Abstract. We describe how closed geodesics lying in a prescribed homology class on a negatively curved manifold split when lifted to a finite cover. This generalizes a result of Zelditch in the case of compact hyperbolic surfaces.

0. Introduction

Given a compact manifold of negative curvature, there are geometric analogues of the Chebotarev Theorem in algebraic number theory due to Sunada [13] (cf. also Parry and Pollicott [8] for the generalization to Axiom A flows). More precisely, given a finite Galois cover of the manifold, these theorems describe the proportion of closed geodesics which lift in a prescribed way to the cover. In this geometric setting, it is also natural to consider infinite covers, and, in particular, the number of closed geodesics lying in a prescribed homology class has been studied by Katsuda and Sunada [4], Phillips and Sarnak [9], Katsuda [3], Lalley [7] and Pollicott [10] (with generalizations to Anosov flows by Katsuda and Sunada [5] and Sharp [12]). In this note we shall combine these points of view, generalizing a result of Zelditch for hyperbolic Riemann surfaces [14].

Let $M$ be a compact smooth Riemannian manifold with negative curvature. Let $\tilde{M}$ be a finite Galois covering of $M$ with covering group $G$. For a closed geodesic $\gamma$ on $M$, let $l(\gamma)$ denote its length, $\langle \gamma \rangle$ its Frobenius class in $G$ and $[\gamma]$ its homology class in $H = H_1(M, \mathbb{Z})$.

We shall examine how the closed geodesics lying in a fixed homology class $\alpha \in H$, split when lifted to $\tilde{M}$. More precisely, for a conjugacy class $C$ in $G$, we study the asymptotics of

$$\pi(T, \alpha, C) = \text{Card}\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, \langle \gamma \rangle = C\}.$$

The problem is complicated by the fact that, in general, $[\gamma]$ and $\langle \gamma \rangle$ are not independent quantities. This occurs if the abelian quotient group $G/[G, G]$ is non-trivial, since this group is also a quotient of $H$, the maximal abelian covering group of $M$. Let $\pi_G : G \to G/[G, G]$ and $\pi_H : H \to G/[G, G]$ be the natural projections. In particular, the image $\pi_G(C)$ of a conjugacy class $C \subset G$ is a single element in $G/[G, G]$ and if $\pi_G(C) \neq \pi_H(\alpha)$, then $\pi(T, \alpha, C) = 0$, for all values of $T$.

On the other hand, we have the following result, which extends work of Zelditch for Riemann surfaces [14].
There is a discrete group of isometries $\Gamma$ where $\lambda M$ has negative curvature and let $X$ be the unique unit-speed geodesic with $x > 0$. Then the fibres above each point in $M$ are compactified Fuchsian groups. Define a homomorphism $\phi : \Gamma \to G$ by setting $\phi(a_1) = i$, $\phi(a_2) = j$ and $\phi(b_1) = \phi(b_2) = 1$ and extending this to $\Gamma$. We can then define a normal subgroup by $\Gamma_0 = \ker(\phi)$. If we set $M = \mathbb{H}^2/\Gamma_0$ and $\tilde{M} = \mathbb{H}^2/\Gamma_0$, then $\tilde{M}$ is a finite cover of $M$ with covering group $G$.

Let us consider a closely related problem. Consider the frame flow $f : FM \to FM$ on the space of orthonormal frames above $M$. This is an $SO(n - 1)$-extension for the geodesic flow. Changing notation slightly, let $\gamma$ be a periodic orbit of the geodesic flow, to which we associate a holonomy $\Theta(\gamma) \in SO(n - 1)$ which comes from a reference frame being transported around $\gamma$. This is defined up to conjugacy. In [N] it was shown that the holonomies were equidistributed on $SO(n - 1)$. The following shows that the corresponding result holds for geodesics in a fixed homology class. (Recall that a class function is a function that is constant on conjugacy classes.)

Theorem 2. Let $F : SO(n - 1) \to \mathbb{R}$ be a class function. Then

$$\frac{1}{\pi(T, \alpha)} \sum_{l(\gamma) \leq T} F(\Theta(\gamma)) \to \int Fd\lambda, \quad as \ T \to +\infty,$$

where $\lambda$ denotes the Haar measure on $SO(n - 1)$.

1. Preliminaries

Let $M$ be a compact smooth manifold equipped with a Riemannian metric of negative curvature and let $X$ denote its universal cover. (In the special case where $M$ is a surface with constant curvature $-1$, $X$ is the hyperbolic plane $\mathbb{H}^2$.) Then there is a discrete group of isometries $\Gamma \cong \pi_1(M)$ of $X$ such that $M = X/\Gamma$. Now let $\Gamma_0$ be a normal subgroup of $\Gamma$ with finite index. Then $\tilde{M} = X/\Gamma_0$ is a finite (Galois) covering of $M$, with covering group $G = \Gamma/\Gamma_0$ (i.e., $G$ acts transitively on the fibres above each point in $M$).

There is a natural dynamical system, the geodesic flow, associated to $M$. Let $SM$ denote the unit-tangent bundle of $M$ and, for $(x, v) \in SM$, let $\gamma : \mathbb{R} \to M$ be the unique unit-speed geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then the geodesic flow $\phi : SM \to SM$ is defined by $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$, and we shall write $h$ for its topological entropy. There is a one-to-one correspondence between periodic $\phi$-orbits and directed closed geodesics on $M$. The fact that $M$ is negatively curved ensures that the geodesic flow is an Anosov flow and that $h > 0$. This will enable us to use results proved in the context of Anosov flows in this setting.

We shall make use of $L$-functions defined with respect to certain representations of $\Gamma = \pi_1(M)$. Let $\rho : \Gamma \to U(d)$ be a unitary representation of $\Gamma$. We define an
The geodesic flow on \( L \), and hence the analytic properties of \( L \), lift to \( \tilde{M} \). There are a countable infinity of closed geodesics on \( M \); we shall denote a typical one by \( \gamma \) and its length by \( l(\gamma) \). Each such \( \gamma \) has \( n = |G| \) lifts \( \gamma_1, \ldots, \gamma_n \) to \( M \). These lifts are not necessarily closed but, for each \( i = 1, \ldots, n \), there is a covering transformation \( g_i \in G \) relating the endpoints of \( \gamma_i \), and, for \( i, j = 1, \ldots, n \), \( g_i \) and \( g_j \) are conjugate. Hence we may associate to \( \gamma \) a well-defined conjugacy class \( \langle \gamma \rangle \subset G \), called the Frobenius class of \( \gamma \). These classes satisfy an analogue to Chebotarev’s Theorem in number theory: for a conjugacy class \( C \subset G \),

\[
\lim_{T \to +\infty} \frac{\#\{\gamma : l(\gamma) \leq T, \langle \gamma \rangle = C\}}{\#\{\gamma : l(\gamma) \leq T\}} = \frac{|C|}{|G|},
\]

The identity (1.2) is proved by considering \( L \)-functions

\[
L(s, R_\chi) = \prod_\gamma \det(I - R_\chi(\langle \gamma \rangle))e^{-sl(\gamma)}^{-1},
\]

where \( R_\chi \) is an irreducible representation of \( G \) with character \( \chi \). Since \( R_\chi \) lifts to a representation of \( \Gamma \), \( L(s, R_\chi) \) is a special case of the \( L \)-functions defined by (1.1). The geodesic flow on \( SM \) is also covered by the geodesic flow on \( S\tilde{M} \), with covering group \( G \), and hence the analytic properties of \( L(s, R_\chi) \) may be deduced from the results in [3].

**Lemma 1.1.**

(i) Let \( 1 \) denote the trivial one-dimensional representation of \( G \). Then \( L(s, 1) \) is analytic and nonzero on a neighbourhood of \( \{ s : \operatorname{Re}(s) \geq h \} \), apart from a simple pole at \( s = h \).

(ii) If \( R_\chi \neq 1 \) is an irreducible representation of \( G \), then \( L(s, 1) \) is analytic and nonzero on a neighbourhood of \( \{ s : \operatorname{Re}(s) \geq h \} \).

In this paper, we shall refine (1.2) by requiring that \( \gamma \) lies in a prescribed homology class in \( H = H_1(M, \mathbb{Z}) \). More precisely, for \( \alpha \in H \), we shall write \( \pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\} \) and \( \pi(T, \alpha, C) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, \langle \gamma \rangle = C\} \) and study the ratio

\[
\pi(T, \alpha, C) \quad \pi(T, \alpha)
\]

where \([\gamma] \in H_1(M, \mathbb{Z})\) denotes the homology class of \( \gamma \). Theorem 1 states that either \( \pi(T, \alpha, C) \) is identically zero or (1.3) has a limit as \( T \to +\infty \). We shall prove this in the next section; however, to do so, we need to first recall how \( \pi(T, \alpha) \) behaves as \( T \to +\infty \).

The asymptotics of \( \pi(T, \alpha) \) are also obtained by considering a family of \( L \)-functions, in this case indexed by the characters of \( H \). We suppose that \( H \) is
infinite and, for simplicity, we consider $H$ modulo torsion. Then these characters may be identified with the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$. For $\theta \in T^d$, we write

$$L(s, \theta) = \prod_{\gamma} \left(1 - e^{-s\langle \gamma \rangle + 2\pi i \theta \cdot [\gamma]}\right)^{-1}.$$ 

Since characters of $H$ lift to $\Gamma$, this is again an $L$-function of the form defined in (1.1).

We also write

$$\eta_\alpha(s) = \int_{T^d} e^{-2\pi i \theta \cdot \alpha} \frac{d^{\nu+1}}{ds^{\nu+1}} \left(\log L(s, \theta)\right) d\theta,$$

where $\nu = \lfloor d/2 \rfloor$. The following lemma is taken from [5] and [12].

**Lemma 1.2.** For each $\alpha \in H$, $\eta_\alpha(s)$ is analytic for $\text{Re}(s) > h$.

(i) If $d$ is even, then, for some constant $c_0 > 0$,

$$\lim_{\sigma \to h^-} \left(\eta_\alpha(\sigma + i \tau) - \frac{(-1)^{\nu+1}c_0}{\sigma + i \tau - h}\right)$$

exists for almost every $\tau \in \mathbb{R}$ and is locally integrable. Moreover, there exists a locally integrable function $f(\tau)$ such that, for $\sigma > h$,

$$\left|\eta_\alpha(\sigma + i \tau) - \frac{(-1)^{\nu+1}c_0}{\sigma + i \tau - h}\right| \leq f(\tau).$$

(ii) If $d$ is odd, then, for some constant $c_0 > 0$,

$$\lim_{\sigma \to h^-} \left(\eta_\alpha(\sigma + i \tau) - \frac{(-1)^{\nu+1}c_0\sqrt{\pi}}{\sqrt{\sigma + i \tau - h}}\right)$$

exists for almost every $\tau \in \mathbb{R}$ and is locally integrable with locally integrable first derivative. Moreover, there exists a locally integrable function $f(\tau)$ such that, for $\sigma > h$,

$$\left|\eta_\alpha(\sigma + i \tau) - \frac{(-1)^{\nu+1}c_0\sqrt{\pi}}{\sqrt{\sigma + i \tau - h}}\right| \leq f(\tau).$$

The constant $c_0$ in (i) and (ii) is independent of $\alpha$.

When combined with appropriate Tauberian theorems, this lemma is enough to ensure that, for some constant $c > 0$, independent of $\alpha \in H$,

$$\lim_{T \to +\infty} T^{1+d/2} e^{-hT} \pi(T, \alpha) = c.$$ (1.4)

(See [5] or [12] for more details.)

### 2. Proof of Theorem 1

It is clear that, in general, (1.3) will depend on the relationship between $G$ and $H$ and, particularly, $C$ and $\alpha$. Write $A = G/[G, G]$, the abelianization of $G$. Clearly, $A$ is a quotient of $\pi_1(M)$ and, since $H$ is the maximal abelian quotient of $\pi_1(M)$, $A$ is also a quotient of $H$. The extreme cases are:

(a) $G$ is abelian. Then $G = A$ and $G$ itself is a quotient of $H$.

(b) $G$ is perfect. Then $G = [G, G]$ and $A$ is trivial.
We shall write \( \pi_G : G \to A \) and \( \pi_H : H \to A \) to denote the respective projections. In particular, it is clear that if \( \pi_G(C) \neq \pi_H(\alpha) \), then \( \pi(T, \alpha, C) = 0 \) for all \( T > 0 \).

The proof of Theorem 1 depends on considering \( L \)-functions defined with respect to unitary representations of \( \Gamma \) of the form \( \theta \otimes R_\chi \), where \( \theta \in \mathbb{T}^d \) and \( R_\chi \) is an irreducible representation of \( G \) (or, more precisely, the lifts of these quantities to \( \Gamma \)). However, as we shall describe below, some of these \( \theta \otimes R_\chi \) are trivial. The corresponding \( L \)-functions take the form
\[
L(s, \theta \otimes R_\chi) = \prod_{\gamma} \det(I - R_\chi((\gamma)))e^{-snl(\gamma) + 2\pi in\theta[\gamma]},
\]
which converge to analytic functions for Re(\( s \)) > \( h \). Taking the logarithm and differentiating \( \nu + 1 \) times gives
\[
\left( \frac{d}{ds} \right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) = \sum_{\gamma} \sum_{n=1}^{\infty} n^{\nu-1} \chi(\langle \gamma \rangle) e^{-snl(\gamma) + 2\pi in\theta[\gamma]}.
\]
Applying the standard orthogonality relations for both for irreducible representations of \( G \) and for \( \mathbb{T}^d \) term-by-term in the above formula we obtain the relation
\[
\sum_{R_\chi_{\text{irred}}} \int_{\mathbb{T}^d} e^{-2\pi i \theta \cdot \alpha} \chi(C) \left( \frac{d}{ds} \right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) d\theta
\]
\[
= -\frac{|G|}{|C|} \sum_{\gamma} n^{\nu-1} \chi(\langle \gamma \rangle) e^{-snl(\gamma)},
\]
where the right-hand side vanishes if \( \pi_G(C) \neq \pi_H(\alpha) \).

The asymptotic behaviour of \( \pi(T, \alpha, C) \) may be deduced from properties of the right-hand side of (2.1): so, to prove Theorem 1, it is enough to study
\[
\sum_{R_\chi_{\text{irred}}} \int_{\mathbb{T}^d} e^{-2\pi i \theta \cdot \alpha} \chi(C) \left( \frac{d}{ds} \right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) d\theta
\]
and understand its meromorphic extension, via that of \( L(s, \theta \otimes R_\chi) \), and, in particular, the nature of the singularities on Re(\( s \)) = \( h \).

First we determine which \( \theta \otimes R_\chi \) are trivial. Let \( m = |G/[G,G]| \) be the cardinality of \( G/[G,G] \) and let \( 1 = \chi_0, \ldots, \chi_{m-1} \) be the characters of \( G/[G,G] \) (i.e., the 1-dimensional representations of \( G/[G,G] \)). These lift to \( G \) via \( \pi_G \) but 1-dimensional characters on \( G \) also descend to \( G/[G,G] \), since any such character annihilates commutators. Thus we may identify 1-dimensional representations of \( G \) with characters of \( G/[G,G] \). Each \( \chi_i \) also lifts to a character of \( H \), which we can denote by \( \theta_i \).

**Lemma 2.1.** The representation \( \theta \otimes R_\chi \) is trivial precisely when it is of the form \( \theta_i^{-1} \otimes R_\chi \), \( i = 0, \ldots, m-1 \).

**Proof.** It is clear from their construction that these representations are trivial. On the other hand, if \( \theta \otimes R_\chi \) is trivial, then \( R_\chi \) is one dimensional and hence it corresponds to one of the characters \( \chi_0, \ldots, \chi_{m-1} \) of \( G/[G,G] \). It is easy to see that, for \( i = 0, \ldots, m-1 \), \( \theta \otimes \chi_i \) is trivial only if \( \theta = \theta_i^{-1} \). \( \square \)
An immediate consequence is the following.

**Lemma 2.2.** For $i = 0, \ldots, m - 1$, the function $\zeta(s) := L(s, \theta^{-1} R_{\chi_i})$ has a simple pole at $s = h$ and no other poles on $\text{Re}(s) = h$.

Next we can consider $L(s, \theta \otimes R_{\chi})$ for $\theta \otimes R_{\chi}$ nontrivial.

**Lemma 2.3.** If $L(s, \theta \otimes R_{\chi})$ has a pole on $\text{Re}(s) = h$, then $R_{\chi}$ is 1-dimensional and $\chi$ is a character of $G/[G, G]$, namely, one of the $\chi_1, \ldots, \chi_{m-1}$.

**Proof.** This follows from the discussion on page 146 of [8]. \qed

If $R_{\chi}$ is 1-dimensional, then, as above, we can lift $\theta \otimes R_{\chi}$ to $\theta + \theta_i \in T^2g$ and rewrite the $L$-function as

$$L(s, \theta \otimes R_{\chi}) = L(s, \theta + \theta_i) = \prod_{\gamma}(1 - e^{-s(\gamma) + 2\pi i(\theta + \theta_i)[\gamma]})^{-1}.$$ 

However, this is again an $L$-function for homology.

**Lemma 2.4.** If $\theta \otimes R_{\chi}$ is nontrivial, then $L(s, \theta \otimes R_{\chi})$ is analytic on $\text{Re}(s) = h$.

**Proof.** By the above, we only need to consider the case when $L(s, \theta \otimes R_{\chi}) = L(s, \theta + \theta_i)$. However, if $\theta \otimes R_{\chi}$ is nontrivial, then $\theta \neq -\theta_i$, so the lemma follows from standard results in [3], [5], [10], [12]. \qed

To proceed, we return to the expression (2.1). We can rewrite this as

$$\frac{|C|}{|G|} \sum_{i=0}^{m-1} \chi_i(C) \int_{T^2g} e^{-2\pi i \theta - \alpha} \left( d \frac{d}{ds} \right)^{g+1} (\log L)(s, \theta \otimes R_{\chi}) d\theta + \phi(s),$$

where $\phi(s)$ is a function analytic in a neighbourhood of $\text{Re}(s) = h$ and, from (2.2), $L(s, \theta \otimes R_{\chi}) = L(s, \theta + \theta_i)$. We also note that, since we are assuming that $\pi_G(C) = \pi_H(\alpha)$, we have that

$$\chi_i(C)e^{-2\pi i \theta - \alpha} = 1, \text{ for } i = 0, \ldots, m - 1.$$ 

Hence one sees that the function in (2.1) satisfies an analogue of Lemma 1.2 in which that $c_0$ is replaced by $c_0|m|G|/|C|$.

From this one deduces, as in [5] or [12], that

$$\lim_{T \to +\infty} T^{1+\epsilon/2} e^{-\frac{1}{d} \pi(T, \alpha, C) = cm |C|/|G|},$$

with $c$ as in (1.4).

Finally, recalling that $m = |G/[G, G]|$, this is enough to prove Theorem 1.

**Remarks.** (i) If $M$ is either a surface or has curvature that is 1/4-pinched, then, working along the lines of [1], [6], [11], one can get an $O(T^{-1})$ error term, as Zelditch obtained for a hyperbolic surface. It is also possible to prove analogous results where a fixed homology class is replaced by one which changes linearly in $T$ (cf. [2], [7]).

(ii) There is a natural extension of Theorem 1 to Anosov flows which are homologically full in the sense of [12], i.e., ones for which every homology class is represented by a periodic orbit. \qed
3. Proof of Theorem 2

We can easily adapt the proof of Theorem 1 to prove Theorem 2. Since we are replacing a finite group $G$ by a compact group $SO(n-1)$ we need to consider a countable family of representations $R_{\chi}$, rather than a finite family. However, by approximation it suffices to consider each representation separately. As in the proof in the last section, one can consider representations $\theta \otimes R_{\chi}$. However, a significant advantage here is that the groups $H$ and $SO(n-1)$ can be treated independently.

In the case of the trivial representation, we have that $F = \chi = 1$ and we see that

$$\left( \frac{d}{ds} \right)^{\nu+1} (\log L)(s, \theta \otimes 1)$$

has a singularity of the form

$$\text{Const.} \times \frac{1}{(s - s(\theta))}.$$  

The analysis reduces to that in [5], [12], from which we get an asymptotic formula

$$\lim_{T \to +\infty} \frac{1}{\pi(T, \alpha)} \sum_{\substack{l(\gamma) \leq T \\
\gamma = \alpha}} F(\Theta(\gamma)) = 1.$$  

However, in the case of nontrivial representations we have that the $L$-function $L(s, \theta \otimes 1)$ is always analytic on $\Re(s) = h$, from which one sees that

$$\sum_{\substack{l(\gamma) \leq T \\
\gamma = \alpha}} F(\Theta(\gamma)) = o\left( \pi(T, \alpha) \right).$$

References

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