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THE 3-LOCAL *tmf*-HOMOLOGY OF $B\Sigma_3$

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Abstract. In this paper, we introduce a Hopf algebra, developed by the author and André Henriques, which is usable in the computation of the *tmf*-homology of a space. As an application, we compute the *tmf*-homology of $B\Sigma_3$ in a manner analogous to Mahowald and Milgram's computation of the *ko*-homology $\mathbb{R}P^\infty$.

1. Introduction

In this paper, we compute the 3-local *tmf*-homology and *tmf* Tate cohomology of the symmetric group Σ_3. This computation is motivated as follows. Mahowald and Milgram's computation of $ko_*(\mathbb{R}P^\infty)$ (see [7]) has proved useful in a variety of contexts. In particular, Mahowald used $ko_*(\mathbb{R}P^n)$ and $ko_*(\mathbb{R}P^\infty/\mathbb{R}P^k)$ to get information about v_1 metastable homotopy theory in the EHP sequence [9]. Mahowald has also used $ko_*(\mathbb{R}P^\infty)$ to detect elements in his η_j family [8]. At the prime 3, the role of the spectrum *ko* is most naturally played by the spectrum *tmf*. To generalize these results of Mahowald's, the initial piece of data needed is the *tmf*-homology of $B\Sigma_3$. Both of the aforementioned results should be generalizable starting from this point.

A theorem of Arone and Mahowald shows that v_n-periodic information is captured by the first p^n stages of the Goodwillie tower [1]. This recasts Mahowald's result from [9] into a more readily generalizable form. To get v_2-periodic information at the prime 3, the initial data needed comes in part from QS^0 and $Q(R_k^\infty)$, where R_k^∞ is a particular Thom spectrum of $B\Sigma_3$. Just as Mahowald uses knowledge of the *ko*-homology of stunted projective spaces to reduce the questions involved to ones of J-homology, we hope that a similar analysis, using Behrens' $Q(2)$ spectrum, will allow an analysis of the v_2-primary Goodwillie tower at 3 [4].

Minami shows that the odd primary η_j family will be detectable in the Hurewicz image of the *tmf*-homology of the n-skeleton of $B\Sigma_3$ for appropriate choices of n [10]. While determining the full Hurewicz image is a trickier task, understanding the groups and simple *tmf* operations on them could help determine if the conjectural η_j elements actually survive at the prime 3.

1.1. **Organization of the paper.** In §2, we introduce the main computational Hopf algebra $\mathcal{A}$, Ext over which is the Adams E_2 term for computing *tmf*-homology.

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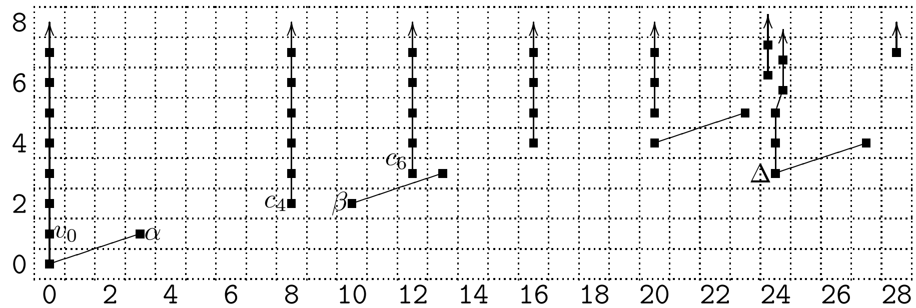

FIGURE 1. The Adams E_2 term for tmf_*

In §3, we carry out one of the steps analogous to Mahowald and Milgram's computation of $ko_*(\mathbb{R}P^\infty)$, computing the *tmf*-homology of the cofiber of the transfer map, and in §4, we complete the computation of $tmf_*(B\Sigma_3)$. Rounding out the computations, in §5, we compute the *tmf*-homology of the finite skeleta of the cofiber of the transfer. Finally, the homotopy of the Σ_3 Tate spectrum for *tmf* is presented in §6.

1.2. **Conventions and notation.** We restrict our attention to the prime 3 and assume that all spaces and spectra are 3-completed. For ease of readability, let H be $H\mathbb{Z}/3$, let P^∞ be $B\Sigma_3$, and let R be the cofiber of the transfer $B\Sigma_3 \to S^0$. If X is a space or spectrum, let $X^{[n]}$ denote its n-skeleton.

Finally, we need some *tmf* specific notation. To describe it, we begin with a picture of the Adams E_2 term which we will derive in §2 in which all of the elements in question will be labeled (Figure 1).

Let I denote the ideal of the Adams E_2 term for tmf_* generated by v_0, c_4 and c_6. Let $\bar{I}$ denote the ideal of tmf_* generated by 3, c_4, c_6, and their Δ and Δ^2 translates. I is the annihilator ideal of the elements α and β. For brevity, the reader is asked to always assume the relations $I\alpha = 0$ and $I\beta = 0$ in all Adams E_2 terms, unless explicitly stated otherwise. Moreover, the relation $c_4^3 - c_6^2 = 27\Delta$ always holds and will not be stated explicitly.

2. An Adams spectral sequence for *tmf*

We begin by quickly stating the variant of the Adams spectral sequence we will use. Full details of the construction and related issues have been worked out by Baker and Lazarev in [2].

Let R be an E_∞ ring spectrum, let E be an E_∞ R-algebra, and let $E_*^R M$ denote $\pi_*(E \wedge_R M)$.

Proposition 2.1 (Baker-Lazarev)**.** *If $E_*^R E$ is flat as an E_*-module, then the pair $(E_*, E_*^R E)$ is a Hopf algebroid, and there is an Adams spectral sequence with E_2 term*

$$\mathrm{Ext}_{(E_*, E_*^R E)}(E_*, E_*^R M)$$

converging to the homotopy of the E-nilpotent completion of M.

We apply this Baker-Lazarev machinery to the case $R = tmf$, $E = H$, and $M = tmf \wedge X$. The spectrum H is made into an E_∞ *tmf*-algebra by composing the zeroth Postnikov section of *tmf* with the reduction modulo 3. Since each of these is a map of E_∞ ring spectra, the composite is. Moreover, since every module is flat over H_*, we need only identify

$$\mathcal{A} := H_*^{tmf} H.$$

Theorem 2.2 (Henriques-Hill)**.** *As a Hopf algebra,*

$$\mathcal{A} = \mathcal{A}(1)_* \otimes E(a_2),$$

where $|a_2| = 9$*, and* $\mathcal{A}(1)_* = \mathbb{F}_3[\xi_1]/\xi_1^3 \otimes E(\tau_0, \tau_1)$ *is dual to the subalgebra of the Steenrod algebra generated by* β *and* $\mathcal{P}^1$*. The elements in* $\mathcal{A}(1)_*$ *have their usual coproducts, and*

$$\Delta(a_2) = 1 \otimes a_2 + \xi_1 \otimes \tau_1 - \xi_1^2 \otimes \tau_0 + a_2 \otimes 1.$$

We begin with a proposition which describes a spectrum which is to tmf at 3 what ku is to ko at 2. Let

$$C = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8.$$

Proposition 2.3 (Hopkins-Mahowald, Behrens [4])**.** *There is a splitting*

$$tmf_0(2) = tmf \wedge C = BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle.$$

This is a ring spectrum, and the generator, a_4*, of* π_8 *corresponding to the second* $BP\langle 2\rangle$ *factor acts as a square root of* v_2*, making*

$$\pi_*\big(tmf_0(2)\big) = \mathbb{Z}_3[v_1, a_4].$$

Proof of Theorem 2.2. We use the spectrum $tmf_0(2)$ as an intermediary. If we let $V(1)$ denote the Smith-Toda complex with which both 3 and v_1 are zero, then the above proposition shows that as a $\pi_*\big(tmf_0(2)\big)$-module,

$$\pi_*\big(tmf \wedge C \wedge V(1)\big) = \mathbb{F}_3[a_4].$$

This allows us to identify H as the cofiber of multiplication by a_4.

To finish the proof, we smash this cofiber sequence with H over tmf, giving the cofiber sequence

$$\Sigma^8 H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big) \xrightarrow{a_4} H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big) \to H \wedge_{tmf} H.$$

We begin by analyzing the homotopy of the first two tmf-modules in this resolution:

$$\pi_*\Big(H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big)\Big) = H_*\big(C \wedge V(1); \mathbb{Z}/3\big).$$

The structure of this as a graded vector space is that of $\mathcal{A}(1)_*$. Since $\mathcal{A}$ is a commutative Hopf algebra, the Borel classification of commutative Hopf algebras over a finite field ensures both that a_4 is zero in homology and that the structure of this as an algebra is as listed [5]. This follows from considering the degrees of the elements, since odd elements must be exterior classes and the element in degree 4 must be the generator of a truncated polynomial algebra.

Since the unit map $S^0 \to tmf$ is a 6-equivalence, the natural map

$$H \wedge_{S^0} H \to H \wedge_{tmf} H$$

is a 6-equivalence. This implies that the induced map in homotopy is a Hopf algebra isomorphism in the same range, and this gives the coproducts on the elements τ_0, τ_1 and ξ.

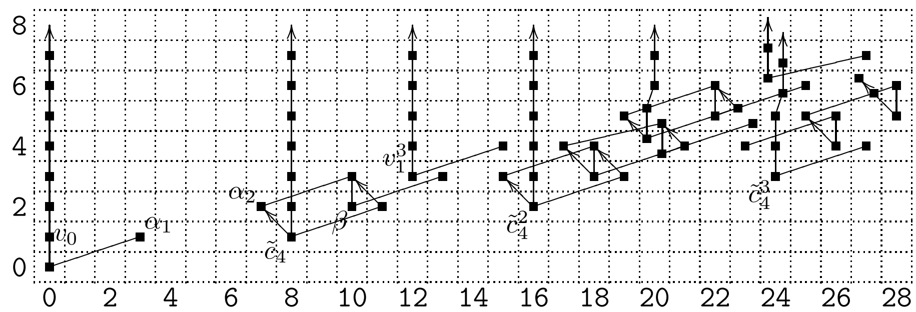

FIGURE 2. $\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3)$

To determine the coproduct on a_2, we will endow $\mathcal{A}$ with a filtration such that a_2 is primitive in the associated graded Hopf algebra. This filtration gives rise to a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$$

converging to the E_2 term of the Adams spectral sequence which computes $\pi_*(tmf)$. We shall use the known computation of $\pi_*(tmf)$ to deduce differentials in this algebraic spectral sequence, and this will determine the coproduct on a_2 [3].

We first filter $\mathcal{A}$ by letting $\mathcal{A}(1)_*$ have filtration 0 and letting a_2 have filtration 1. The initial piece of data needed is the cohomology of $\mathcal{A}(1)_*$. As an algebra

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3[v_0, v_1^3, \beta] \otimes E(\alpha_1, \alpha_2)/\big(v_0\alpha_1 = v_0\alpha_2 = 0,\, \alpha_1\alpha_2 = v_0\beta\big),$$

where the bidegrees, written as $(t-s, s)$, are $|v_0| = (0, 1)$, $|v_1^3| = (12, 3)$, $|\beta| = (10, 2)$, and $|\alpha_i| = (2i(p-1) - 1, 1)$.

Since a_2 is primitive in the associated graded Hopf algebra, we know that

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3) = \mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3)[\tilde{c}_4].$$

This Ext group is the E_1 page of a spectral sequence converging to the Adams E_2 term $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$. Since there is nothing in dimension 7 in tmf_*, we know that the element α_2 must be killed. The only possible way to achieve this is for $d_1(\tilde{c}_4) = \alpha_2$. This E_1 page is given together with this necessary d_1 differential in Figure 2.

At this point, we rename some of the remaining elements:

$$c_4 = v_0\tilde{c}_4, \quad c_6 = v_1^3, \quad \Delta = \tilde{c}_4^3.$$

Algebraic manipulation gives the d_2 differentials:

$$d_2([\alpha_2\tilde{c}_4^2]) = v_1^3\beta \text{ and } d_2([v_0\tilde{c}_4^2]) = v_1^3\alpha,$$

and for degree reasons, the spectral sequence collapses at this point.

For the d_1 to have the appropriate form, we must have

$$\psi(a_2) = 1 \otimes a_2 + a_2 \otimes 1 \pm (\xi_1 \otimes \tau_1 - \xi_1^2 \otimes \tau_0).$$

If the sign is negative, then we can simply replace a_2 by $-a_2$ to correct this. □

Corollary 1. *There is a spectral sequence with E_2 term*

$$\mathrm{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(X)\right)$$

converging to the 3*-completed* tmf*-homology of a space or spectrum X.*

This corollary complements the known results for primes different from 3.

Proposition 2.4 (Hopkins-Mahowald, Rezk [11]). *At $p = 2$, there is a spectral sequence with E_2 term*

$$\mathrm{Ext}_{\mathcal{A}(2)_*}\big(\mathbb{F}_2, H_*(X)\big)$$

converging to the 2-completed tmf-homology of a spectrum X.

For $p > 3$, tmf splits as a wedge of copies of $BP\langle 2\rangle$.

If $X = S^0$, then the spectral sequence has a simple form, depicted through dimension 28 as Figure 1.

Proposition 2.5. *As an algebra, the E_2 term for tmf_* is*

$$\mathbb{F}_3[v_0, \beta, c_4, c_6, \Delta] \otimes E(\alpha)/\big(I(\alpha, \beta), c_4^3 - c_6^2 = v_0^3\Delta\big).$$

There are two non-trivial differentials:

$$d_2(\Delta) = \alpha\beta^2 \quad \text{and} \quad d_3(\alpha\Delta^2) = \beta^5.$$

It is worth pointing out here the difference between the Adams spectral sequence herein derived and the Adams-Novikov spectral sequence worked out by Bauer [3]. The only difference is in the filtration of the elements. In this spectral sequence, v_0, c_4, c_6 and Δ all have positive filtration. These elements form the Adams-Novikov 0 line. This change is the only one, and recognizing this allows us to conclude that the differentials are, but for a change in index, those of the Adams-Novikov spectral sequence.

3. The *tmf*-homology of the cofiber of the transfer $P^\infty \to S^0$

Let R denote the cofiber of the transfer map $P^\infty \to S^0$. The homology of R sits as an extension of the homology of ΣP^∞ by the homology of S^0. Let e_i denote the generator of $H_i(R)$. The coaction of the dual Steenrod algebra on $H_*(R)$ is determined by the comodule structure on $H_*(\Sigma P^\infty)$ and the coaction formula

$$\psi(e_4) = -\xi_1 \otimes e_0 + 1 \otimes e_4.$$

Let M be the comodule $\mathcal{A}(1)_* \square_{\mathcal{A}(0)_*} \mathbb{F}_3$, where $\mathcal{A}(0)$ is the exterior algebra on the Bockstein.

Lemma 3.1. *$H_*(R)$ admits a filtration for which the associated graded module is*

$$Gr\big(H_*(R)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{12k} M.$$

Proof. The $-k^{\text{th}}$ stage of the filtration is given by taking the subcomodule generated by the classes in dimensions $12n + 1$ for all $n > k$. An elementary computation in the cohomology of the symmetric group shows that the associated graded module is exactly what is claimed. □

Lemma 3.2. *As an $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$-module,*

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, M) = \mathbb{F}_3\left[v_0, \tilde{c}_4\right].$$

Proof. First filter $\mathcal{A}$ as before by letting $\mathcal{A}(1)_*$ have filtration 0 and a_2 have filtration 1. This filtration extends to a filtration of M by letting M have filtration 0, and we have a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, M) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, M).$$

Since a_2 is primitive in $Gr(\mathcal{A})$, we have a Cartan-Eilenberg spectral sequence of the form

$$\operatorname{Ext}_{E(a_2)}\left(\mathbb{F}_3, \operatorname{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, M)\right) \Rightarrow \operatorname{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, M).$$

A change-of-rings argument shows that

$$\operatorname{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, M) = \operatorname{Ext}_{\mathcal{A}(0)_*}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3[v_0],$$

and this forces the result in question, since the target of any differential on the polynomial generator is zero for degree reasons. □

Since this algebra is concentrated in even degrees and since each of the graded pieces starts an even number of steps apart, the spectral sequence starting with Ext of the associated graded module for $H_*(R)$ collapses. There are non-trivial extensions.

Proposition 3.3. *As an* $\operatorname{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$*-module,*

$$\operatorname{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(R)\right) = \left(\bigoplus_{k=0}^{\infty} \mathbb{F}_3\left[v_0, \tilde{c}_4\right] e_{12k}\right) / c_6 e_{12k} = v_0^3 e_{12(k+1)}.$$

Proof. This is a routine computation in the bar complex. □

Theorem 3.4. *The Adams spectral sequence for the tmf-homology of R collapses, and as a tmf_*-module,*

$$tmf_*(R) = \mathbb{Z}_3\left[\frac{c_4}{3}, \frac{c_6}{27}\right].$$

Proof. The Adams E_2 term is concentrated in even topological degrees, and this implies the collapse of the Adams spectral sequence. The previous lemma solved the extension problem, and the proof of Theorem 2.2 shows that $3\tilde{c}_4$ is c_4. □

4. The *tmf*-homology of P^∞

With the most difficult of the computations now behind us, we can compute the *Stmf*-homology of P^∞ by considering the long exact sequence induced by applying tmf_* to the cofiber sequence

$$S^0 \to R \to \Sigma P^\infty.$$

The first map is the inclusion of the zero cell into R, and so this map in *tmf*-homology just takes 1 to 1. Since this is a map of tmf_*-modules, we see immediately that this map is injective on elements of Adams-Novikov filtration 0.

Additionally, since α and β act as zero on all of the classes in $tmf_*(R)$, the kernel of this first map is the submodule of tmf_* generated by α, β and their Δ translates. These together establish the following theorem about the *tmf* homology of ΣP^∞.

Theorem 4.1. *The tmf-homology of ΣP^∞ sits in a short exact sequence*

$$0 \to G_n \to tmf_n(\Sigma P^\infty) \to \widehat{tmf}_{n-1} \to 0,$$

where $\widehat{tmf}_{n-1}$ is the subgroup of tmf_{n-1} of Adams-Novikov filtration at least 1, and G_n, the cokernel of the map $tmf_n \to tmf_n(R)$, is given by

$$G_{24k+12j+8i} = \begin{cases} \mathbb{Z}/3 \oplus \bigoplus_{m=1}^{k} \mathbb{Z}/3^{6m}, & k \equiv 1,2 \bmod 3, i+j=0, \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i}, & k \equiv 0 \bmod 3, \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i}, & k \equiv 1,2 \bmod 3, i+j>0, \\ 0, & \textit{otherwise,} \end{cases}$$

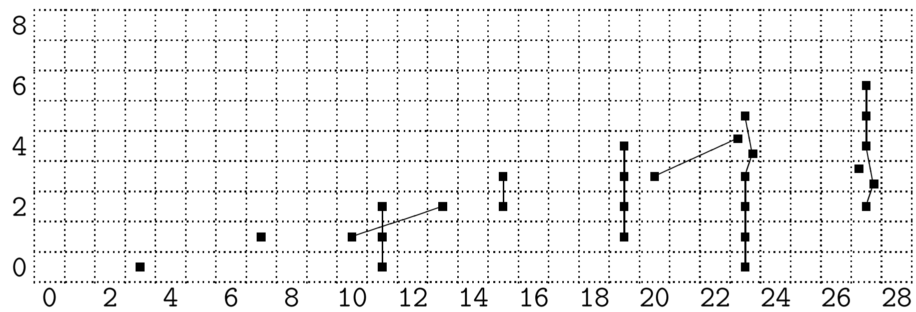

FIGURE 3. The *tmf* homology of $B\Sigma_3$

where $j < 2$ and $i < 3$. The sequence is split as a sequence of groups. There is a hidden α extension originating on the copy of β^2 in $\widehat{tmf}_{20}$ and hitting the $\mathbb{Z}/3$ summand of G_{24}.

To make it easier to understand the statement of the theorem, we include a picture of the homotopy through dimension 28 as Figure 3.

Proof. This short exact sequence is just a restatement of the earlier comments about the long exact sequence in *tmf*-homology. It is split because the elements coming from G_n have Adams-Novikov filtration 0, and the convergence of the Adams-Novikov spectral sequence ensures a map of groups from $tmf_*(\Sigma P^\infty)$ to G_n which is a left inverse to this inclusion.

The structure of the groups G_n is easy to show. A basis for $tmf_*(R)$ is given by the collection of monomials of the form $\Delta^k \tilde{c}_6^j \tilde{c}_4^i$, where $i < 3$, and $27\tilde{c}_6 = c_6$, $3\tilde{c}_4 = c_4$. This is simply because we can solve the relation on Δ in $tmf_*(R)$. A basis for the Adams-Novikov filtration 0 subring of tmf_* is given by the monomials

$$\Delta^k c_6^j c_4^i \text{ for } k \equiv 0 \mod 3 \text{ or } k \equiv 1, 2 \mod 3,\ i + j > 0, \quad 3\Delta^{k+1},\ \text{and } 3\Delta^{k+2}.$$

Recalling that

$$\Delta^k c_6^j c_4^i = 3^{3j+i} \Delta^k \tilde{c}_6^j \tilde{c}_4^i$$

and collecting all terms of the same degree yield G_n.

The hidden extension can most readily be seen by considering the long exact sequence in Ext induced by the cofiber sequence. In this situation, Δ from the ground sphere kills Δ in the Adams E_2 term for $tmf_*(R)$, and $\alpha\beta^2$ on the ground sphere survives. □

5. THE *tmf*-HOMOLOGY OF THE FINITE SKELETA OF R

For completeness, we include the *tmf*-homology of the finite skeleta of R. These computations serve as starting points for the program of Minami to detect the 3-primary η_j family [10].

5.1. The Skeleta of R.

Let $n = 12k + i$, for $0 < i \leq 12$.

Lemma 5.1. *There is a filtration of $H_*(R^{[12k+i]})$ such that the associated graded module is*

$$Gr\big(H_*(R^{[12k+i]})\big) = \left(\bigoplus_{n=0}^{k-1} \Sigma^{12n} M\right) \oplus \Sigma^{12k} M_i,$$

where M is $\mathcal{A}(1)_*\Box_{\mathcal{A}(0)_*}\mathbb{F}_3$, and where M_i is the subcomodule of M generated by all classes of degree at most i for $i < 12$, and M_{12} is M_9 plus a primitive class in dimension 12.

Proof. The required filtration is just the restriction of the filtration used in the proof of Lemma 3.1 to the subcomodule $H_*(R^{[12k+i]})$. □

The comodules M_i are the homology of $R^{[i]}$, and this splitting result and the following theorem demonstrate that knowing their *tmf*-homology gives that of all finite skeleta. The proof of Theorem 3.4 shows the following.

Theorem 5.2. *As a module over tmf_*,*

$$tmf_*(R^{[12k+i]}) = \left(\mathbb{Z}_3\left[\frac{c_4}{3}\right]\{e_0, e_{12}, \ldots, e_{12(k-1)}\} \oplus \widetilde{M}_i e_{12k}\right)/(c_6e_{12j} - 27e_{12(j+1)}),$$

where $\widetilde{M}_i$ is the tmf-homology of spectrum $R^{[i]}$.

The remainder of the section will be spent computing the modules $\widetilde{M}_i$. To save space, in what follows we use two indices: δ, which ranges from 0 to 2, and ϵ, which ranges from 0 to 1. When these appear, it means that all possible values of the index are actually present. Many of the elements are also named in reference to other elements that do not survive the Adams spectral sequence. These classes are often indecomposable, and we name the element by enclosing the original name in square brackets. Additionally, since the steps in the proofs are all nearly identical to the first two, we omit proofs beyond these.

Proposition 5.3. *Since $R^{[1]}$, $R^{[2]}$, and $R^{[3]}$ are S^0, $\widetilde{M}_i = tmf_*$ for $1 \le i \le 3$.*

Proposition 5.4. *The spectrum $R^{[4]}$ is the cofiber of α_1. The tmf-homology of this is the extension of the tmf_*-module generated by $[\Delta^\epsilon e_0]$ and $[\alpha e_4]$ and subject to the relations*

$$\alpha[\alpha e_4] = \beta e_0,\ \alpha[\Delta e_0] = \beta^2[\alpha e_4],\ \alpha e_0 = \beta^3[\Delta^\epsilon e_0] = I[\alpha e_4] = \beta^4[\alpha e_4]$$

by the module

$$\mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_4], [c_4e_4], [c_6e_4]\}.$$

The extension is determined by the two relations

$$c_4[3e_4] = 3[c_4e_4] \pm c_6e_0, \quad c_6[3e_4] = 3[c_6e_4] \pm c_4^2e_0.$$

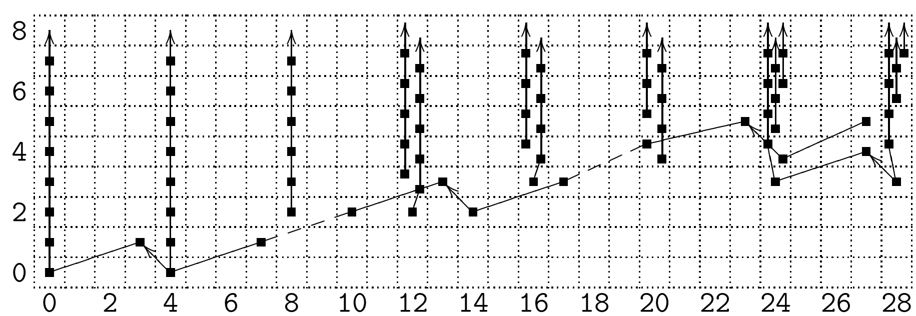

FIGURE 4. The long exact sequence for $\mathrm{Ext}(M_4)$

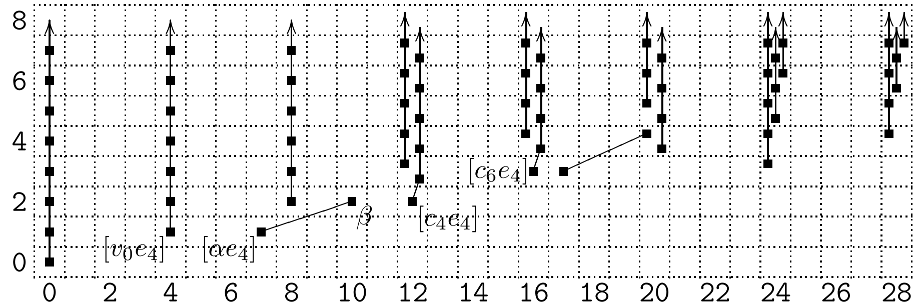

FIGURE 5. The Adams E_2 term for $tmf_*(R^{[4]})$

Proof. Since the spectrum M_4 is the cone on α_1, we can use the long exact sequence in Ext to compute the Adams E_2 term (Figure 4).

As a module over the Adams E_2 term for tmf_*, this E_2 term is the extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[v_0e_4], [c_4e_4], [c_6e_4]\} \oplus \mathbb{F}_3[\Delta, \beta]\{[\alpha e_4]\},$$

subject to the relations

$$c_4[v_0e_4] = v_0[c_4e_4] \pm c_6e_0, \quad c_6[v_0e_4] = v_0[c_6e_4] \pm c_4^2e_0, \quad \alpha[\alpha e_4] = \beta$$

and depicted in Figure 5.

The Adams differentials for the sphere imply that Δe_0 and $\Delta^2 e_0$ are d_2 cycles and that the following differentials hold:

$$d_2(\Delta[\alpha e_4]) = \beta^3 e_0, \quad d_3(\alpha\Delta^2[\alpha e_4]) = \beta^5[\alpha e_4].$$

This last d_3 implies also that

$$d_3(\Delta^2 e_0) = \beta^4[\alpha e_4],$$

using the relation involving α multiplication on $[\alpha e_4]$. □

Proposition 5.5. *The spectra $R^{[5]}$, $R^{[6]}$, and $R^{[7]}$ are the cofiber of the extension of α_1 over the mod 3 Moore spectrum. The tmf_*-module $\widetilde{M_i}$ is generated by*

$$[\tfrac{c_4}{3}\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0], [\Delta^\epsilon e_0], [\alpha e_4], [\beta e_5]$$

and is subject to the relations

$$\alpha[\beta e_5] = \beta[\tfrac{c_4}{3}e_0], \alpha[\alpha e_4] = \beta e_0, \alpha[\Delta e_0] = \beta^2[\alpha e_4],$$
$$(\alpha, \beta^3)e_0 = I([\alpha e_4], [\beta e_5]) = \beta^4[\alpha e_4] = 0.$$

Proof. In the long exact sequence in Ext induced by the inclusion of the 4-skeleton into $R^{[5]}$, the inclusion of the 5-cell kills the element $[v_0e_4]$ (Figure 6).

The elements $[c_4e_4]$ and $[c_6e_4]$ survive, and the relations in the Ext term for the 4-skeleton ensure that in the Adams E_2 term for $\widetilde{M_5}$,

$$v_0[c_4e_4] = c_6e_0, \quad v_0[c_6e_4] = c_4^2e_0.$$

Moreover, since α and β multiplications on the class $[v_0e_4]$ are trivial, the classes $[\alpha e_5]$ and $[\beta e_5]$ survive to the Adams E_2 page (Figure 7).

A computation in the bar complex establishes that $v_0[\alpha e_5] = c_4e_0$.

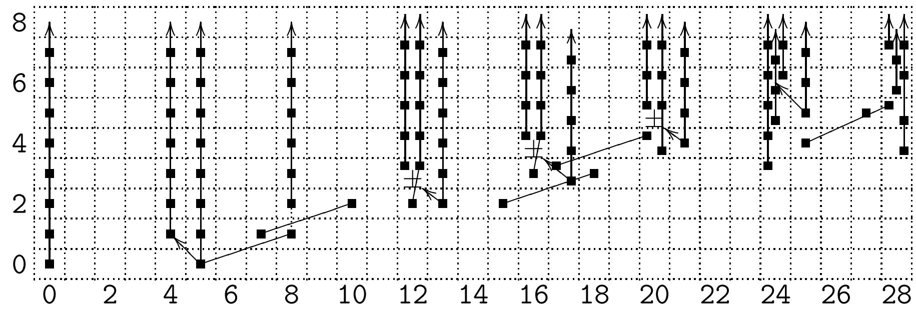

FIGURE 6. The long exact sequence for $\mathrm{Ext}(M_5)$

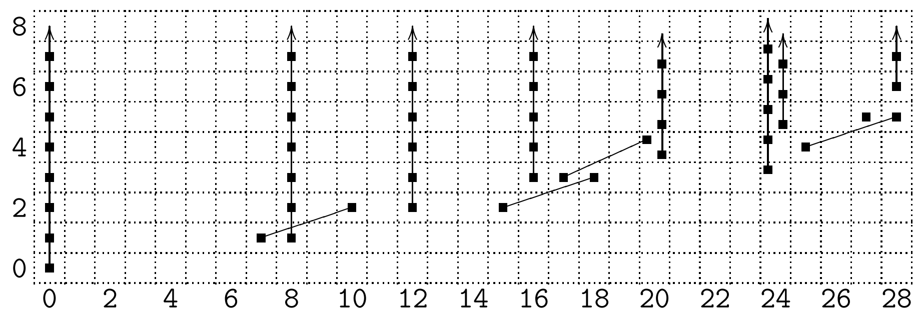

FIGURE 7. The Adams E_2 term for $tmf_*(R^{[5]})$

This shows that the Adams E_2 page, as a module over that for tmf_*, is

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0, [\tfrac{c_4}{v_0}e_0], [\tfrac{c_6}{v_0}e_0], [\alpha e_4], [\beta e_5]\}/ \big(\alpha[\alpha e_4] - \beta e_0, \beta[\tfrac{c_4}{v_0}e_0] - \alpha[\beta e_5], \alpha e_0, I([\beta e_5], [\alpha e_4])\big).$$

The differentials again follow from those in the Adams spectral sequence of tmf_*. □

Proposition 5.6. *The spectrum $R^{[8]}$ is the spectrum C from §2, where the middle cell is replaced by the mod 3 Moore spectrum. The module $\widetilde{M}_8$ sits in a short exact sequence of tmf_*-modules*

$$0 \to tmf_*\{[\tfrac{c_4}{3}\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0], [\Delta^\delta e_0], [\beta e_5]\}/\big((\alpha,\beta)\big([\tfrac{c_4}{3}{}^\epsilon\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0]\big), I[\beta e_5]\big) \to \widetilde{M}_8 \to \mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_8], [c_4e_8], [c_6e_8]\} \to 0,$$

where the extension is determined by the two relations

$$c_4[3e_8] = 3[c_4e_8] \pm c_4[\tfrac{c_4}{3}e_0], \quad c_6[3e_8] = 3[c_6e_4] \pm c_4[\tfrac{c_6}{3}e_0].$$

Proposition 5.7. *The spectra $R^{[9]}$, $R^{[10]}$, and $R^{[11]}$ are the cofiber of the map from $\Sigma^4C(\alpha_1)$ to C which is multiplication by 3 on the 4 and 8 cells. The module $\widetilde{M}_9$ can be expressed via the short exact sequence*

$$0 \to tmf_*\{[\alpha e_9]\} \to \widetilde{M}_9 \to \mathbb{Z}_3\left[\frac{c_4}{3}\right]e_0 \to 0,$$

where the only extension is given by

$$c_6 e_0 = 9[\alpha e_9].$$

Proposition 5.8. *As a* tmf_**-module,*

$$\widetilde{M}_{12} = \widetilde{M}_9 \oplus \Sigma^{12} tmf_*.$$

6. The Σ_3 Tate homology of *tmf*

The analysis used to compute the *tmf*-homology of R applies to compute the homotopy of the Σ_3 Tate spectrum of *tmf* [6],

$$tmf^{t\Sigma_3} = \Sigma\,(tmf \wedge P^\infty)_{-\infty} = \Sigma \varprojlim \left(tmf \wedge P^\infty_{-n}\right).$$

A mod 3 form of James periodicity shows that as $\mathcal{A}(1)_*$-comodules,

$$H_*\left(P^\infty_{-12k+3}\right) = \Sigma^{-12k} H_*\left(P^\infty\right).$$

The Adams spectral sequence argument in §4 shows that the map

$$\pi_*\left(tmf \wedge P^\infty_{-12(k+1)+3}\right) \to \pi_*\left(tmf \wedge P^\infty_{-12k+3}\right)$$

is surjective on the G_* summand and zero on the $\widehat{tmf}_*$ summand. This implies that there are no $\lim^1$ terms coming from the inverse system of homotopy groups. Moreover, this is a system of tmf_*-modules, and considering the action of c_4 and c_6 in each of the modules in the inverse system allows us to conclude with the following theorem.

Theorem 6.1. *The homotopy of the* Σ_3 *Tate spectrum of tmf is an indecomposable* tmf_* *module, and*

$$\pi_*(tmf^{t\Sigma_3}) = \mathbb{Z}_3\left[\frac{c_4}{3}, \left(\frac{c_6}{27}\right)^{\pm 1}\right]^\wedge_I,$$

where I is the ideal in $\pi_0(tmf^{t\Sigma_3})$ *generated by elements of positive Adams filtration.*

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