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THE KAC JORDAN SUPERALGEBRA: AUTOMORPHISMS AND MAXIMAL SUBALGEBRAS

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Abstract. In this note the group of automorphisms of the Kac Jordan superalgebra is described and used to classify the maximal subalgebras.

1. Introduction

Finite dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero were classified by V. Kac in 1977 [9] (see also Kantor [10], where a missing case is added). Among these superalgebras we find the ten dimensional Kac Jordan superalgebra, K_{10}, which is exceptional (see [12] and [17]) and plays a significant role (see [13]).

Here we are interested in describing the group of automorphisms of K_{10} and, as a consequence, in classifying the maximal subalgebras of the Kac superalgebra over an algebraically closed field of characteristic zero.

In two previous works (see [6], [7]), the authors have given a description of maximal subalgebras of finite dimensional central simple superalgebras which are either associative or associative with superinvolution and also a description of maximal subalgebras of finite dimensional special simple Jordan superalgebras over an algebraically closed field of characteristic zero. These papers are thus in the spirit of previous work by E. Dynkin ([3], [4]), M. Racine ([14], [15]) or A. Elduque ([5]) on maximal subalgebras of different classes of algebras.

Let F be a field of characteristic not two. This assumption will be kept throughout the paper. Recall that a superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ ($A_\alpha A_\beta \subseteq A_{\alpha+\beta}\ \forall \alpha, \beta \in \mathbb{Z}_2$). If $a \in A_\alpha$ we say that a is a *homogeneous element* and we use the notation $\bar{a} = \alpha$. Elements belonging to $A_{\bar{0}}$ are called *even* elements and the ones in $A_{\bar{1}}$ *odd* elements. A superalgebra A is said to be *nontrivial* if $A_{\bar{1}} \neq 0$, and it is called *simple* if it contains no proper nontrivial graded ideals and $A^2 \neq 0$.

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A Jordan algebra is an algebra satisfying the following identities:

$$xy = yx, \tag{1}$$

$$(x^2y)x = x^2(yx). \tag{2}$$

The last identity can be written as $(x^2, y, x) = 0$, where (x, y, z) denotes the associator $(xy)z - x(yz)$. Following the standard procedure, a superalgebra is a *Jordan superalgebra* if its Grassmann envelope is a Jordan algebra. In particular, a Jordan superalgebra satisfies the identities

$$xy = (-1)^{\bar{x}\bar{y}}yx, \tag{3}$$

$$\sum_{\substack{\text{cyclic}\\ x,y,t}} (-1)^{(\bar{x}+\bar{z})\bar{t}}(xy, z, t) = 0 \tag{4}$$

for homogeneous elements x, y, z, t. (If the characteristic is $\neq 2, 3$, these two identities characterize Jordan superalgebras.)

The even part of a Jordan superalgebra is a Jordan algebra, while the odd part is a Jordan bimodule for the even part.

Associative superalgebras are just $\mathbb{Z}_2$-graded associative algebras, but note that Jordan superalgebras are not $\mathbb{Z}_2$-graded Jordan algebras.

The following examples of Jordan algebras will be useful in the sequel:

i) Let A be an associative algebra over a ground field of characteristic $\neq 2$. The new operation $a \cdot b = \frac{1}{2}(ab + ba)$ defines a structure of a Jordan algebra on A. It is denoted by A^+.

ii) Let V be a vector space over F with a nondegenerate symmetric bilinear form $(\ ,\ ) \colon V \times V \to F$. The direct sum $F1 + V$ with the product $(\lambda 1 + v)(\mu 1 + w) = (\lambda\mu + (v, w))1 + (\lambda w + \mu v)$ is a Jordan algebra, and it is called the Jordan algebra of a nondegenerate bilinear form.

Simple finite dimensional Jordan algebras over an algebraically closed field were classified by P. Jordan, J. von Neumann, E. Wigner ([8]) and by A. Albert ([1]). The examples given above are two of the four types of algebras given in that classification.

Also the following examples of Jordan superalgebras will be needed later on:

1) $J = K_3$, the *Kaplansky superalgebra*:
$J_{\bar{0}} = Fe, \quad J_{\bar{1}} = Fx + Fy,$
$e^2 = e, \quad e \cdot x = \frac{1}{2}x, \quad e \cdot y = \frac{1}{2}y, \quad x \cdot y = e.$

2) The family of superalgebras $J = D_t$, with $t \in F \setminus \{0\}$:
$J_{\bar{0}} = Fe + Ff, \quad J_{\bar{1}} = Fu + Fv,$
$e^2 = e, \quad f^2 = f, \quad e \cdot f = 0, \quad e \cdot u = \frac{1}{2}u, \quad e \cdot v = \frac{1}{2}v, \quad f \cdot u = \frac{1}{2}u,$
$f \cdot v = \frac{1}{2}v, \quad u \cdot v = e + tf.$

3) $J = K_{10}$, the *Kac superalgebra*, which will be described in more detail in the following section.

4) The *superalgebra of a superform*: Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a graded vector space over F, and let $(\ ,\ )$ be a nondegenerate supersymmetric bilinear superform on V, that is, a nondegenerate bilinear map that is symmetric on $V_{\bar{0}}$, skewsymmetric on $V_{\bar{1}}$, and $V_{\bar{0}}, V_{\bar{1}}$ are orthogonal relative to $(\ ,\ )$. Now consider $J_{\bar{0}} = Fe + V_{\bar{0}}$, $J_{\bar{1}} = V_{\bar{1}}$ with $e \cdot x = x$, $v \cdot w = (v, w)e$ for every $x \in J$, $v, w \in V$. This superalgebra, J, is called the superalgebra of a superform.

5) A^+, with A an associative superalgebra over a field of characteristic not 2, where the product operation in A has been changed to $a_i \cdot b_j = \frac{1}{2}(a_i b_j + (-1)^{ij} b_j a_i)$.

As we have mentioned at the beginning of this Introduction, V. Kac and I. Kantor gave the classification of nontrivial simple Jordan superalgebras. The four examples given above are also examples of simple Jordan superalgebras, assuming in 3) that the characteristic is not 3, and in 4) that A is a simple associative superalgebra (see [18]).

In this paper, our purpose is to describe the group of automorphisms of K_{10} and also to classify its maximal subalgebras.

2. Kac superalgebra

The Kac superalgebra, $K_{10} = J_{\bar{0}} \oplus J_{\bar{1}}$, is a Jordan superalgebra with 6-dimensional even part and 4-dimensional odd part:

$$J_{\bar{0}} = (Fe + Fa + Fb + \sum_{i=1}^{2} Fc_i) \oplus Ff, \qquad J_{\bar{1}} = \sum_{i=1}^{2} (Fp_i + Fq_i)$$

and product given in Table 1.

Table 1. K_{10}

	e	a	b	c_1	c_2	f	p_1	p_2	q_1	q_2
e	e	a	b	c_1	c_2	0	$\frac{1}{2}p_1$	$\frac{1}{2}p_2$	$\frac{1}{2}q_1$	$\frac{1}{2}q_2$
a	a	$4e$	0	0	0	0	p_1	p_2	$-q_1$	$-q_2$
b	b	0	$-4e$	0	0	0	q_1	q_2	$-p_1$	$-p_2$
c_1	c_1	0	0	0	$2e$	0	0	q_1	0	p_1
c_2	c_2	0	0	$2e$	0	0	q_2	0	p_2	0
f	0	0	0	0	0	f	$\frac{1}{2}p_1$	$\frac{1}{2}p_2$	$\frac{1}{2}q_1$	$\frac{1}{2}q_2$
p_1	$\frac{1}{2}p_1$	p_1	q_1	0	q_2	$\frac{1}{2}p_1$	0	$a+2(e-3f)$	$2c_1$	b
p_2	$\frac{1}{2}p_2$	p_2	q_2	q_1	0	$\frac{1}{2}p_2$	$-a-2(e-3f)$	0	$-b$	$-2c_2$
q_1	$\frac{1}{2}q_1$	$-q_1$	$-p_1$	0	p_2	$\frac{1}{2}q_1$	$-2c_1$	b	0	$a-2(e-3f)$
q_2	$\frac{1}{2}q_2$	$-q_2$	$-p_2$	p_1	0	$\frac{1}{2}q_2$	$-b$	$2c_2$	$-a+2(e-3f)$	0

This basis of K_{10} is obtained by using the one in [16, p. 378] with $a = v_1 + v_2$, $b = v_1 - v_2$, $c_1 = v_3$, $c_2 = v_4$, $p_1 = x_1 - y_2$, $p_2 = x_2 + y_1$, $q_1 = x_1 + y_2$, $q_2 = x_2 - y_1$.

From [11] we know that $J_{\bar{0}} = J(V,Q) \oplus Ff$, where $J(V,Q)$ is the Jordan algebra of a nondegenerate bilinear form Q, e is the identity element in $J(V,Q)$, and V is the vector space with basis $\{a, b, c_1, c_2\}$. We will use this presentation in Section 4.

In [2] G. Benkart and A. Elduque gave a realization of K_{10} which enables us to check directly that it is indeed a Jordan superalgebra. We introduce this construction, which will be very useful for us in Section 3.

First we pick up the Kaplansky superalgebra, K_3, over the field F and we define the following supersymmetric bilinear form (that is, the even and odd parts are

orthogonal and the form is symmetric in the even part and alternating in the odd part):

$$(e|e) = \frac{1}{2}, \quad (x|y) = 1, \quad ((K_3)_{\bar{0}}|(K_3)_{\bar{1}}) = 0.$$

Consider now the vector space over F: $F \cdot 1 \oplus (K_3 \otimes_F K_3)$ and define on it the product:

$$(a \otimes b)(c \otimes d) = (-1)^{\bar{b}\bar{c}}(ac \otimes bd - \frac{3}{4}(a|c)(b|d)1)$$

with $a, b, c, d \in K_3$ homogeneous elements, and where 1 is a formal identity element. Then,

Theorem 2.1 (Benkart, Elduque)**.** *K_{10} is isomorphic to $F \cdot 1 \oplus (K_3 \otimes K_3)$ by means of the linear map given by:*

$$\begin{aligned}
e &\mapsto \frac{3}{2} \cdot 1 - 2e \otimes e, & f &\mapsto -\frac{1}{2} \cdot 1 + 2e \otimes e,\\
a &\mapsto -4x \otimes x - y \otimes y, & b &\mapsto -4x \otimes x + y \otimes y,\\
c_1 &\mapsto 2x \otimes y, & c_2 &\mapsto -2y \otimes x,\\
p_1 &\mapsto 4x \otimes e - 2e \otimes y, & p_2 &\mapsto -4e \otimes x - 2y \otimes e,\\
q_1 &\mapsto 4x \otimes e + 2e \otimes y, & q_2 &\mapsto -4e \otimes x + 2y \otimes e.
\end{aligned}$$

If we denote by W the vector space $(K_3)_{\bar{1}}$, we notice that $V = W \otimes W$ is endowed with the bilinear nondegenerate symmetric form given by $b(s \otimes t, u \otimes v) = (s|u)(t|v)$. We note also that the corresponding decomposition of $(K_{10})_{\bar{0}}$ into the direct sum of two simple ideals according to the one given above is $(K_{10})_{\bar{0}} = I \oplus J$, where $I = F \cdot (-1/2 + 2e \otimes e)$ and $J = F \cdot (3/2 - 2e \otimes e) \oplus V$.

3. Automorphisms of K_{10}

Here by automorphism we mean automorphism of graded algebras.

It is easy to compute the group of automorphisms of the examples 1), 2) and 4) of Jordan superalgebras given in Section 1. So, with the notation of Section 2, $Aut(K_3) = Sp(W)$, the symplectic group (or also $Aut(K_3) = SL(W)$, the special linear group). In addition, $Aut(D_t) = Sp(U)$, where U is the vector space generated by $\{u, v\}$. If J is the superalgebra of a superform, then $Aut(J) = O(n, F) \oplus Sp(m, F)$, where $O(n, F)$ denotes the orthogonal group of $V_{\bar{0}}$, with $\dim V_{\bar{0}} = n$, related to $(\ ,\ )|_{V_{\bar{0}} \times V_{\bar{0}}}$, and $Sp(m, F)$ denotes the symplectic group of $V_{\bar{1}}$, with $\dim V_{\bar{1}} = m$, related to $(\ ,\ )|_{V_{\bar{1}} \times V_{\bar{1}}}$.

In what follows in this section it will be assumed that F is a field of characteristic $\neq 2, 3$ such that $F^2 = F$.

Let $C_2 = \{1, \epsilon\}$ be the cyclic group of order 2 ($\epsilon^2 = 1$) and consider the wreath product $G = Sp(W) \wr C_2$ (that is, G is the semidirect product $(Sp(W) \times Sp(W)) \rtimes C_2$ and $(f, g)\epsilon = \epsilon(g, f)$ for any $f, g \in Sp(W)$).

The following maps are clearly group homomorphisms:

$$\begin{array}{llll}
\Psi: & Aut(K_{10}) & \longrightarrow O(V,b) & \\
 & \varphi & \longmapsto \varphi|V & \\
 & & & \\
\tilde{\Psi}: & G & \longrightarrow O(V,b) & \\
 & (f,g) & \longmapsto f\otimes g : W\otimes W \to & W\otimes W \\
 & & \qquad\qquad s\otimes t & \mapsto f(s)\otimes g(t) \\
 & \epsilon & \longmapsto \quad \hat{\epsilon} \quad : W\otimes W \to & W\otimes W \\
 & & \qquad\qquad s\otimes t & \mapsto \quad -t\otimes s \\
 & & & \\
\Phi: & G & \longrightarrow Aut(K_{10}) & \\
 & (f,g) & \longmapsto \quad \Phi_{(f,g)}: & \begin{cases} 1 & \to 1 \\ a\otimes b & \mapsto \tilde{f}(a)\otimes\tilde{g}(b) \end{cases} \\
 & \epsilon & \longmapsto \quad \delta: & \begin{cases} 1 & \to 1 \\ a\otimes b & \mapsto (-1)^{\bar{a}\bar{b}} b\otimes a \end{cases}
\end{array}$$

where for any $f \in Sp(W)$, $\tilde{f}$ denotes the automorphism of K_3 such that $\tilde{f}(e) = e$, $\tilde{f}(w) = f(w)\ \forall w \in W$.

Note that $\Psi \circ \Phi = \tilde{\Psi}$.

Lemma 3.1. *$\tilde{\Psi}$ is onto with $Ker\tilde{\Psi} = \{\pm(id, id)\}$.*

Proof. Let $\{u, v\}$ be a symplectic basis of W (so $(u \mid v) = 1$). Then if $\alpha = u\otimes s + v\otimes t$ is an isotropic vector in V (that is, $b(\alpha,\alpha) = 0$), then $0 = b(\alpha,\alpha) = (s \mid t)$, so s, t are linearly dependent. This shows that

$$\{\text{isotropic vectors of } V\} = \{s\otimes t \mid s, t \in W\}.$$

Now let us show that $\tilde{\Psi}$ is onto. For any $\varphi \in O(V,b)$, $\varphi(u\otimes u)$ is isotropic ($u\otimes u$ is), so there are $s, t \in W$ with $\varphi(u\otimes u) = s\otimes t \neq 0$.

Take $f, g \in Sp(W)$ with $f(s) = u$, $g(t) = u$. Hence $\varphi \in im\tilde{\Psi}$ if and only if

$$(f\otimes g)\circ\varphi = \tilde{\Psi}(f,g)\circ\varphi \in im\tilde{\Psi},$$

and, therefore, it can be assumed that $\varphi(u\otimes u) = u\otimes u$.

Now $\varphi(v\otimes v) = s\otimes t$ for some $s, t \in W$ with

$$1 = b(u\otimes u, v\otimes v) = b(\varphi(u\otimes u), \varphi(v\otimes v)) = b(u\otimes u, s\otimes t) = (u \mid s)(u \mid t),$$

and there exists $0 \neq \alpha \in F$ such that $(u \mid s) = \alpha$, $(u \mid t) = \alpha^{-1}$ or $(u \mid \alpha^{-1}s) = 1$, $(u \mid \alpha t) = 1$. Thus there exist $f, g \in Sp(W)$ such that $f(u) = u$, $f(v) = \alpha^{-1}s$, $g(u) = u$, $g(v) = \alpha t$, and changing φ to $(f^{-1}\otimes g^{-1})\circ\varphi$ we may assume that $\varphi(u\otimes u) = u\otimes u$, $\varphi(v\otimes v) = v\otimes v$.

Then $\varphi(u\otimes v)$ is isotropic and orthogonal to both $u\otimes u$ and $v\otimes v$, so it is a scalar multiple of either $u\otimes v$ or $v\otimes u$. By using $\tilde{\Psi}(\epsilon) = \hat{\epsilon}$ we may assume that $\varphi(u\otimes v) = \gamma u\otimes v$ for some $0 \neq \gamma \in F$ and then, necessarily, $\varphi(v\otimes u) = \gamma^{-1}v\otimes u$. Let $\mu \in F = F^2$ with $\mu^2 = \gamma$. Then $\varphi = f\otimes g$ where $f(u) = \mu u$, $f(v) = \mu^{-1}v$, $g(u) = \mu^{-1}u$, $g(v) = \mu v$, so that $\varphi \in im\tilde{\Psi}$.

The assertion about the kernel is clear. $\square$

Lemma 3.2. *Ψ is onto with $Ker\Psi = \{id, \tau\}$, where τ is the grading automorphism ($\tau(z) = (-1)^{\bar{z}}z$) for any homogeneous $z \in K_{10}$.*

Proof. Since $\Psi \circ \Phi = \tilde{\Psi}$ and $\tilde{\Psi}$ is onto, so is Ψ. Moreover, if $\varphi \in Aut(K_{10})$ belongs to the kernel of Ψ, then $\varphi|(K_{10})_{\bar{0}} = Id$, so $\varphi(z_{\bar{0}} \cdot z_{\bar{1}}) = z_{\bar{0}} \cdot \varphi(z_{\bar{1}})$ for any $z_{\bar{0}} \in (K_{10})_{\bar{0}}$ and $z_{\bar{1}} \in (K_{10})_{\bar{1}}$.

For any $s \in W$, $s \otimes e$ is, up to scalars, the unique odd element annihilated by $s \otimes W$ ($\subseteq (K_{10})_{\bar{0}}$); hence $\varphi(s \otimes e) = \beta s \otimes e$ for some $0 \neq \beta \in F$.

Take $t \in W$ with $(s \mid t) = 1$. Then $\forall z \in W$, $(t \otimes z) \cdot (s \otimes e) = \frac{1}{2} e \otimes z$, so $\varphi(e \otimes z) = 2(t \otimes z) \cdot \varphi(s \otimes e) = \beta(e \otimes z)$ for any $z \in W$. It follows that $\varphi|(K_{10})_{\bar{1}} = \varphi|(e \otimes W \oplus W \otimes e) = \beta Id$ and, since φ is an automorphism, $\beta^2 = 1$, so φ is either the identity or the grading automorphism τ. □

Theorem 3.3. *Φ is an isomorphism.*

Proof. Since $\Phi(-id, -id) = \tau$, $\Phi(Ker\tilde{\Psi}) = Ker\Psi$. Now the assertion follows from the fact that $\Psi \circ \Phi = \tilde{\Psi}$, together with Lemmata 1 and 2. □

4. Maximal subalgebras of K_{10}

In what follows the word subalgebra will be used in the graded sense, so any subalgebra is graded. Also in this paragraph we consider that K_{10} is a superalgebra over F, an algebraically closed field of characteristic $\neq 2, 3$.

Theorem 4.1. *Any maximal subalgebra of K_{10} is, up to an automorphism of K_{10}, one of the following:*

(i) $(K_{10})_{\bar{0}}$.
(ii) *The subalgebra with basis* $\{e, f, a, p_1, p_2\}$.
(iii) *The subalgebra with basis* $\{e, f, a + b, c_1, p_1, q_1, p_2 + q_2\}$.
(iv) *The subalgebra with basis* $\{e, f, a, b, c_1, p_1, q_1\}$.

Proof. Note that if B is a maximal subalgebra of K_{10}, $1 = e + f \in B$, as $F1 + B$ is a subalgebra of K_{10} and B is an ideal in $F1 + B$. If $f \notin B$, then $B_{\bar{0}} \oplus Ff = \pi_I(B_{\bar{0}}) \oplus \pi_J(B_{\bar{0}})$, which is a subalgebra of $\big(K_{10}\big)_{\bar{0}}$ (recall that $\big(K_{10}\big)_{\bar{0}}$ is the direct sum of the ideals I and $J = Ff$; here π_I and π_J denote the corresponding projections), and so the subalgebra generated by B and f equals $B \oplus Ff$. Since B is maximal, $B \oplus Ff = K_{10}$, so, in particular, $B_{\bar{1}} = \big(K_{10}\big)_{\bar{1}}$, but $\big(K_{10}\big)_{\bar{1}}$ generates K_{10}, so $B = K_{10}$, a contradiction. Therefore $f \in B$, and hence $e = 1 - f \in B$ too.

We now use the description of K_{10} given at the beginning of Section 2 and due to D. King. Then $B_{\bar{0}} = Fe + Ff + V_0$, with V_0 a vector subspace of V and so, in order to study the maximal subalgebras of K_{10}, we analyze the possible dimensions of V_0.

(1) If $V_0 = V$, then $B = (K_{10})_{\bar{0}}$ is a maximal subalgebra of K_{10}.

(2) If $\dim V_0 = 1$, we have two possible situations: either the rank of Q is 1 or it is 0.

If $\operatorname{rank}(Q|_{V_0}) = 1$, using Witt's Theorem and Theorem 3.3 we can suppose that $V_0 = Fa$. Now $B_{\bar{1}} = P_B + Q_B$ with $P_B = \{p \in B : pa = p\}$, $Q_B = \{q \in B : qa = -q\}$. If $\dim P_B = 2$ we can check that $B = \langle e, f, a, p_1, p_2 \rangle$ is a maximal subalgebra of K_{10}. Likewise if $\dim Q_B = 2$, $B = \langle e, f, a, q_1, q_2 \rangle$ is a maximal subalgebra of K_{10}, but there is an automorphism of K_{10} applying $\langle e, f, a, p_1, p_2 \rangle$ into $\langle e, f, a, q_1, q_2 \rangle$ (the automorphism given by the isometry that applies a to $-a$ and fixes b, c_1, c_2). If $\dim P_B = \dim Q_B = 1$, then we can check that $\dim V_0 > 1$, a contradiction. If

$P_B = 0$, then $B \subseteq \langle e, f, a, q_1, q_2\rangle$, which is a maximal subalgebra, and we are in the case above.

If $\operatorname{rank}(Q|_{V_0}) = 0$, using Witt's Theorem and Theorem 3.3 we can suppose that $V_0 = Fc_1$. But $c_1 B_{\bar{1}} \subseteq Fp_1 + Fq_1$ and $B_{\bar{1}} B_{\bar{1}} \subseteq Fc_1 + Fe + Ff$, so $B_{\bar{1}} \subseteq \langle p_1, q_1\rangle$. It follows that $B = \langle e, f, c_1, p_1, q_1\rangle$ is not maximal since $B \subsetneqq \langle e, f, a, b, c_1, p_1, q_1\rangle$.

(3) If $\dim V_0 = 2$, then $\operatorname{rank}(Q|_{V_0}) = 0, 1$ or 2.

If $\operatorname{rank}(Q|_{V_0}) = 2$, we can suppose that $V_0 = Fa + Fb$, again by Witt's Theorem and Theorem 3.3. As in (2), we have that $B_{\bar{1}} = P_B + Q_B$. If $\dim P_B = 2$, then since $bP_B = Q_B$, it follows that $\dim Q_B = 2$, and then $B = K_{10}$, a contradiction. If $\dim P_B = 1$, then $\dim Q_B = 1$, but since $B_{\bar{1}} B_{\bar{1}} \subseteq B_{\bar{0}}$, then either $B_{\bar{1}} = \langle p_1, q_2\rangle$ or $B_{\bar{1}} = \langle p_2, q_1\rangle$. This is a contradiction with $bB_{\bar{1}} \subseteq B_{\bar{1}}$. If $\dim P_B = 0$, then $\dim Q_B = 0$ and B is not a maximal subalgebra because $B \subseteq (K_{10})_{\bar{0}}$.

If $\operatorname{rank}(Q|_{V_0}) = 1$, then $V_0 = Fa + Fc_1$ and again $B_{\bar{1}} = P_B + Q_B$. If $\dim P_B = 2$, then $\dim Q_B \geq 1$ (because $c_1 p_1 = 0$ and $c_1 p_2 = q_1$) and so $B = K_{10}$, a contradiction. If $\dim P_B = 1$, then $\dim Q_B = 1$ or 0. If $\dim Q_B = 1$, from $B_{\bar{1}} B_{\bar{1}} \subseteq B_{\bar{0}}$ we have that $Q_B = Fq_1$ and for $0 \neq p \in P_B$ it follows that $pq_1 \in V_0 = Fa + Fc_1$, and therefore $p \in Fp_1$. But in this case B is not maximal since $B \subsetneqq \langle e, f, a, b, c_1, p_1, q_1\rangle$. Also, if $\dim Q_B = 0$, then $P_B = Fp_1$ and again B is not maximal. Now if $\dim P_B = 0$, then $\dim Q_B = 1$, with $Q_B = \langle q_1\rangle$. But again B is not maximal because $B \subsetneqq \langle e, f, a, b, c_1, p_1, q_1\rangle$.

Finally, if $\operatorname{rank}(Q|_{V_0}) = 0$, we can suppose that $V_0 = F \cdot (a + b) + Fc_1$, by Witt's Theorem and Theorem 3.3. If $x = \alpha p_1 + \beta p_2 + \sigma q_1 + \gamma q_2 \in B_{\bar{1}}$, then $c_1 x = \beta q_1 + \gamma p_1 \in B_{\bar{1}}$ and $(a+b)(c_1 x) = (\gamma - \beta)(p_1 - q_1) \subseteq B_{\bar{1}}$. Therefore if $\gamma \neq \beta$, then $\{p_1, q_1\} \subseteq B_{\bar{1}}$. So either $\{p_1, q_1\} \subseteq B_{\bar{1}}$ or $B_{\bar{1}} \subseteq Fp_1 + Fq_1 + F(p_2 + q_2)$. Moreover with $\gamma \neq \beta$, $q_1 x \notin B_{\bar{0}}$. Therefore $\gamma = \beta$ and $B = \langle e, f, a+b, c_1, p_1, q_1, p_2 + q_2\rangle$, which is a maximal subalgebra.

(4) If $\dim V_0 = 3$, then $\operatorname{rank}(Q|_{V_0}) = 2$ or 3.

If $\operatorname{rank}(Q|_{V_0}) = 3$, then we can suppose by Witt's Theorem and Theorem 3.3 that $V_0 = Fa + Fb + F \cdot (c_1 + c_2)$. Again $B_{\bar{1}} = P_B + Q_B$. If $\dim P_B = 2$, then $\dim Q_B = 2$ (because $bP_B \subseteq Q_B$) and $B_{\bar{1}} = (K_{10})_{\bar{1}}$, a contradiction. So $\dim P_B = 1 = \dim Q_B$, but then, since $(c_1 + c_2) B_{\bar{1}} \subseteq B_{\bar{1}}$ and $B_{\bar{1}} B_{\bar{1}} \subseteq B_{\bar{0}}$, it follows that either $B_{\bar{1}} = \langle p_2, q_1\rangle$ or $B_{\bar{1}} = \langle p_1, q_2\rangle$, and this contradicts $bB_{\bar{1}} \subseteq B_{\bar{1}}$.

If $\operatorname{rank}(Q|_{V_0}) = 2$, then $V_0 = Fa + Fb + Fc_1$ and $B_{\bar{1}} = P_B + Q_B$, as usual. If $\dim P_B = 2$, then $\dim Q_B = 2$, a contradiction. So $\dim P_B = 1$ and $\dim Q_B = 1$. Since $P_B Q_B \subseteq V_0$ we have that $B_{\bar{1}} = \langle p_1, q_1\rangle$ and $B = \langle e, f, a, b, c_1, p_1, q_1\rangle$, which is a maximal subalgebra of K_{10}. □

Next we describe the structure of each one of the types of maximal subalgebras that we have obtained for K_{10}:

(i) $(K_{10})_{\bar{0}}$ is semisimple, and we have seen that $K_{10} \cong J(V, Q) \oplus F$.

(ii) $\langle e, f, a, p_1, p_2\rangle$ is semisimple and isomorphic to $F \oplus D_{-6}$, where
$$F \cong \langle 2e - a\rangle \quad \text{and} \quad \langle f, 2e + a\rangle + \langle p_1, p_2\rangle \cong D_{-6}.$$

(iii) $B = \langle e, f, a + b, c_1, p_1, q_1, p_2 + q_2\rangle$ has the following radical:
$$R = \langle c_1, a + b, p_1 + q_1\rangle,$$
and the quotient of the subalgebra B by its radical is isomorphic to D_{-3}:
$$B/R = \langle \bar{e}, \bar{f}, \overline{p_1}, \overline{\frac{p_2 + q_2}{2}}\rangle \cong D_{-3}.$$

(iv) $B = \langle e, f, a, b, c_1, p_1, q_1\rangle$ has as radical $R = \langle c_1, p_1, q_1\rangle$ with $B/R \cong F \oplus T$ with T the Jordan algebra of a bilinear form and

$$B/R \cong \langle \bar{e}, \bar{f}, \bar{a}, \bar{b}\rangle.$$

Finally, we will give the description of the maximal subalgebras of K_{10} in terms of the realization shown in Section 2, given by Benkart and Elduque. First consider the map

$$\begin{array}{rccc} \varphi\colon & K_{10} & \longrightarrow & K_{10} \\ & \lambda 1 + a\otimes b & \longmapsto & \lambda 1 + (-1)^{\bar{a}\bar{b}} b\otimes a, \end{array}$$

with a, b homogeneous elements of K_3. We remark that φ is an automorphism because

$$\begin{aligned} \varphi(a\otimes b)\varphi(c\otimes d) &= ((-1)^{\bar{a}\bar{b}} b\otimes a)\cdot((-1)^{\bar{c}\bar{d}} d\otimes c) \\ &= (-1)^{\bar{a}\bar{b}+\bar{c}\bar{d}+\bar{a}\bar{d}}(bd\otimes ac - \frac{3}{4}(b\,|\,d)(a\,|\,c)1) \\ &= (-1)^{\bar{b}\bar{c}}((-1)^{(\bar{a}+\bar{c})(\bar{b}+\bar{d})}(bd\otimes ac - \frac{3}{4}(a\,|\,c)(b\,|\,d)1)) \\ &= \varphi((a\otimes b)(c\otimes d)). \end{aligned}$$

Corollary 4.2. *The maximal subalgebras of $K_{10} = F\cdot 1 \oplus (K_3\otimes K_3)$ are, up to conjugation by automorphisms, the following:*

(i) $(K_{10})_{\bar{0}}$.
(ii) *The fixed subalgebra by the automorphism φ.*
(iii) *The subalgebra $F\cdot 1\oplus(M\otimes K_3)$, where $M = Fe+Fx$ is a maximal subalgebra of K_3.*
(iv) *The subalgebra $F\cdot 1 + F(e\otimes e) + (x\otimes K_3) + (K_3\otimes y)$.*

Proof. It is easy to compute the fixed elements by φ in terms of the original basis of K_{10} given in the introduction. This subalgebra is $\langle e, f, c_1+c_2, p_1-q_2, p_2-q_1\rangle$, and now, using Proposition 3.1, we know that there exists an automorphism of K_{10} that applies $\langle e, f, a, p_1, p_2\rangle$ into $\langle e, f, c_1+c_2, p_1-q_2, p_2-q_1\rangle$ (notice that $F(c_1+c_2)$ and Fa are isometric relative to the bilinear form Q of V, and moreover $\langle p_1-q_2, p_2-q_1\rangle = \{x\in(K_{10})_{\bar{0}} \mid (c_1+c_2)x = -x\}$). So the maximal subalgebras of type (ii) in Theorem 4.1 are conjugate to the subalgebra of fixed elements by φ.

Also the maximal subalgebras of type (iii) correspond, through the isomorphism in Theorem 2.1, to $\langle 1, e\otimes e, x\otimes x, x\otimes y, e\otimes x, x\otimes e, e\otimes y\rangle$, that is, to subalgebras $F\cdot 1\oplus(M\otimes K_3)$ with M a maximal subalgebra of K_3, while the maximal subalgebras of type (iv) correspond, through the isomorphism in Theorem 2.1, to $F\cdot 1 + F(e\otimes e) + (x\otimes K_3) + (K_3\otimes y)$. □

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