HOMOGENEOUS POLYNOMIALS
ON STRICTLY CONVEX DOMAINS

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Abstract. We consider a circular, bounded, strictly convex domain \( \Omega \subset \mathbb{C}^d \)
with boundary of class \( \mathcal{C}^2 \). For any compact subset \( K \) of \( \partial \Omega \) we construct a sequence of homogeneous polynomials on \( \Omega \) which are big at each point of \( K \).

As an application for any \( E \subset \partial \Omega \) circular subset of type \( G_\delta \) we construct a holomorphic function \( f \) which is square integrable on \( \Omega \setminus D \) and such that \( E = E_{D}^2(f) := \{ z \in \partial \Omega : \int_{D} |f|^2 \, d\mathcal{L}^2_{D} = \infty \} \) where \( D \) denotes unit disc in \( \mathbb{C} \).

1. Introduction

Let \( \Omega \) denote a bounded, convex and circular domain with a defining function \( \eta \) of class \( \mathcal{C}^2 \). We also denote by \( D \) the unit disc in \( \mathbb{C} \) and define the exceptional set \( E_{D}^2(f) := \{ z \in \partial \Omega : \int_{D} |f|^2 \, d\mathcal{L}^2_{D} = \infty \} \) for a holomorphic function \( f \in \mathcal{O}(\Omega) \). For more information about exceptional sets see [1, 2, 3, 4, 5, 6, 7].

In the paper [12] a natural number \( K \) and a sequence \( \{ p_n \}_{n=0}^\infty \) of homogeneous polynomials on \( \mathbb{C}^d \) were constructed so that \( |p_n(z)| \le 2 \) and \( \sum_{j=Km}^{K(m+1)-1} |p_n(z)| \ge 0.5 \) for all \( z \) belonging to the boundary of the unit ball \( \partial B^d \). In the paper [7] we introduced some additional arguments in such a way that for any circular set \( E \subset \partial B^d \) of type \( G_\delta \) and \( F_\sigma \) we could construct a holomorphic function \( f \) on the unit ball \( B^d \) so that \( E_{B^d}^2(f) = E \).

In this paper we construct similar homogeneous polynomials as in [7, 10, 11, 12]. Whilst these papers dealt with homogeneous polynomials on the unit ball, in this paper we construct homogeneous polynomials on \( \Omega \) which is a bounded, circular and strictly convex domain with boundary of class \( \mathcal{C}^2 \).

1.1. Geometric notions. In the complex \( d \)-dimensional space \( \mathbb{C}^d \) we consider the natural scalar product \( \langle \cdot, \cdot \rangle \). We also consider rotation invariant pseudometrics

\[ \rho(z, w) = \min_{|\lambda|=1} \| z - \lambda w \| . \]

As usual, by \( B(\xi; r) \) we denote the open ball with center \( \xi \) and radius \( r \), i.e.

\[ B(\xi; r) := \{ z \in \mathbb{C}^d : \rho(\xi, z) < r \} . \]
Let us recall that for \( \xi < q \) we may calculate:

\[
q_0 t^{d-1} \leq \mathcal{L}^d(B(\xi); r) \leq q_1 t^{d-1}
\]

for \( \xi \in \partial \Omega \) and \( 0 \leq r \leq 2R := 2 \sup_{z,w \in \partial \Omega} \rho(z, w) \).

Since rotation does not change \( \mathcal{L}^d(B(\xi); r) \) we can assume that \( \xi = (a, 0) \) for some \( a \in \mathbb{R}_+ \). In particular we can calculate

\[
\rho(\lambda, (a, 0)) = \min_{|\eta|=1} \| \eta \lambda, \eta w \| - (a, 0) \| = \sqrt{(|\lambda| - a)^2 + \| w \|^2}.
\]

Assume for a moment that \( r \leq a \). Since \( \xi = (a, 0) \) we can observe that \( B(\xi; r) = B_+ (\xi; r) \cup B_- (\xi; r) \) where:

\[
B_+(\xi; r) := \{(a + s)e^{is}, w) \in \mathbb{C}^d : 0 < s < r, \phi \in [0, 2\pi], \| w \| < \sqrt{r^2 - s^2}\} \\
B_- (\xi; r) := \{(a - s)e^{is}, w) \in \mathbb{C}^d : 0 < s < r, \phi \in [0, 2\pi], \| w \| < \sqrt{r^2 - s^2}\}.
\]

We may calculate\(^1\)

\[
\mathcal{L}^d(B(\xi; r)) = \int_{B_+(\xi; r)} 1d\mathcal{L}^d + \int_{B_- (\xi; r)} 1d\mathcal{L}^d \\
= \int_0^r \int_0^{2\pi} \int_{\| w \| < \sqrt{r^2 - s^2}} (a + s) + (a - s)d\mathcal{L}^d - 2(w)d\phi \\
= 4\pi a \tau_{2d-2} \int_0^r (r^2 - s^2)^{d-1} ds \\
= 4\pi a \tau_{2d-2} \int_0^r r^{d-1} \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k r^{2(d-k)} s^{2k} ds \\
= 4\pi a \tau_{2d-2} \int_0^r r^{d-1} \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k (2k + 1)^{-d-1}.
\]

Since \( 0 < r < 2R \) and \( 0 \in \Omega \) the above equality implies that there exist constants \( 0 < q_0 < q_1 \) such that \( \mathcal{L}^d(B(\xi; r)) = r^{d-1} \| \xi \| \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k (2k + 1)^{-d-1} \) for \( \xi \in \partial \Omega \). However for us it suffices to use \( \Omega \).

Additionally let us assume that \( \mathcal{L}^d(B(0; 2R)) \leq q_1 q_0. \)

A subset \( A \subset \mathbb{C}^d \) is called \( \alpha \)-separated if \( \rho(z_1, z_2) > \alpha \) for all distinct elements \( z_1 \) and \( z_2 \) of \( A \). It is clear that for \( \alpha > 0 \) each \( \alpha \)-separated subset of \( \partial \Omega \) is finite.

If \( g : \mathbb{C}^d \to \mathbb{C} \) is a function of class \( C^2 \), then we denote \( g_\xi = \left( \frac{\partial g}{\partial z_1} (\xi), ..., \frac{\partial g}{\partial z_d} (\xi) \right) \)

and

\[
H_g(P, w) := \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 g}{\partial z_j \partial z_k} (P) w_j w_k + \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 g}{\partial z_j \partial \overline{z_k}} (P) \overline{w}_j w_k + \sum_{j,k=1}^d \frac{\partial^2 g}{\partial z_j \partial \overline{z_k}} (P) w_j \overline{w}_k.
\]

Let us recall that \( \eta \) is a defining function of class \( C^2 \) for \( \Omega \). Let \( X \) be a compact, circular set. Assume that \( X \) contains only strictly convex points of \( \partial \Omega \), i.e. if \( \xi \in X \), then \( H_\eta (\xi, w) > 0 \) when \( w \neq 0 \) and \( \Re ((w, \overline{\eta})) = 0. \)

\(^1\pi_m := \mathcal{L}^m(\{w \in \mathbb{C}^m : \| w \| < 1\})\).
2. Homogeneous polynomials

All homogeneous polynomials of degree \( n \) constructed in this paper have the following form:

\[
p_n(z) = \sum_{\xi \in A} \langle z, \nu_\xi \rangle^n
\]

where \( A \) is a finite subset of \( \partial \Omega \) and \( \nu_\xi = \frac{1}{\langle \xi, \nu_\xi \rangle} \).

We begin with a very important estimation of \( |\langle z, \nu_\xi \rangle| \).

**Lemma 2.1.** There exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \rho^2(z, \xi) \leq 1 - |\langle z, \nu_\xi \rangle| \leq c_2 \rho^2(z, \xi)
\]

for \( \xi \in X, z \in \partial \Omega \).

**Proof.** Since \( \Omega \) is a circular and convex domain

\[
|\langle z, \nu_\xi \rangle| - |\langle \xi, \nu_\xi \rangle| \leq |\langle z, \nu_\xi \rangle| - \Re \langle \xi, \nu_\xi \rangle \leq \max_{|\lambda| = 1} \Re \langle \lambda z - \xi, \nu_\xi \rangle \leq 0.
\]

First we prove that for \( \xi \in \partial \Omega \) we have the following property:

\[
\langle \xi, \nu_\xi \rangle \in \mathbb{R}_+.
\]

Let \( \lambda_0 \) be such that \( \langle \xi, \lambda_0 \nu_\xi \rangle \in \mathbb{R}_+ \) and \( |\lambda_0| = 1 \). Observe that

\[
\Re \langle z - \xi, \lambda_0 \nu_\xi \rangle \leq |\langle z, \nu_\xi \rangle| - |\langle \xi, \nu_\xi \rangle| \leq 0
\]

for \( z \in \partial \Omega \). Since \( \partial \Omega \) is of class \( C^2 \) we have \( \nu_\xi = \lambda_0 \nu_\xi \). In particular \( \lambda_0 = 1 \).

In the next step we prove that there exist constants \( c_3, c_4 > 0 \) such that for \( z \in \partial \Omega \) and \( \xi \in X \) we have:

\[
c_3 \|z - \xi\|^2 \leq |\Re \langle z - \xi, \nu_\xi \rangle| \leq c_4 \|z - \xi\|^2.
\]

Due to [9] Lemma 3.1.6 there exist a defining function \( \tilde{\eta} \) of class \( C^2 \) for \( \Omega \) and constants \( c_5, c_6 > 0 \) such that \( c_5 \|w\|^2 \leq H\tilde{\eta}(\xi, w) \leq c_6 \|w\|^2 \) for \( \xi \in X \) and \( w \in \mathbb{C}^d \).

Let

\[
\phi(\xi, h) := \langle h, \tilde{\eta}_\xi \rangle + \Re \langle h, \nu_\xi \rangle - \Re \langle h, \eta_\xi \rangle.
\]

Since \( \tilde{\eta} \) is of class \( C^2 \) we have \( \tilde{\eta}(\xi + h) = \tilde{\eta}(\xi) + \langle h, \tilde{\eta}_\xi \rangle + \Re \langle h, \nu_\xi \rangle + H\tilde{\eta}(\xi, h) + f(\xi, h) \|h\|^2 \) where \( f \) is a continuous function such that \( \lim_{h \to 0} f(\xi, h) = 0 \). Observe that

\[
2\Re \langle z - \xi, \tilde{\eta}_\xi \rangle = \Re \phi(\xi, z - \xi) \quad \text{for} \quad z \in \partial \Omega \quad \text{and} \quad \xi \in X.
\]

In particular we may estimate

\[
\frac{2\Re \langle z - \xi, \tilde{\eta}_\xi \rangle}{\|z - \xi\|^2} = \frac{-H\tilde{\eta}(\xi, z - \xi) - f(\xi, z - \xi) \|z - \xi\|^2}{\|z - \xi\|^2} \leq -c_5 - f(\xi, z - \xi)
\]

and

\[
\frac{2\Re \langle z - \xi, \tilde{\eta}_\xi \rangle}{\|z - \xi\|^2} \geq -c_6 - f(\xi, z - \xi).
\]

The above inequalities imply that there exist constants \( c_7, c_8 > 0 \) such that

\[
c_7 \|z - \xi\|^2 \leq |\Re \langle z - \xi, \tilde{\eta}_\xi \rangle| \leq c_8 \|z - \xi\|^2.
\]
Since \( \eta, \tilde{\eta} \) are defining functions for \( \Omega \) there exists a continuous, positive function \( g \) such that
\[
\left( \frac{\partial \eta}{\partial x_1}, \frac{\partial \eta}{\partial y_1}, \ldots, \frac{\partial \eta}{\partial x_d}, \frac{\partial \eta}{\partial y_d} \right) = g(\xi) \left( \frac{\partial \tilde{\eta}}{\partial x_1}, \frac{\partial \tilde{\eta}}{\partial y_1}, \ldots, \frac{\partial \tilde{\eta}}{\partial x_d}, \frac{\partial \tilde{\eta}}{\partial y_d} \right).
\]
In particular \( \Re \langle z - \xi, \eta \rangle = g(\xi) \Re \langle z - \xi, \eta \rangle \) and there exist constants \( c_3, c_4 > 0 \) such that (5) holds.

Now we prove the main conclusion. Let constants \( c_3, c_4 > 0 \) be such that (5) holds. Let \( \lambda_1, \lambda_2 \) be such that \( \min_{|\lambda|=1} |\Re \langle \lambda z - \xi, \eta \rangle| = |\Re \langle \lambda_1 z - \xi, \eta \rangle| \) and \( \rho(z, \xi) = \min_{|\lambda|=1} |\lambda z - \xi| = |\lambda_2 z - \xi| \). By (5) we may estimate
\[
c_3 \|\lambda_2 z - \xi\|^2 \leq c_3 \|\lambda_1 z - \xi\|^2 \leq |\Re \langle \lambda_1 z - \xi, \eta \rangle| \leq |\Re \langle \lambda_2 z - \xi, \eta \rangle| \leq c_4 \|\lambda_2 z - \xi\|^2.
\]
By (4) we have \( |\Re \langle \lambda_1 z - \xi, \eta \rangle| = |\langle \xi, \eta \rangle| - |\langle z, \eta \rangle| \). In particular we may estimate
\[
c_3 \frac{\langle \xi, \eta \rangle}{|\langle \xi, \eta \rangle|} \rho^2(z, \xi) \leq 1 - |\langle z, \eta \rangle| \leq \frac{c_4}{|\langle \xi, \eta \rangle|} \rho^2(z, \xi).
\]
Since \( X \) is a compact set and \( |\langle \xi, \eta \rangle| > 0 \) it is enough to define \( c_1 = \inf_{\xi \in X} \frac{c_3}{|\langle \xi, \eta \rangle|} \) and \( c_2 = \sup_{\eta \in X} \frac{c_4}{|\langle \xi, \eta \rangle|} \). □

In order to control the values of the constructed polynomials we need some information about \( \alpha \)-separated sets.

Lemma 2.2. Suppose that \( A = \{\xi_1, \ldots, \xi_s\} \) is a \( 2\alpha t \)-separated subset of \( \partial \Omega \). For \( z \in \partial \Omega \) let
\[
A_k(z) := \{ \xi \in A : \alpha k t \leq \rho(z, \xi) \leq \alpha (k + 1)t \}.
\]
Then the set \( A_k(z) \) has at most \( q(k + 2)^{2d-1} \) elements. The set \( A_0 \) has at most 1 element and \( s \leq q_1(\alpha t)^{1-2d} \).

Proof. Observe that \( B(\xi_1; \alpha t) \cap B(\xi_2; \alpha t) = \emptyset \) for \( \xi_1 \neq \xi_2 \in A \). Moreover
\[
\bigcup_{\xi \in A_k(z)} B(\xi; \alpha t) \subset B(z; \alpha (k + 2)t).
\]
Let \( d_k \) be a number of elements in \( A_k(z) \). In particular
\[
d_k q_0(\alpha t)^{2d-1} \leq \sum_{\xi \in A_k(z)} \mathcal{L}^d(B(\xi; \alpha t)) \leq \mathcal{L}^d(B(z; \alpha (k + 2)t)) \leq q_1 q_0(\alpha (k + 2)t)^{2d-1}.
\]
We conclude that \( d_k \leq q_1 (k + 2)^{2d-1} \). Moreover if \( \xi_k \in A_0(z) \), then \( \rho(\xi_j, \xi_k) \leq \rho(z, \xi_j) + \rho(z, \xi_k) < 2\alpha t \) so \( \xi_j = \xi_k \) and \( d_0 \leq 1 \).

Since \( \Omega \subset B(0, R) \) (see section 1.1) we may assume that \( \alpha t \leq R \). In particular
\[
\bigcup_{\xi \in A_k(z)} B(\xi; \alpha t) \subset B(0; 2R)
\]
and we may estimate (see section 1.1)
\[
s q_0(\alpha t)^{2d-1} \leq \sum_{\xi \in A} \mathcal{L}^d(B(\xi; \alpha t)) \leq \mathcal{L}^d(B(0; 2R)) \leq q_1 q_0. \quad \square
\]

Lemma 2.3. If \( A \subset \partial \Omega \) is \( \alpha \)-separated, then for each \( \beta > \alpha \) there exists an integer \( K = K(\alpha, \beta) \) such that \( A \) can be partitioned into \( K \) disjoint \( \beta t \)-separated sets.
Lemma 2.5. \textbf{Proof.} Let us select from $A$ a maximal $\beta t$-separated subset $A_1$. Next from $A \setminus A_1$ we select a maximal $\beta t$-separated subset $A_2$. We continue this way until we exhaust $A$. Let $A_x$ be the last non-empty set in this procedure. Let $\xi \in A_x$. Observe that $B(\xi; \beta t) \cap A_k \neq \emptyset$ for $k = 1, \ldots, s - 1$. In particular $B(\xi; \beta t)$ contains at least $s$ different elements $\{\xi_1, \ldots, \xi_s\}$ from $A$. Since $B(\xi_j; \alpha t) \cap B(\xi_k; \alpha t) = \emptyset$ for $j \neq k$ and $B(\xi_j; \alpha t) \subset B(\xi; (\beta + \alpha)t)$, then

\begin{equation*}
    s q_0(\alpha t)^{2d-1} \leq \sum_{j=1}^{s} N^{2d} (B(\xi_j; \alpha t)) \leq N^{2d} (B(\xi; (\beta + \alpha)t)) \leq q_0((\beta + \alpha)t)^{2d-1}.
\end{equation*}

We can conclude that $s \leq q_1 \left( \frac{\alpha}{\alpha + 1} \right)^{2d-1}$. Now it suffices to choose a natural number $K$ so that $q_1 \left( \frac{\alpha}{\alpha + 1} \right)^{2d-1} \leq K$. \hfill \qed

Proposition 2.4. We have the following inequalities for $0 < x < 1$:

\begin{equation}
(1 - x)^{\frac{1}{2}} < e^{-1} < (1 - x)^{\frac{1}{2}}. \tag{6}
\end{equation}

\textbf{Proof.} Let $x \in (0, 1)$. To prove the left inequality let $f(x) := x + \ln (1 - x)$. Since $f'(x) = \frac{1}{1-x} < 0$ we have $f(x) < f(0) = 0$ and $\frac{1}{2} \ln (1 - x) < -1$.

To prove right inequality let $g(x) = \frac{1}{1-x} - \ln (1 - x)$. Since $g'(x) = \frac{1}{(1-x)^2} < 0$, then $g(x) < g(0) = 0$ and $-1 < \frac{1}{2} \ln (1 - x)$. \hfill \qed

Now we are ready to state some estimations for polynomials of the form (2).

Lemma 2.5. Let $0 < c_1 < c_2$ be constants from Lemma 2.1. For a given $a \in (0, 0.5)$ there exist constants $C > 2$ and $N_0 \in \mathbb{N}$ such that for all integers $N \geq N_0$, for each $C/\sqrt{c_1}N$-separated subset $A$ of $X$ and each integer $m$ with $N \leq m \leq 2N$ the polynomial $p_m(z) := \sum_{\xi \in A} \langle z, \nu_\xi \rangle^m$ satisfies

1. If $z \in \partial \Omega$, $Q(z) := \left\{ \xi \in A : \rho(z, \xi) \geq \frac{C}{2\sqrt{c_1}N} \right\}$, then $\sum_{\xi \in Q(z)} |\langle z, \nu_\xi \rangle|^m < a$.
2. If $z \in \partial \Omega$, then $Q(z) \setminus A$ has at most one element.
3. If $\xi_0 \in A$, $z \in \partial \Omega$ are such that $\rho(z, \xi_0) \leq \frac{a}{\sqrt{c_2}N}$, then
   a. $Q(z) = A \setminus \{\xi_0\}$,
   b. $|\langle z, \nu_{\xi_0} \rangle|^m > 1 - 2a^2$,
   c. $|p_m(z)| > 1 - 2a^2 - a$.
4. $|p_m(z)| \leq \sum_{\xi \in A} |\langle z, \nu_\xi \rangle|^m < 1 + a$ for all $z \in \partial \Omega$.

\textbf{Proof.} There exists a constant $C > 2$ large enough that for $k \in \mathbb{N}_+$ we have

\begin{equation}
\sum_{k=1}^{\infty} q_1(k + 2)^{2d-1} \exp \left( \frac{k^2 c_2}{4} \right) < a. \tag{7}
\end{equation}

Due to Proposition 2.4 we can estimate:

\[ \lim_{N \to \infty} \exp \left( \frac{-2a^2}{1 - a^2 N^{-1}} \right) = \exp (-2a^2) > 1 - 2a^2. \]

In particular we can choose $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

\begin{equation}
\exp \left( \frac{-2a^2}{1 - a^2 N^{-1}} \right) > 1 - 2a^2. \tag{8}
\end{equation}
There exists Theorem 2.6.

Let \( z \in \partial \Omega \), \( Q(z) := \{ \xi \in A : \rho(z, \xi) \geq \frac{C}{2\sqrt{c_1 N}} \} \) and
\[
A_k(z) = \{ \xi \in A : \frac{kC}{2\sqrt{c_1 N}} \leq \rho(z, \xi) < \frac{(k + 1)C}{2\sqrt{c_1 N}} \}.
\]

Due to Lemma 2.2 the set \( A_0(z) \) has at most 1 element and
\[
\#A_k(z) \leq q_1(k + 2)^{2d-1}.
\]

Since \( Q(z) \setminus A = A_0(z) \) we have the property (2). Due to Lemma 2.1 for \( \xi \in A_k(z) \) we have
\[
|\langle z, \nu_\xi \rangle| \leq 1 - c_1 \rho^2(z, \xi) \leq 1 - \frac{k^2C^2}{4N}.
\]

Now we may obtain the property (1):
\[
\sum_{\xi \in Q(z)} |\langle z, \nu_\xi \rangle|^m \leq \sum_{k=1}^{\infty} \sum_{\xi \in A_k(z)} |\langle z, \nu_\xi \rangle|^m \leq \sum_{k=1}^{\infty} \sum_{\xi \in A_k(z)} \left( 1 - \frac{k^2C^2}{4N} \right)^N
\]
\[
\leq \sum_{k=1}^{\infty} \#A_k(z) \exp \left( -\frac{k^2C^2}{4} \right) \leq \sum_{k=1}^{\infty} q_1(k + 2)^{2d-1} \exp \left( \frac{C^2}{24N} \right) < a.
\]

Since \( A_0(z) \) has at most one element we obtain the property (4):
\[
|p_m(z)| \leq \sum_{\xi \in A} |\langle z, \nu_\xi \rangle|^m \leq 1 + \sum_{\xi \in Q(z)} |\langle z, \nu_\xi \rangle|^m < 1 + a.
\]

Now let \( \xi_0 \in A, z \in \partial \Omega \) be such that \( \rho(z, \xi_0) \leq \frac{a}{\sqrt{c_2 N}} < \frac{C}{2\sqrt{c_1 N}} \). Since \( A_0(z) \) has at most 1 element we have \( A_0(z) = \{ \xi_0 \} \), which gives the property (3a). Moreover we have:
\[
|\langle z, \nu_{\xi_0} \rangle| \geq 1 - c_2 \rho^2(z, \xi_0) \geq 1 - \frac{a^2}{N}.
\]

Now we observe the property (3b) for \( N \geq N_0 \):
\[
|\langle z, \nu_{\xi_0} \rangle|^m \geq \left( 1 - \frac{a^2}{N} \right)^{2N} \geq \exp \left( \frac{-a^2N^{-1}2N}{1 - a^2N^{-1}} \right) > 1 - 2a^2.
\]

Moreover we may conclude the property (3c):
\[
|p_m(z)| \geq |\langle z, \nu_{\xi_0} \rangle|^m - \sum_{\xi \in Q(z)} |\langle z, \nu_\xi \rangle|^m > 1 - 2a^2 - a,
\]
which finishes the proof.

We are ready for main result of this paper.

**Theorem 2.6.** There exists \( K \in \mathbb{N} \) such that for \( 0 < \epsilon < 1 \) and for each pair of compact, circular and disjoint sets \( D, T \) such that \( T \subset X, D \subset \partial \Omega \), we can choose \( m_0 = m_0(D, T, \epsilon) \in \mathbb{N} \) and a sequence \( p_m \) of homogeneous polynomials of degree \( m \) which satisfy

1. \( |p_m(z)| \leq 2 \) for all \( z \in \partial \Omega, m > m_0 \),
2. \( \sum_{k=K_m}^{K(m+1)-1} |p_k(z)|^2 \geq 0.25 \) for all \( z \in T, m > m_0 \),
3. \( \sum_{k=K_m}^{K(m+1)-1} |p_k(z)|^2 \leq 2^{-((Km)^{1-\epsilon}} \) for all \( z \in D, m > m_0 \).
Proof. Let $0 < c_1 < c_2$ be from Lemma 2.1. For $a = \frac{1}{4}$ we can choose $C$ from Lemma 2.5. Let $K = K(\alpha, \beta)$ be from Lemma 2.3 for $\alpha = \frac{1}{4} \sqrt{c_2}$ and $\beta = \frac{C}{\sqrt{c_2}}$. For $N = Km$ fix a maximal $1/(4\sqrt{c_2N})$-separated subset $A \subset T$. Using Lemma 2.3 we can divide $A$ into at most $K$ disjoint $C/\sqrt{c_1N}$-separated subsets $A_0, A_1, \ldots, A_{K-1}$. We define

$$p_{Km+j}(z) := \sum_{\xi \in A_j} \langle z, \nu_\xi \rangle^{Km+j}$$

for $j = 0, 1, \ldots, K - 1$. From Lemma 2.5 we infer that there exists $m_0$ so high that for $m > m_0$ we have $|p_{Km+j}(z)| < 1 + a = \frac{5}{4} < 2$ for all $z \in \partial \Omega$ and $|p_{Km+j}(z)| > 1 - 2a^2 - a = \frac{1}{4} > 0.5$ for

$$z \in \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right).$$

Since $A = \bigcup_{j=0}^{K-1} A_j$ is a maximal $1/(4\sqrt{c_2N})$-separated subset of $T$ we conclude that

$$\bigcup_{j=0}^{K-1} \bigcup_{\xi \in A_j} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right) = \bigcup_{\xi \in A} B\left(\xi; \frac{1}{4\sqrt{c_2N}}\right) \supset T,$$

and from this follows that

$$\sum_{j=Km}^{K(m+1)-1} |p_j(z)|^2 \geq 0.25 \text{ for all } z \in T, m > m_0.$$

Without loss of generality we can assume that $m_0$ is so large that $\rho(z, w) > \sqrt{1/c_1N^\varepsilon}$ for all $z \in D$ and $w \in T$. Due to Lemma 2.2 we have

$$\# A_j \leq q_1 \left(\frac{\sqrt{c_1N}}{C}\right)^{2d-1}.$$

If $\xi \in A$ and $z \in D$, then on the basis of Lemma 2.1 we have

$$|\langle z, \nu_\xi \rangle| \leq 1 - c_1 \rho^2(z, \xi) \leq 1 - \frac{1}{N^\varepsilon}.$$

Now for $m_0$ large enough, $m > m_0$, $N = Km$ and $z \in D$ we may estimate

$$\sum_{j=0}^{K-1} |p_{Km+j}(z)|^2 \leq \sum_{j=0}^{K-1} \sum_{\xi \in A_j} |\langle z, \nu_\xi \rangle|^{2Km+j} \leq \sum_{\xi \in A} |\langle z, \nu_\xi \rangle|^N \leq \sum_{\xi \in A} \left(1 - \frac{1}{N^\varepsilon}\right)^N \leq q_1 K \left(\frac{\sqrt{c_1N}}{C}\right)^{2d-1} \left(1 - \frac{1}{N^\varepsilon}\right)^{N^\varepsilon N^{1-\varepsilon}} \leq \frac{1}{2^{N^{1-\varepsilon}}}. \tag*{\square}$$

As an application we can present the following result:

**Theorem 2.7.** Assume that $\Omega$ is a circular, bounded and strictly convex domain with the boundary of class $C^2$. Then for any circular subset $E \subset \partial \Omega$ of type $G_6$ there exists a holomorphic function $f$ which is square integrable on $\Omega \setminus \mathbb{D}E$ and such that $E = E_0(f) := \{z \in \partial \Omega : \int_{\mathbb{D}z} |f|^2 d\mathcal{O}_{\mathbb{D}z} = \infty\}$. 


Proof. Let σ be the natural measure on ∂Ω. On the basis of [8, Theorem 2.6] there exist sequences \(\{D_i\}_{i \in \mathbb{N}}\), \(\{T_i\}_{i \in \mathbb{N}}\) of compact, circular sets in ∂Ω such that:

1. \(\bigcup_{i \in \mathbb{N}} D_i = \partial \Omega \setminus E\) and \(D_j \subset D_{j+1}\) for \(j \in \mathbb{N}\),
2. \(T_j \cap D_j = \emptyset\) for \(j \in \mathbb{N}\),
3. \(E = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} T_i\),
4. \(\sigma(\partial \Omega \setminus (E \cup D_j)) \leq 2^{-j}\).

Since \(\Omega\) is a strictly convex domain then \(X = \partial \Omega\). Let \(K\) be a natural number from Theorem 2.6. We can use Theorem 2.6 once again and conclude that there exist a sequence of natural numbers \(\{m_j\}_{j \in \mathbb{N}}\) and a sequence of homogeneous polynomials \(\{p_m\}_{m \in \mathbb{N}}\) such that

1. \(m_j < m_{j+1}\) for \(j \in \mathbb{N}\),
2. \(|p_m(z)| \leq 2\) for all \(z \in \partial \Omega\), \(m > m_0\),
3. \(\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \geq 0.25\) for all \(z \in T_j\),
4. \(\sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 \leq 2^{-j}\) for all \(z \in D_j\).

Now we can define

\[
f = \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} \sqrt{k+1} p_k.
\]

Observe that for \(z \in \partial \Omega\) we have

\[
\int_{\mathbb{D}} |f|^2 d\Omega^2_{\mathbb{D}} = \sum_{j=1}^{\infty} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2.
\]

In particular for \(z \in E\) we have

\[
\int_{\mathbb{D}} |f|^2 d\Omega^2_{\mathbb{D}} \geq \sum_{j \in T_j} 0.25 = \infty.
\]

If \(z \in \partial \Omega \setminus E\), then there exists \(j_0\) such that \(z \in D_j\) for \(j \geq j_0\). In particular

\[
\int_{\mathbb{D}} |f|^2 d\Omega^2_{\mathbb{D}} \leq \sum_{j=1}^{j_0-1} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k(z)|^2 + \sum_{j=j_0}^{\infty} 2^{-j} < \infty.
\]

Now we prove that \(f\) is square integrable on \(\Omega \setminus \mathbb{D}\). There exists \(M > 0\) such that

\[
\int_{\Omega \setminus \mathbb{D}} |f|^2 \, d\Omega^{2d} \leq M \int_{\partial \Omega \setminus E} \int_{\mathbb{D}} |f|^2 \, d\Omega^2_{\mathbb{D}} \, d\sigma(z).
\]

In particular we may estimate

\[
\int_{\Omega \setminus \mathbb{D}} |f|^2 \, d\Omega^{2d} \leq M \sum_{j=1}^{\infty} \int_{\partial \Omega \setminus E} \sum_{k=Km_j}^{K(m_j+1)-1} |p_k|^2 \, d\sigma
\]

\[
\leq M \sum_{j=1}^{\infty} 2^{-j} \sigma(D_j) + M \sum_{j=1}^{\infty} 4K \sigma(\partial \Omega \setminus (E \cup D_j))
\]

\[
\leq M \sigma(\partial \Omega) + 4KM < \infty.
\]

References


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