)_G$ is connected for all $b \in NA$.

We consider the complete graph $S_b$ with vertices the connected components $G(b)_i$ of $G(b)$, and we let $T_b$ be a spanning tree of $S_b$; for every edge of $T_b$ joining the components $G(b)_i$ and $G(b)_j$, we choose a binomial $x^u - x^v$ with $x^u \in G(b)_i$ and $x^v \in G(b)_j$. We call $F_{T_b}$ the collection of these binomials. Note that if $b$ is not a Betti $A$-degree, then $F_{T_b} = \emptyset$.

**Theorem 2.6.** The set $F = \bigcup_{b \in NA} F_{T_b}$ is a minimal generating set of $I_A$.

**Proof.** First we will prove that $F$ is a generating set of $I_A$. From Theorem 2.5 it is enough to prove that $\Gamma(b)_F$ is connected for every $b$. We will prove the theorem by induction on $b$. If $b$ is a minimal binomial $A$-degree, the vertices of $\Gamma(b)_F$, which are also the vertices and the connected components of $G(b)$, and the tree $T_b$ give a path between any two vertices of $G(b)$. Next, let $b$ be nonminimal binomial $A$-degree. Suppose that $\Gamma(b)_F$ is connected for all $c \not\leq b$ and let $x^u, x^v$ be two vertices of $\Gamma(b)_F$. We will show that there is a path between these two vertices.
Corollary 2.8. The monomial $x^u$ is an indispensable monomial of $A$-degree $b$ if and only if $\{x^u\}$ is a component of $G(b)$.

We will consider two cases, depending on whether the vertices are in the same connected component of $G(b)$ or not.

1. If $x^u$, $x^v$ are in the same connected component $G(b)_i$ of $G(b)$, then $x^u - x^v = \sum_i x^d(u^i - v^i)$ where $u^i$, $v^i$ have $A$-degree $b_i \leq b$. From the inductive hypothesis $\Gamma(b)_i$ is connected and there is a path from $x^{b_i}$ to $x^{v_i}$. This gives a path from $x^d(u^i)$ to $x^d(v^i)$ and joining these paths we find a path from $x^u$ to $x^v$ in $\Gamma(b)_i$.

2. If $x^u$, $x^v$ belong to different components of $G(b)$, we use the tree $T_b$ to find a path between the two components. In each component we use the previous case and/or the induction hypothesis to move between vertices if needed. The join of these paths provides a path from $x^u$ to $x^v$ in $\Gamma(b)_i$.

Next, we will show that no proper subset $\mathcal{F}'$ of $\mathcal{F}$ generates $I_A$. Let $B = x^u - x^v \in \mathcal{F} \setminus \mathcal{F}'$, and let $\deg_A(B) = b$. Since $B$ is an element of $\mathcal{F}_b$, it corresponds to an edge $\{G(b)_i, G(b)_j\}$ of $T_b$, and the monomials $x^u$, $x^v$ belong to different components of $G(b)$. Suppose that there was a path $\{x^{u_1} = x^u, x^{u_2}, \ldots, x^{u_i} = x^v\}$ in $\Gamma(b)_j$ joining the vertices $x^u$ and $x^v$. Certainly there are monomials $x^{u_i}, x^{u_{i+1}}$ that are in different connected components of $G(b)$. Since $\gcd(x^{u_i}, x^{u_{i+1}}) \neq 1$ implies that the monomials $x^{u_i}$, $x^{u_{i+1}}$ are in the same connected component of $G(b)$, we conclude that $\gcd(x^{u_i}, x^{u_{i+1}}) = 1$ for some $i$. In this case the binomial $x^{u_i} - x^{u_{i+1}}$ is in $\mathcal{F}'$, it has $A$-degree $b$, and it corresponds to an edge of $T_b$. By considering these binomials and corresponding edges, we obtain a path in $T_b$ joining the components $G(b)_i, G(b)_j$ and not containing $\{G(b)_i, G(b)_j\}$ of $T_b$. This is a contradiction since $T_b$ is a tree.

The converse is also true: let $G = \bigcup_{b \in NA} G_b$ be a minimal generating set for $I_A$ where $G_b$ consists of the binomials in $G$ of $A$-degree $b$. We will show that $G_b$ determines a spanning tree of $S_b$.

Theorem 2.7. Let $G = \bigcup_{b \in NA} G_b$ be a minimal generating set for $I_A$. The binomials of $G_b$ determine a spanning tree $T_b$ of $S_b$.

Proof. Let $B = x^u - x^v \in G_b$. The monomials $x^u$, $x^v$ are in different connected components of $G(b)$; otherwise $B$ is not a part of a minimal generating set of $I_A$. Therefore $B$ indicates an edge in $S_b$. Let $T_b$ be the union over $B \in G_b$ of these edges. Here $T_b$ is tree of $S_b$, since if $T_b$ contains a cycle, we can delete a binomial from $G$ and still generate the ideal $I_A$, contradicting the minimality of $G$. Theorem 2.3 guarantees that the tree $T_b$ is spanning.

An immediate corollary of Theorems 2.6 and 2.7 concerns the indispensable monomials.

Corollary 2.8. The monomial $x^u$ is an indispensable monomial of $A$-degree $b$ if and only if $\{x^u\}$ is a component of $G(b)$.
Theorem 2.9. For a toric ideal \( I_A \) we have that
\[
\nu(I_A) = \prod_{b \in \mathbb{N}^A} t_1(b) \cdots t_{n_b}(b)(t_1(b) + \cdots + t_{n_b}(b))^{n_b-2}
\]
where \( n_b \) is the number of connected components of \( G(b) \) and \( t_i(b) \) is the number of vertices of the connected component \( G(b)_i \) of the graph \( G(b) \).

Proof. Let \( d_i \) be the degree of \( G(b)_i \) in a spanning tree \( T_b \), i.e. the number of edges of \( T_b \) incident with \( G(b)_i \). We have that \( \sum_{i=1}^{n_b} d_i = 2n_b - 2 \). There are
\[
\frac{(n_b - 2)!}{(d_1 - 1)!(d_2 - 1)! \cdots (d_{n_b} - 1)!}
\]
such spanning trees; see for example the proof of Cayley’s formula in [8]. For fixed \( T_b \) with degrees \( d_i \), there are \( (t_i(b))^{d_i} \) choices for the monomials for the edges involving the vertex \( G(b)_i \). This implies that the number of possible binomial sets \( \mathcal{F}_{T_b} \) is \( (t_1(b))^{d_1} \cdots (t_{n_b}(b))^{d_{n_b}} \). Therefore the total number of all possible \( \mathcal{F}_{T_b} \) is
\[
\sum_{d_1 + \cdots + d_{n_b} = 2n_b - 2} \frac{(n_b - 2)!}{(d_1 - 1)!(d_2 - 1)! \cdots (d_{n_b} - 1)!}(t_1(b))^{d_1} \cdots (t_{n_b}(b))^{d_{n_b}}
= t_1(b) \cdots t_{n_b}(b)(t_1(b) + \cdots + t_{n_b}(b))^{n_b-2}. \]

We point out that if \( t_i(b) = 1 \) for all \( i \), then the number of possible spanning trees of \( G(b) \) is \( n_b^{n_b-2} \) (Cayley’s formula; see [5]). We also note that if \( n_b = 1 \), for some \( b \in \mathbb{N}^A \), then the factor \( t_1(b)(t_1(b))^{-1} \) in the above product has value 1. Thus the contributions to \( \nu(I_A) \) come only from Betti \( A \)-degrees \( b \in \mathbb{N}A \). On the other hand we have a unique choice for a generator of degree \( b \) when \( n_b = 2 \) and \( t_1(b) = t_2(b) = 1 \). Thus in these cases \( G(b) \) consists of two isolated vertices and by Proposition 2.4 \( b \) is minimal. These remarks prove the following:

Corollary 2.10. Let \( B = x^u - x^v \in I_A \) with \( A \)-degree \( b \). Here \( B \) is indispensable if and only if the graph \( G(b) \) consists of two connected components, \( \{x^u\} \) and \( \{x^v\} \). Moreover \( b \) is a minimal binomial \( A \)-degree.

Corollary 2.11. Suppose that the Betti \( A \)-degrees \( b_1, \ldots, b_q \) of \( I_A \) are minimal binomial \( A \)-degrees. Then
\[
\nu(I_A) = (\beta_{0,b_1} + 1)\beta_{0,b_1}^{-1} \cdots (\beta_{0,b_q} + 1)\beta_{0,b_q}^{-1}.
\]

Proof. By Proposition 2.4 the connected components of \( G(b_i) \) are singletons. It follows that \( t_j(b_i) = 1 \) and that \( n_{b_i} = \sum t_j(b_i) \). Moreover \( \beta_{0,b_i} = |\mathcal{F}_{b_i}| = n_{b_i} - 1 \).

The next theorem provides a necessary and sufficient condition for a toric ideal to be generated by its indispensable binomials. It is a generalization of Corollary 2.1 in [14].

Theorem 2.12. The ideal \( I_A \) is generated by its indispensable binomials if and only if the Betti \( A \)-degrees \( b_1, \ldots, b_q \) of \( I_A \) are minimal binomial \( A \)-degrees and \( \beta_{0,b_i} = 1 \).

Proof. Suppose that \( I_A \) is generated by indispensable binomials. Then \( \nu(I_A) = 1 \) and therefore, from Theorem 2.9 \( t_j(b_i) = 1 \) and \( n_{b_i} = 2 \), for all \( j, i \). Thus \( \beta_{0,b_i} = 1 \). Now Proposition 2.4 together with the fact that \( t_j(b_i) = 1 \) implies that all \( b_i \) are minimal binomial \( A \)-degrees.
We point out that the above theorem implies that in the case that a toric ideal $I_A$ is generated by indispensable binomials, no two minimal generators can have the same $A$-degree. We compute $\nu(I_A)$ in the following example.

**Example 2.13.** Let $A = \{a_0 = k, a_1 = 1, \ldots, a_n = 1\} \subset \mathbb{N}$ be a set of $n+1$ natural numbers with $k > 1$ and $I_A \subset K[x_0, x_1, \ldots, x_n]$, the corresponding toric ideal. The ideal $I_A$ is minimally generated by the binomials $x_0 - x_1^k, x_1 - x_2, \ldots, x_{n-1} - x_n$. The Betti $A$-degrees are $b_1 = 1$ and $b_2 = k$, while the $A$-graded Betti numbers are $\beta_{0,1} = n - 1$ and $\beta_{0,k} = 1$. Also $G(1)$ consists of $n$ vertices, each one being a connected component, and $G(k)$ has two connected components, the singleton $\{x_0\}$ and the complete graph on the $\binom{k+n-1}{n-1}$ vertices $x_1^k, x_2^k, \ldots, x_n^k$. Thus

$$\nu(I_A) = n^{n-2}\binom{k+n-1}{n-1}.$$

**Example 2.14.** Generic toric ideals were introduced in [12] by Peeva and Sturmfels. The term generic is justified due to a result of Barany and Scarf (see [2]) in integer programming theory which shows that, in a well defined sense, almost all fels. The term generic is justified due to a result of Barany and Scarf (see [2]) in integer programming theory which shows that, in a well defined sense, almost all toric ideals are generic. Given a vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the support of $\alpha$, denoted by $\text{supp}(\alpha)$, is the set $\{i \in \{1, \ldots, m\} \mid \alpha_i \neq 0\}$. For a monomial $x^u$ we define $\text{supp}(x^u) := \text{supp}(u)$. A toric ideal $I_A \subset K[x_1, \ldots, x_m]$ is called generic if it is minimally generated by binomials with full support, i.e.

$$I_A = (x^{u_1} - x^{v_1}, \ldots, x^{u_r} - x^{v_r})$$

where $\text{supp}(u_i) \cup \text{supp}(v_i) = \{1, \ldots, m\}$ for every $i \in \{1, \ldots, r\}$. In [12] it was shown that the minimal binomial generating set of $I_A$ is unique and thus $I_A$ is generated by its indispensable binomials. This also follows as an easy application of Theorem 2.12.

### 3. The indispensable complex of a vector configuration

Consider a vector configuration $A = \{a_1, \ldots, a_m\}$ in $\mathbb{Z}^n$ with $A$A pointed and the toric ideal $I_A \subset K[x_1, \ldots, x_m]$. In [11] it is proved that a binomial $B$ is indispensable if and only if either $B$ or $-B$ belongs to the reduced Gröbner base of $I_A$ for any lexicographic term order on $K[x_1, \ldots, x_m]$. In [11] it is shown that a monomial $M$ is indispensable if the reduced Gröbner base of $I_A$, with respect to any lexicographic term order on $K[x_1, \ldots, x_m]$, contains a binomial $B$ such that $M$ is a monomial of $B$. We are going to provide a more efficient way to check if a binomial is indispensable and, respectively, for a monomial. Namely we will give a criterion that provides the indispensable binomials and monomials with only the information from one specific generating set of $I_A$.

We let $M_A$ be the monomial ideal generated by all $x^u$ for which there exists a nonzero $x^u - x^v \in I_A$; in other words given a vector $u = (u_1, \ldots, u_m) \in \mathbb{N}^m$, the monomial $x^u$ belongs to $M_A$ if and only if there exists $v = (v_1, \ldots, v_m) \in \mathbb{N}^m$ such that $v \neq u$, i.e. $v_i \neq u_i$ for some $i$, and $\deg_A(x^u) = \deg_A(x^v)$. We note that if $\{B_1 = x^{u_1} - x^{v_1}, \ldots, B_s = x^{u_s} - x^{v_s}\}$ is a generating set of $I_A$, then $M_A = (x^{u_1}, x^{u_2}, x^{u_3}, \ldots, x^{u_s})$. Let $T_A := \{M_1, \ldots, M_k\}$ be the unique minimal monomial generating set of $M_A$.

**Proposition 3.1.** The indispensable monomials of $I_A$ are precisely the elements of $T_A$. 

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Proof. First we will prove that the elements of $T_A$ are indispensable monomials. Let \( \{B_1, \ldots, B_s\} \) be a minimal generating set of \( I_A \). Set \( M_j := x^u \) for \( j \in \{1, \ldots, k\} \). Since \( x^u - x^v \) is in \( I_A \) for some \( v \), it follows that there is an \( i \in \{1, \ldots, s\} \) and a monomial \( N \) of \( B_i \) such that \( N \) divides \( x^u \) and thus \( x^u = N \).

Conversely consider an indispensable monomial \( x^u \) of \( I_A \) and assume that it is not an element of \( T_A \). Then \( x^u = M_j x^c \) for some \( j \in \{1, \ldots, k\} \) and \( c \neq 0 \). By our previous argument \( M_j \) is indispensable. Without loss of generality we may assume that \( B_1 = M_j - x^u \). If \( B_j = x^u - x^v \), then \[ B'_j := x^e x^u - x^v = B_j - x^e B_1 \in I_A \]
and therefore \( I_A = (B_1, \ldots, B_{j-1}, B'_j, B_{j+1}, \ldots, B_s) \). This way we can eliminate \( x^u \) from all the elements of the generating set of \( I_A \), a contradiction to the fact that \( x^u \) is indispensable. \( \square \)

Definition 3.2. We define the indispensable complex \( \Delta_{\text{ind}(A)} \) to be the simplicial complex with vertices the elements of \( T_A \) and faces all subsets of \( T_A \) consisting of monomials with the same \( A \)-degree.

By Proposition 3.1 the indispensable monomials are the vertices of \( \Delta_{\text{ind}(A)} \). The connected components consist of the vertices of the same \( A \)-degree and are simplices of \( \Delta_{\text{ind}(A)} \); actually they are facets. Different connected components have different \( A \)-degrees. We compute \( \Delta_{\text{ind}(A)} \) in the following example.

Example 3.3. In Example 2.13 we have that \( M_A = (x_0, x_1, \ldots, x_n) \) and also the facets of \( \Delta_{\text{ind}(A)} \) are \( \{x_0\} \) and \( \{x_1, \ldots, x_n\} \).

It follows easily that whenever \( \deg_A(x^u) \) is a minimal binomial \( A \)-degree, then \( x^u \in T_A \). The converse is not true in general. Indeed in Example 2.13 \( x_0 \) belongs to \( T_A \) but \( \deg_A(x_0) \) is not minimal. Next we give a criterion that determines the indispensable binomials.

Theorem 3.4. A binomial \( B = x^u - x^v \in I_A \) is indispensable if and only if \( \{x^u, x^v\} \) is a 1-dimensional facet of \( \Delta_{\text{ind}(A)} \) and \( \deg_A(B) \) is a minimal binomial \( A \)-degree.

Proof. Let \( b = \deg_A(B) \). Suppose that \( \{x^u, x^v\} \) is a 1-dimensional facet of \( \Delta_{\text{ind}(A)} \) and \( b \) is a minimal binomial \( A \)-degree. By Proposition 2.4 minimality of \( b \) implies that the elements of \( \deg_A^{-1}(b) \) which are the vertices of \( G(b) \) are vertices of \( \Delta_{\text{ind}(A)} \) and the connected components of \( G(b) \) are singletons. Since \( \Delta_{\text{ind}(A)} \) contains only two vertices of \( A \)-degree \( b \), \( G(b) \) consists of two connected components, \( \{x^u\} \) and \( \{x^v\} \), and \( B \) is indispensable by Corollary 2.10. The other direction is done by reversing the implications. \( \square \)

Theorem 2.4 shows that the toric ideal \( I_A \) of Example 2.13 has no indispensable binomials for \( n > 2 \). Indeed in this case the indispensable complex of \( I_A \) contains no 1-simplices that are facets.

We remark that to check the minimality of the \( A \)-degree \( b \) of the binomial \( B \in I_A \), it is enough to compare \( b \) with the \( A \)-degrees of the vertices of \( \Delta_{\text{ind}(A)} \). Thus given any generating set of \( I_A \), one can compute \( T_A \) and construct the simplicial complex \( \Delta_{\text{ind}(A)} \). The elements of \( T_A \) are the indispensable monomials and the 1-dimensional facets of \( \Delta_{\text{ind}(A)} \) of minimal binomial \( A \)-degree are the indispensable binomials.
Example 3.5. Let

\[ A = \{(2, 1, 0), (1, 2, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2)\}. \]

Using CoCoA [8], we see that \( I_A \) is minimally generated by \( x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_2x_6, x_1x_2x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2 \). Moreover \( T_A = \{ M_1 = x_1x_6, M_2 = x_2x_4, M_3 = x_3x_5, M_4 = x_3x_6^2, M_5 = x_2^2x_5, M_6 = x_2x_6^2, M_7 = x_4x_5^2, M_8 = x_2^2x_6, M_9 = x_1x_4^2, M_{10} = x_2^2x_6, M_{11} = x_1x_5^2, M_{12} = x_1^2x_5, M_{13} = x_2^2x_3, M_{14} = x_1^2x_4, M_{15} = x_2x_3^2, M_{16} = x_2x_3x_6, M_{17} = x_1x_4x_5 \}. \) It follows that \( \Delta_{\text{ind}}(A) \) is a simplicial complex on 17 vertices and its connected components are the facets

\[
\begin{align*}
&M_1, M_2, M_3; M_4, M_5; M_6, M_7; M_8, M_9; \\
&M_{10}, M_{11}; M_{12}, M_{13}; M_{14}, M_{15}; M_{16}, M_{17}.
\end{align*}
\]

The \( A \)-degrees of the components are accordingly

\[(2, 2, 2), (2, 2, 5), (1, 4, 4), (4, 1, 4),\]
\[(2, 5, 2), (4, 4, 1), (5, 2, 2), (3, 3, 3).\]

All of them are minimal binomial \( A \)-degrees and thus \( I_A \) has seven indispensable binomials corresponding to the 1-dimensional facets. We see that all nonzero \( A \)-graded Betti numbers equal 1, except from \( \beta_{0,(2,2,2)} \), which equals 2. From Corollary 2.11 we take that \( \nu(I_A) = 3 \).

The next corollary gives a necessary condition for a toric ideal to be generated by the indispensable binomials.

Corollary 3.6. Let \( A = \{a_1, \ldots, a_m\} \) be a vector configuration in \( \mathbb{Z}^n \). If \( I_A \) is generated by the indispensable binomials, then every connected component of \( \Delta_{\text{ind}}(A) \) is 1-simplex.

Proof. Let \( \{B_1, \ldots, B_s\} \) be a minimal generating set of \( I_A \) consisting of indispensable binomials \( B_i = x^{u_i} - x^{v_i} \). We note that the monomials of the \( B_i \) are all indispensable and form \( T_A \). Thus if a face of \( \Delta_{\text{ind}}(A) \) contains \( x^{u_i} \), it also contains \( x^{v_i} \). By Theorem 3.4 \( \{x^{u_i}, x^{v_i}\} \) is a facet of \( \Delta_{\text{ind}}(A) \). \(\square\)

The next example shows that the converse of Corollary 3.6 does not hold.

Example 3.7. We return to Example 2.3. The simplicial complex \( \Delta_{\text{ind}}(A) \) consists of only two 1-simplices \( \{x_1x_2, x_2x_3\}, \{x_5x_6, x_7x_8\} \), the indispensable binomials are \( x_1x_2 - x_3x_4, x_5x_6 - x_7x_8 \) and \( I_A \neq (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8) \).

When \( \Delta_{\text{ind}}(A) \) is a 1-simplex, the next proposition shows that \( I_A \) is principal and therefore generated by an indispensable binomial.

Proposition 3.8. The simplicial complex \( \Delta_{\text{ind}}(A) \) is a 1-simplex if and only if \( I_A \) is a principal ideal.

Proof. One direction of this proposition is trivial. For the converse assume that \( \Delta_{\text{ind}}(A) = \{x^{u_i}, x^{v_i}\} \) and let \( B_1 := x^{u_1} - x^{v_1} \). We will show that \( I_A = (B_1) \). Let \( B = x^{u} - x^{v} \) be the binomial of minimal binomial \( A \)-degree such that \( B \in I_A \setminus (B_1) \).

Since \( x^e = x^{d}x^{v_1} \) and \( x^v = x^{d}x^{u_1} \), where \( x^e \neq x^d \), and none of them equals 1, we have that

\[ x^{v_1}(x^e - x^d) = x^eB_1 - B. \]

Therefore \( 0 \neq x^e - x^d \in I_A \), while \( \deg_A(x^e) \leq \deg_A(x^u) \), a contradiction. \(\square\)
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