

## SOLUTIONS FOR A NONLOCAL CONSERVATION LAW WITH FADING MEMORY

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ABSTRACT. Global entropy solutions in  $BV$  for a scalar nonlocal conservation law with fading memory are constructed as the limits of vanishing viscosity approximate solutions. The uniqueness and stability of entropy solutions in  $BV$  are established, which also yield the existence of entropy solutions in  $L^\infty$  while the initial data is only in  $L^\infty$ . Moreover, if the memory kernel depends on a relaxation parameter  $\varepsilon > 0$  and tends to a delta measure weakly as measures when  $\varepsilon \rightarrow 0+$ , then the global entropy solution sequence in  $BV$  converges to an admissible solution in  $BV$  for the corresponding local conservation law.

### 1. INTRODUCTION AND MAIN THEOREMS

We study global entropy solutions to a scalar nonlocal conservation law with fading memory:

$$(1.1) \quad u_t + f(u)_x + \int_0^t k(t-\tau) f(u(\tau))_x d\tau = 0, \quad x \in \mathbb{R},$$

and initial data

$$(1.2) \quad u(0, x) = u_0(x),$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function and  $u_0 \in BV(\mathbb{R})$ . For simplicity, we sometimes use the notation  $u(t) := u(t, x)$  to emphasize the state at  $t > 0$  as in (1.1).

In one-dimensional viscoelasticity, hyperbolic conservation laws

$$(1.3) \quad U_t + F(U)_x = 0$$

correspond to the constitutive relations of an elastic medium when the value of the flux function  $F$  at  $(t, x)$  is solely determined by the value of  $U(t, x)$ . However, this model (1.3) is inadequate when viscosity and relaxation phenomena are present. In that case, the flux function depends also on the past history of the material, i.e. on  $U(\tau, x)$  for  $\tau < t$ . Under these circumstances, we say that the material has memory.

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An important class of media of this type are materials with fading memory, which correspond to the constitutive relations with flux functions of the form

$$(1.4) \quad F(U(t, x)) + \int_0^t k(t - \tau)G(U(\tau, x)) d\tau,$$

where  $F$ ,  $G$  are smooth functions and  $k$  is a smooth kernel, integrable over  $\mathbb{R}_+ := [0, \infty)$ . When the kernel satisfies appropriate conditions motivated by physical considerations, the influence of the memory term in (1.4) reflects a damping effect. Consequently, global smooth solutions exist for given small initial data (cf. Renardy, Hrusa, and Nohel [15]), in contrast to the situation with elastic media in which classical solutions in general break down in finite time even when the initial data is small. However, when the initial data is large, the destabilizing action of nonlinearity of the flux function  $f$  prevails over the damping, and solutions break down in a finite time; see Dafermos [3] and Malek-Madani and Nohel [10].

In this paper we first construct global entropy solutions in  $BV$  to the nonlocal conservation law (1.1) with fading memory via the vanishing viscosity approximation by adding the artificial viscosity term as follows:

$$(1.5) \quad u_t^\nu + f(u^\nu)_x + \int_0^t k(t - \tau)f(u^\nu(\tau))_x d\tau = \nu(u^\nu + \int_0^t k(t - \tau)u^\nu(\tau) d\tau)_{xx}.$$

For a scalar local conservation law modeling elastic materials, such a result was first established in [8, 13, 17].

The main motivation for the vanishing viscosity approximation (1.5) is that the conservation law (1.1) can be viewed as a linear Volterra equation, which was first observed by MacCamy [11] and later employed in Dafermos [4] and Nohel, Rogers, and Tzavaras [12] (also see [1]). In this way, it is easy to extract the damping character of the memory term. Let  $r(t)$  be the resolvent kernel associated with  $k(t)$ :

$$(1.6) \quad r + k * r = -k.$$

Then we can write (1.1) as

$$-f(u)_x = u_t + \int_0^t r(t - \tau)u_t(\tau) d\tau.$$

Integrating by parts yields

$$(1.7) \quad u_t + f(u)_x + r(0)u = r(t)u_0 - \int_0^t r'(t - \tau)u(\tau) d\tau,$$

which is equivalent to (1.1). The vanishing viscosity approximation (1.5) is equivalent to the following artificial viscosity approximation to (1.7):

$$(1.8) \quad u_t^\nu + f(u^\nu)_x + r(0)u^\nu = r(t)u_0 - \int_0^t r'(t - \tau)u^\nu(\tau) d\tau + \nu u_{xx}^\nu.$$

Note that the above argument applies only if the nonlinearity  $f$  in the instantaneous response is the same as in the memory term chosen in (1.1); also see [4, 11, 12] for the same restriction. The artificial viscosity term in (1.5) is chosen so that (1.8) has the standard artificial viscosity term to ensure the  $L^\infty$  and  $BV$  estimates of  $u^\nu$  (also see [1, 4, 12]).

The existence of a unique, regular local solution  $u^\nu(t, x)$  of (1.8) when the initial data  $u_0$  is smooth can be established through the standard Banach Fixed Point

Theorem. The local solution may be extended to a global solution with the help of the a priori  $L^\infty$  estimate established in Section 2.1.

A function  $u = u(t, x)$  is called an *entropy solution* to the Cauchy problem (1.1)–(1.2) if it satisfies that, for any test function  $\varphi \in C_0^1(\mathbb{R}_+^2)$  with  $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}$ ,  $\varphi \geq 0$ ,

$$(1.9) \quad \iint_{\mathbb{R}_+^2} (\eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(r(t)u_0 - r(0)u - \int_0^t r'(t-\tau)u(\tau)d\tau)\varphi) dt dx + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0, x) dx \geq 0,$$

for any convex entropy  $\eta(u)$ , where  $q(u)$  is the corresponding entropy flux satisfying  $q'(u) = \eta'(u)f'(u)$ .

Before we state the results, we introduce some notation. Let  $\varrho$  be the standard mollifier. We define the mollification of  $u_0$  to be

$$(1.10) \quad u_0^\nu := (u_0\chi_\nu) * \varrho_\nu,$$

where, for each  $\nu > 0$ ,  $\varrho_\nu(x) := \frac{1}{\nu} \varrho(\frac{x}{\nu})$  and  $\chi_\nu(x) := 1$  for  $|x| \leq 1/\nu$  and 0 otherwise. The main result is the following.

**Theorem 1.1.** *Consider the Cauchy problem (1.5) with Cauchy data:*

$$(1.11) \quad u^\nu(0, x) = u_0^\nu(x),$$

where the initial data  $u_0^\nu$  is given by (1.10) and  $u_0 \in BV(\mathbb{R})$ . Let the resolvent kernel  $r$  associated with  $k$  as defined in (1.6) be a nonnegative, nonincreasing function in  $L^1(\mathbb{R}_+)$ . Then, for each  $\nu > 0$ , the Cauchy problem (1.5) and (1.11) has a unique solution  $u^\nu$  defined globally with a uniform BV bound. Moreover, as  $\nu \rightarrow 0$ ,  $u^\nu$  converges in  $L_{loc}^1$  to an entropy solution  $u \in BV$  to (1.1)–(1.2), which satisfies

$$(1.12) \quad \|u\|_{L^\infty(\mathbb{R}_+^2)} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

$$(1.13) \quad TV\{u(t)\} + \int_0^t r(t-\tau)TV\{u(\tau)\} d\tau \leq CL,$$

$$(1.14) \quad \|u(t) - u(s)\|_{L^1(\mathbb{R})} \leq C|t - s|,$$

where  $L = 1 + \|r\|_{L^1(\mathbb{R}_+)}$ , and  $C = C(TV\{u_0\}, \|u_0\|_{L^\infty})$  is a positive constant independent of  $r(t)$ .

Furthermore, we have

**Theorem 1.2.** *Let the resolvent kernel  $r(t)$  associated with  $k$  be a nonnegative and nonincreasing function in  $L^1(\mathbb{R}_+)$ . Let  $u, v \in BV(\mathbb{R}_+^2)$  be entropy solutions to (1.1) with initial data  $u_0, v_0 \in BV(\mathbb{R})$ , respectively. Then*

$$(1.15) \quad \|u(t) - v(t)\|_{L^1(\mathbb{R})} + \int_0^t r(t-\tau)\|u(\tau) - v(\tau)\|_{L^1(\mathbb{R})} d\tau \leq L\|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

That is, any entropy solution in BV to (1.1)–(1.2) is unique and stable in  $L^1$ . As a consequence, if  $u_0$  is only in  $L^\infty$ , not necessarily in  $BV(\mathbb{R})$ , there exists a global entropy solution  $u \in L^\infty$  to (1.1)–(1.2).

Having established the above results, we then analyze the case when the kernel in the scalar equation (1.1) is a relaxation kernel  $k_\varepsilon$  that depends on a small parameter

$\varepsilon > 0$  so that  $k_\varepsilon(t) \rightharpoonup (\alpha - 1)\delta(t)$  weakly as measures when  $\varepsilon \rightarrow 0+$ , where  $\delta(t)$  denotes the Dirac mass centered at the origin. That is,

$$(1.16) \quad u_t^\varepsilon + f(u^\varepsilon)_x + \int_0^t k_\varepsilon(t - \tau)f(u^\varepsilon(\tau))_x d\tau = 0,$$

$$(1.17) \quad u^\varepsilon(0, x) = u_0(x) \in BV(\mathbb{R})$$

with  $\sup_{\varepsilon > 0} \|k_\varepsilon\|_{L^1(\mathbb{R}_+)} < \infty$ .

We denote the entropy solution to the above problem by  $u^\varepsilon(t, x)$ . Let  $r_\varepsilon$  be the resolvent kernel associated with  $k_\varepsilon$  via (1.6). Hence, (1.16) reduces to

$$(1.18) \quad u_t^\varepsilon + f(u^\varepsilon)_x + r_\varepsilon(0)u^\varepsilon = r_\varepsilon(t)u_0 - \int_0^t r_\varepsilon'(t - \tau)u(\tau) d\tau$$

with  $\sup_{\varepsilon > 0} \|r_\varepsilon\|_{L^1(\mathbb{R}_+)} < \infty$ .

By Theorems 1.1–1.2, we conclude that the unique entropy solution sequence  $u^\varepsilon \in BV$  to (1.16)–(1.17) is uniformly bounded in  $L^\infty$  and uniformly  $L^1$ -stable, independent of  $\varepsilon$ . Then the solution sequence  $\{u^\varepsilon\}$  is a compact set in  $L^1_{loc}$  so that we can extract a subsequence  $\{u^{\varepsilon_k}\}$  that converges in  $L^1_{loc}$  to an admissible weak solution of the local conservation law:

$$(1.19) \quad u_t + \alpha f(u)_x = 0,$$

with Cauchy data  $u_0 \in BV$ .

**Theorem 1.3.** *Consider the Cauchy problem (1.16)–(1.17) with  $u_0 \in BV(\mathbb{R})$ . Let the resolvent kernel  $r_\varepsilon$  associated with  $k_\varepsilon$  as defined in (1.6) be a nonnegative, nonincreasing function with uniform  $L^1$ -norm independent of  $\varepsilon$ . Then the entropy solutions  $u^\varepsilon$  to (1.16)–(1.17) are uniformly bounded in  $L^\infty$  and stable in  $L^1$ :*

$$\begin{aligned} \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \|u_0\|_{L^\infty(\mathbb{R})}, \\ TV\{u^\varepsilon(t)\} &\leq CL, \\ \|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1(\mathbb{R})} &\leq C|t - s|, \\ \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1(\mathbb{R})} &\leq L\|u_0 - v_0\|_{L^1(\mathbb{R})}, \end{aligned}$$

where  $L := 1 + \sup_{\varepsilon > 0} \|r_\varepsilon\|_{L^1} < \infty$ ,  $C = C(TV\{u_0\}, \|u_0\|_{L^\infty}) > 0$  is independent of  $\varepsilon$ , and  $v^\varepsilon(t, x)$  is the entropy solution to (1.16)–(1.17) with initial data  $v_0 \in BV$ . Furthermore, if  $k_\varepsilon(t) \rightharpoonup (\alpha - 1)\delta(t)$  weakly as measures when  $\varepsilon \rightarrow 0+$ , then  $u^\varepsilon$  converges in  $L^1_{loc}$  to an admissible weak solution  $u$  of the Cauchy problem (1.19) and (1.2) with initial data  $u_0 \in BV$ .

In Theorems 1.1–1.3, the assumptions on  $r_\varepsilon(t)$ , or  $r(t)$ , can easily be converted to the assumptions on  $k_\varepsilon(t)$ , or  $k(t)$ , because of their symmetry between the kernel and the resolvent through (1.6). For example, such kernels  $k_\varepsilon(t)$  especially include the following set of kernels (i)–(vi):

- (i)  $k'_\varepsilon(t) \geq 0$  and  $\|k_\varepsilon\|_{L^1(\mathbb{R}_+)} \leq K$  for some constant  $K$  independent of  $\varepsilon > 0$ ;
- (ii)  $|1 + \hat{k}_\varepsilon(z)| \neq 0$  for any  $z$  with  $Re(z) \geq 0$ , and  $\hat{k}_\varepsilon(z)(1 + \hat{k}_\varepsilon(z)) \leq 0$  for the Laplace transform  $\hat{k}_\varepsilon$  of  $k_\varepsilon$ ;
- (iii)  $\sup_{\omega \in \mathbb{R}} |(1 + \hat{k}_\varepsilon(i\omega))^{-1}| \leq q$  for some constant  $q$  independent of  $\varepsilon$ ;
- (vi) there exist positive numbers  $T \sim \varepsilon$  and  $\eta \sim \varepsilon$  such that

$$\int_{|s| \geq T} |k_\varepsilon(t)| \leq \frac{1}{12q}, \quad \sup_{0 < s < \eta} \int_{\mathbb{R}} |k_\varepsilon(t) - k_\varepsilon(t - s)| dt \leq \frac{1}{4}.$$

The prototype is

$$(1.20) \quad k_\varepsilon(t) = -\frac{1-\alpha}{\varepsilon} \exp\left(-\frac{t}{\varepsilon}\right), \quad 0 < \alpha < 1,$$

for which the corresponding family of resolvent kernels is

$$(1.21) \quad r_\varepsilon(t) = \frac{1-\alpha}{\varepsilon} \exp\left(-\frac{\alpha t}{\varepsilon}\right).$$

In Section 2, we develop techniques for the nonlocal case, motivated by Vol'pert-Kruzkov's techniques [17, 8] and the  $L^\infty$ -estimate techniques for the local case, to establish uniform  $L^\infty$  and  $BV$  estimates of the vanishing viscosity approximate solutions, independent of  $\nu$ , by using the damping nature of the memory term.

As a corollary of these estimates, we establish the convergence of the vanishing viscosity approximate solutions to obtain the existence of entropy solutions in  $BV$ . In Section 3, we show that the entropy solution in  $BV$  is unique and stable in  $L^1$  with respect to the initial perturbation. In Section 4, we prove Theorem 1.3 and discuss the hypotheses of the theorems. Finally we give an example and show the relation of the fading memory limit with the zero relaxation limit as first considered systematically in Chen, Levermore, and Liu [2]; also see [9, 16, 18] for the model.

Last, we refer the reader to Christoforou [19] for an extension of the results of Theorem 1.1 on the existence of a global entropy solution to systems of conservation laws with fading memory.

## 2. PROOF OF THEOREM 1.1

In this section, we establish the uniform  $L^\infty$  and  $BV$  estimates, as well as the uniformly continuous dependence on time in  $L^1$ , which are used not only for the global existence of the vanishing viscosity approximate solutions, but also for their compactness. We also establish the existence and regularity of entropy solutions.

**2.1.  $L^\infty$  estimate.** We first obtain a uniform  $L^\infty$  estimate. Note that, by employing the resolvent kernel  $r$ , equation (1.5) can be written in the form (1.8), i.e. we study the Cauchy problem

$$(2.1) \quad u_t^\nu + f(u^\nu)_x + r(0)u^\nu = r(t)u_0^\nu - \int_0^t r'(t-\tau)u^\nu(\tau) d\tau + \nu u_{xx}^\nu,$$

$$(2.2) \quad u(0, x) = u_0^\nu(x), \quad x \in \mathbb{R},$$

where  $u_0^\nu(x)$  is defined in (1.10). By rescaling the coordinates,  $(t, x) \rightarrow (s, y) = (r_0 t, r_0 x)$ , we rewrite (1.8) as

$$(2.3) \quad \bar{u}_s^\nu + f(\bar{u}^\nu)_y + \bar{u}^\nu = \frac{1}{r_0} r\left(\frac{s}{r_0}\right) u_0^\nu - \frac{1}{r_0^2} \int_0^s r'\left(\frac{s-\tau}{r_0}\right) \bar{u}^\nu(\tau) d\tau + \nu r_0 \bar{u}_{yy}^\nu,$$

where  $r_0 > r(0)$  is any positive constant and  $\bar{u}^\nu(s, y) := u^\nu\left(\frac{s}{r_0}, \frac{y}{r_0}\right)$ . For any even integer  $p$ , multiplying (2.3) by  $p|\bar{u}^\nu|^{p-1}$  and integrating over  $[0, S] \times \mathbb{R}$ , we obtain

$$(2.4) \quad \begin{aligned} & \int_{-\infty}^{\infty} |\bar{u}^\nu(S)|^p dy + p \int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(s, y)|^p dy ds \\ & \leq \int_{-\infty}^{\infty} |\bar{u}_0^\nu(y)|^p dy + p \int_0^S \int_{-\infty}^{\infty} \frac{1}{r_0} r\left(\frac{s}{r_0}\right) |\bar{u}_0^\nu(y)| |\bar{u}^\nu(s, y)|^{p-1} dy ds \\ & \quad - p \int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(s, y)|^{p-1} \int_0^s \frac{1}{r_0^2} r'\left(\frac{s-\tau}{r_0}\right) |\bar{u}^\nu(\tau, y)| d\tau dy ds. \end{aligned}$$

By employing the standard inequality  $ab \leq \varepsilon_0 \frac{a^p}{p} + \varepsilon_1 \frac{b^q}{q}$  with  $\varepsilon_0 = (\frac{4(p-1)}{p})^{p-1}$ ,  $\varepsilon_1 = \frac{p}{4(p-1)}$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , the second term on the right-hand side of (2.4) is estimated as

$$(2.5) \quad p \int_0^S \int_{-\infty}^{\infty} \frac{1}{r_0} r\left(\frac{s}{r_0}\right) |\bar{u}_0^\nu(y)| |\bar{u}^\nu(s, y)|^{p-1} dy ds \leq \int_0^S \int_{-\infty}^{\infty} \varepsilon_0 \left(\frac{1}{r_0} r\left(\frac{s}{r_0}\right)\right)^p |\bar{u}_0^\nu(y)|^p dy ds + \frac{1}{4} p \int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(s, y)|^p dy ds,$$

so that the last term in (2.5) is dominated by the damping in (2.4). Similarly, we treat the last term in (2.4) to obtain

$$\int_{-\infty}^{\infty} |\bar{u}^\nu(S, y)|^p dy \leq \int_{-\infty}^{\infty} |\bar{u}_0^\nu(y)|^p dy + \varepsilon_0 \int_0^S \left(\frac{1}{r_0} r\left(\frac{s}{r_0}\right)\right)^p ds \int_{-\infty}^{\infty} |\bar{u}_0^\nu(y)|^p dy + \beta(S) \int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(\tau, y)|^p dy d\tau,$$

where  $\beta(S) := \varepsilon_0 \int_0^S \left(\int_0^s \left|\frac{1}{r_0} r'\left(\frac{s-\tau}{r_0}\right)\right|^q d\tau\right)^{p/q} ds$ . By Gronwall's inequality, we have

$$(2.6) \quad \int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(\tau)|^p dy d\tau \leq e^{\int_0^S \beta(w) dw} \int_0^S W(s) ds$$

for  $W(s) := \left(1 + \varepsilon_0 \int_0^s \left(\frac{1}{r_0} r\left(\frac{\tilde{s}}{r_0}\right)\right)^p d\tilde{s}\right) \int |\bar{u}_0^\nu(y)|^p dy$ . Let

$$(2.7) \quad K_S = \sup \left\{ \frac{1}{r_0^2} \left| r'\left(\frac{\tilde{s}}{r_0}\right) \right| : \tilde{s} \in [0, S] \right\}.$$

Then, for all  $s \in [0, S]$ , we have

$$(2.8) \quad \beta(S) \leq \varepsilon_0 K_S \int_0^S \left(-\frac{1}{r_0^2} \int_0^s r'\left(\frac{s-\tau}{r_0}\right) d\tau\right)^{p/q} ds \leq \varepsilon_0 K_S S,$$

since  $r$  is nonincreasing, and hence  $\lim_{p \rightarrow \infty} \frac{1}{p} \int_0^S \beta(w) dw = 0$ . Also,

$$\left(\int_0^S W(s) ds\right)^{\frac{1}{p}} \leq S^{\frac{1}{p}} \left(1 + \varepsilon_0^{\frac{1}{p}} \left(\int_0^S \left(\frac{1}{r_0} r\left(\frac{\tilde{s}}{r_0}\right)\right)^p d\tilde{s}\right)^{\frac{1}{p}}\right) \left(\int |\bar{u}_0^\nu|^p dy\right)^{\frac{1}{p}}.$$

Thus, if we raise (2.6) to  $1/p$ , take the limit as  $p \rightarrow \infty$ , and note that  $\varepsilon_0(p)^{\frac{1}{p}} \rightarrow 4$  as  $p \rightarrow \infty$ , we conclude that  $\|\bar{u}^\nu\|_{L^\infty([0, S] \times \mathbb{R})} \leq \|(1 + \frac{4}{r_0} r(\frac{s}{r_0}))|\bar{u}_0(y)|\|_{L^\infty([0, S] \times \mathbb{R})}$ . That is,

$$(2.9) \quad \|\bar{u}^\nu\|_{L^\infty(\mathbb{R}_+^2)} \leq \left(1 + \frac{4}{r_0} \|r\|_{L^\infty(\mathbb{R}_+)}\right) \|\bar{u}_0\|_{L^\infty(\mathbb{R})}.$$

Since  $r(t) \geq 0$  is nonincreasing so that  $0 \leq r(t) \leq r(0) < \infty$  and  $r_0 > r(0)$  is an arbitrary constant, then, as  $r_0 \rightarrow \infty$ , we conclude  $\|\bar{u}^\nu\|_{L^\infty(\mathbb{R}_+^2)} \leq \|u_0\|_{L^\infty(\mathbb{R})}$ . By rescaling the coordinates backwards, we obtain the uniform  $L^\infty$ -bound on  $u^\nu$  to (1.8) independent of  $\nu$ ;

$$(2.10) \quad \|u^\nu\|_{L^\infty(\mathbb{R}_+^2)} \leq \|u_0\|_{L^\infty(\mathbb{R})}.$$

**2.2. BV-regularity and estimates.** First note that  $u'_0 \in C^\infty$  has the following bounds:

$$(2.11) \quad \|(u'_0)_x\|_{L^1} \leq M(u_0), \quad \|(u'_0)_{xx}\|_{L^1} \leq \frac{C_0}{\nu} M(u_0),$$

for some  $C_0 > 0$  independent of  $\nu$ , where  $M(u_0) := TV\{u_0\} + 2\|u_0\|_{L^\infty}$ .

Set  $v = u'_x$  and  $w = u'_t$ . Then the evolution equations of  $v$  and  $w$  are

$$(2.12) \quad v_t + (f'(u^\nu)v)_x + r(0)v = r(t)(u'_0)_x - \int_0^t r'(t-\tau)v(\tau) d\tau + \nu v_{xx},$$

$$(2.13) \quad v(0, x) = (u'_0)_x,$$

and

$$(2.14) \quad w_t + (f'(u^\nu)w)_x + r(0)w = - \int_0^t r'(t-\tau)w(\tau) d\tau + \nu w_{xx},$$

$$(2.15) \quad w(0, x) = u'_t(0, x) = (u'_0)_{xx} - f'(u'_0)(u'_0)_x.$$

Multiplying (2.12) by  $\text{sgn}(v(t, x))$  and integrating with respect to  $x$ , we get

$$\frac{d}{dt} \|v(t)\|_{L^1} + r(0)\|v(t)\|_{L^1} \leq r(t)M(u_0) - \int_0^t r'(t-\tau)\|v(\tau)\|_{L^1} d\tau$$

since  $r(0) > 0$  and  $r$  is nonincreasing. Integrating over  $t \in [0, T]$  yields

$$(2.16) \quad \begin{aligned} & \|v(T)\|_{L^1} + r(0) \int_0^T \|v(t)\|_{L^1} dt \\ & \leq (1 + \int_0^T r(t) dt)M(u_0) - \int_0^T \int_0^t r'(t-\tau)\|v(\tau)\|_{L^1} d\tau dt. \end{aligned}$$

Changing the order of integration in the last term, we arrive at

$$\begin{aligned} & \|v(T)\|_{L^1} + r(0) \int_0^T \|v(t)\|_{L^1} dt \\ & \leq (1 + \int_0^T r(t) dt)M(u_0) - \int_0^T (r(T-\tau) - r(0))\|v(\tau)\|_{L^1} d\tau. \end{aligned}$$

Thus, we have

$$\|v(T)\|_{L^1} + \int_0^T r(T-\tau)\|v(\tau)\|_{L^1} d\tau \leq (1 + \int_0^T r(\tau) d\tau)M(u_0).$$

Because  $r(\cdot)$  is bounded in  $L^1(\mathbb{R}_+)$ , we obtain the following uniform bound on the gradient  $v = u_x$ :

$$(2.17) \quad \|u'_x(t)\|_{L^1} + \int_0^t r(t-\tau)\|u'_x(\tau)\|_{L^1} d\tau \leq L M(u_0),$$

where  $L := 1 + \|r\|_{L^1(\mathbb{R}_+)}$ . Similarly, we have

$$(2.18) \quad \|w(t)\|_{L^1} + \int_0^t r(t-\tau)\|w(\tau)\|_{L^1} d\tau \leq \|w(0)\|_{L^1}.$$

Using (2.1) and the bounds in (2.11) for the initial data, we find from (2.15) that

$$\|w(0)\|_{L^1} \leq \nu\|(u'_0)_{xx}\|_{L^1} + \|f'(u'_0)\|_{L^\infty} M(u_0) \leq C_1 M(u_0)$$

for some  $C_1 > 0$  independent of  $\nu$ . Hence, by (2.18),  $u_t^\nu$  is uniformly bounded in  $L^1$ . Thus, for  $0 < s < t$ , we get the uniformly continuous dependence on time for the solutions to (2.1)–(2.2):

$$(2.19) \quad \|u^\nu(t) - u^\nu(s)\|_{L^1} \leq \int_s^t \|w(\tau)\| d\tau \leq C |t - s|$$

with  $C = \max(C_1, 1) M(u_0)$  independent of  $\nu$ .

**2.3. Existence of entropy solutions in  $BV$  to (1.1)–(1.2).** Using (2.10), (2.17), and (2.19), Helly’s Compactness Theorem yields that a convergent subsequence  $\{u^{\nu_m}\}$  may be extracted with  $\nu_m \downarrow 0$  as  $m \rightarrow \infty$ , whose limit is denoted by  $u$ , i.e.,

$$(2.20) \quad u^{\nu_m}(t) \rightarrow u(t) \quad \text{in } L^1_{loc} \quad \text{for all } t > 0.$$

The limit  $u(t, \cdot)$  is a  $BV$  function satisfying (1.12)–(1.14) for all  $t, s > 0$ . By construction, it is easy to check that the limit function  $u(t, x)$  is an entropy solution to (1.1)–(1.2).

### 3. PROOF OF THEOREM 1.2

In this section, we prove the uniqueness of entropy solutions in  $BV$  as stated in Theorem 1.2. For any  $u \in BV$ , the whole space  $\mathbb{R}^2_+$  can be decomposed into three parts (see [5, 6, 17]):

$$\mathbb{R}^2_+ = J(u) \cup C(u) \cup I(u),$$

where  $J(u)$  is the set of points of approximate jump discontinuity,  $C(u)$  is the set of points of approximate continuity of  $u$ , and  $I(u)$  is the set of irregular points of  $u$  whose one-dimensional Hausdorff measure is zero.

First, the entropy inequality (1.9) implies that, on a shock  $x = x(t)$  in  $J(u)$ ,

$$(3.1) \quad \sigma[\eta(u)] - [q(u)] \geq 0,$$

where  $[\eta(u)] = \eta(u(t, x(t) + 0)) - \eta(u(t, x(t) - 0))$  and  $\sigma = x'(t)$  is the shock speed.

Now assume that  $u, v \in BV(\mathbb{R}^2_+)$  are the entropy solutions with initial data  $u_0, v_0 \in BV(\mathbb{R})$ , respectively. Then it can be easily checked that, on  $J(u) \cup J(v)$ ,

$$\sigma[|u - v|] - [\text{sign}(u - v)(f(u) - f(v))] \leq 0.$$

In the continuous region  $C(u) \cap C(v)$ , since  $r'(t) \leq 0$ ,

$$\begin{aligned} \mu(t, x) &:= |u(t) - v(t)|_{t+} (\text{sign}(u(t) - v(t))(f(u(t)) - f(v(t))))_x + r(0) |u(t) - v(t)| \\ &\quad - r(t) |u_0 - v_0| + \int_0^t r'(t - \tau) |u(\tau) - v(\tau)| d\tau \\ &= - \int_0^t |r'(t - \tau)| (|u(\tau) - v(\tau)| - \text{sgn}(u(t) - v(t))(u(\tau) - v(\tau))) d\tau \leq 0. \end{aligned}$$

Therefore,  $\mu$  as a measure on  $\mathbb{R}^2_+$  satisfies

$$\mu(\mathbb{R}^2_+) = - \sum_{J(u) \cup J(v)} (\sigma[|u - v|] - [\text{sign}(u - v)(f(u) - f(v))]) + \mu(C(u) \cap C(v)) \leq 0.$$

Then we follow the same steps as for the  $BV$  estimates in Section 2.2 to conclude (1.15). When  $u_0 \in L^\infty$ , let  $u_0^k$  be a sequence of initial data in  $BV$  for which  $u_0^k \rightarrow u_0$  as  $k \rightarrow \infty$ . Then the  $L^1$ -stability result (1.15) implies that the corresponding entropy solution sequence  $u^k \in BV$  to (1.1) with data  $u_0^k$  is a Cauchy sequence in



$L^1$  which yields a subsequence converging to  $u(t, x) \in L^\infty$ . It is easy to check that the limit  $u(t, x)$  is an entropy solution.

#### 4. PROOF OF THEOREM 1.3

Let  $u^\varepsilon \in BV$  denote the unique entropy solution to (1.18) with initial data  $u_0 \in BV$ . Then the solution sequence  $\{u^\varepsilon\}$  is uniformly bounded and is uniformly stable in  $L^1$  with respect to the initial data since

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+^2)} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} \leq L \|u_0 - v_0\|_{L^1}$$

and it satisfies the following a priori uniform bounds:

$$(4.1) \quad TV\{u^\varepsilon(t)\} \leq CL, \quad \|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1(\mathbb{R})} \leq C|t - s|.$$

This implies that there exists a convergent subsequence  $\{u^{\varepsilon_m}\}$  with  $\varepsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ , whose limit is denoted by  $u$ , i.e.,  $u^{\varepsilon_m}(t, x) \rightarrow u(t, x)$  in  $L^1_{loc}$ . Then, since  $k_\varepsilon(t) \rightarrow (\alpha - 1)\delta(t)$  weakly as measures when  $\varepsilon \rightarrow 0$ , we conclude that  $u$  is an admissible weak solution of the Cauchy problem (1.19) and (1.2). The proof of Theorem 1.3 is complete.

Finally we discuss some families of kernels  $\{k_\varepsilon\}$  that satisfy the assumptions stated in Theorem 1.3.

Suppose that  $k_\varepsilon \in L^1(\mathbb{R}_+)$  for all  $\varepsilon > 0$ . Then, by the Paley-Wiener Theorem [14], the resolvent  $r_\varepsilon$  of  $k_\varepsilon$  is in  $L^1(\mathbb{R}_+)$  if and only if

$$|1 + \hat{k}_\varepsilon(z)| \neq 0 \quad \text{for all } \operatorname{Re}(z) \geq 0$$

for the Laplace transform  $\hat{k}_\varepsilon$  of  $k_\varepsilon$ . By extending  $k_\varepsilon$  as zero to the negative real axis, we choose the number  $q$  such that

$$q \geq \sup_{\omega \in \mathbb{R}} |(1 + \hat{k}_\varepsilon(i\omega))^{-1}|,$$

and we choose positive numbers  $T$  and  $\eta$  satisfying

$$\int_{|s| \geq T} |k_\varepsilon(t)| dt \leq \frac{1}{12q}, \quad \sup_{0 < s < \eta} \int_{-\infty}^{\infty} |k_\varepsilon(t) - k_\varepsilon(t - s)| dt \leq \frac{1}{4}.$$

Then

$$(4.2) \quad \|r_\varepsilon\|_{L^1(\mathbb{R}_+)} \leq (8[6qT\|k_\varepsilon\|_{L^1(\mathbb{R}_+)}][8\|k_\varepsilon\|_{L^1(\mathbb{R}_+)}/\eta] + 6)q\|k_\varepsilon\|_{L^1(\mathbb{R}_+)},$$

where  $\lceil s \rceil$  denotes the smallest integer  $\geq s$ .

With this, for each  $\varepsilon$ , we can take  $k_\varepsilon \in L^1$  such that  $|1 + \hat{k}_\varepsilon(z)| \neq 0$  for all  $\operatorname{Re}(z) \geq 0$  and we take the numbers defined above to be  $q$  independent of  $\varepsilon$ ,  $T \sim \varepsilon$  and  $\eta \sim \varepsilon$ . Then, by (4.2),  $r_\varepsilon$  is uniformly bounded in  $L^1(\mathbb{R})$ . Furthermore, any kernel  $k_\varepsilon(t)$  in the set of kernels (i)–(vi) satisfies the assumptions in Theorem 1.2.

A prototype is the family of kernels  $k_\varepsilon(t)$  in (1.20) that satisfies these assumptions. Then the corresponding family of resolvent kernels is  $r_\varepsilon(t)$  in (1.21) which fulfills the assumptions of Theorem 1.3 when  $0 < \alpha < 1$ . For this example, the scalar nonlocal equation (1.1), i.e.,

$$u_t + f(u)_x - \frac{1 - \alpha}{\varepsilon} \int_0^t e^{-\frac{t-\tau}{\varepsilon}} f(u(\tau))_x d\tau = 0,$$

can also be written as a system of two equations:

$$(4.3) \quad \begin{cases} u_t + (f(u) - v)_x = 0, \\ v_t = \frac{(1 - \alpha)f(u) - v}{\varepsilon}. \end{cases}$$

Then the range of  $\alpha \in (0, 1)$  is the sub-characteristic condition. Thus, the result of Theorem 1.3 applied to this special case is equivalent to establishing the convergence of the relaxation limit (4.3) as considered in [2, 9, 16, 18].

*Remark 4.1.* In order to obtain that the resolvent  $r_\varepsilon$  of  $k_\varepsilon$  is integrable and  $\|r_\varepsilon\|_{L^1(\mathbb{R}_+)} \leq 20$ , it also suffices by the Shea-Wainger Theorem [7] to require that a family of kernels  $\{k_\varepsilon\}$  satisfy that, for each  $\varepsilon > 0$ ,  $k_\varepsilon \in L^1_{loc}(\mathbb{R}_+)$  and is nonnegative, nonincreasing, and convex on  $\mathbb{R}_+$ .

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#### REFERENCES

- [1] G.-Q. Chen and C. M. Dafermos, Global solutions in  $L^\infty$  for a system of conservation laws of viscoelastic materials with memory, *J. Partial Diff. Eqs.* **10** (1997), 369–383. MR1486717 (99b:35131)
- [2] G.-Q. Chen, D. Levermore, and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* **47** (1994), 787–830. MR1280989 (95h:35133)
- [3] C. M. Dafermos, Development of singularities in the motion of materials with fading memory, *Arch. Rational Mech. Anal.* **91** (1986), 193–205. MR806001 (87a:73033)
- [4] C. M. Dafermos, Solutions in  $L^\infty$  for a conservation law with memory, *Analyse Mathématique et Applications*, Gauthier-Villars, Paris, 1988, 117–128. MR956955 (89j:35083)
- [5] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Second Edition, Springer-Verlag, Berlin, 2005. MR2169977 (2006d:35159)
- [6] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969. MR0257325 (41:1976)
- [7] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge, Cambridge University Press, 1990. MR1050319 (91c:45003)
- [8] S. Kruzkov, First-order quasilinear equations with several space variables, *Mat. Sbornik*, **123** (1970), 228–255; *Math. USSR Sbornik*, **10** (1970), 217–273 (in English). MR0267257 (42:2159)
- [9] T. Luo and R. Natalini, *BV solutions and relaxation limit for a model in viscoelasticity*, *Proc. Roy. Soc. Edinburgh*, **128A** (1998), 775–795. MR1635428 (99e:35218)
- [10] R. Malek-Madani and A. J. Nohel, Formation of singularities for a conservation law with memory, *SIAM J. Math. Anal.* **16** (1985), 530–540. MR783978 (86e:35090)
- [11] R. C. MacCamy, A model for one-dimensional, nonlinear viscoelasticity, *Quart. Appl. Math.* **35** (1977), 21–33. MR0478939 (57:18395)
- [12] J. A. Nohel, R. C. Rogers, and A. E. Tzavaras, Weak solutions for a nonlinear system in viscoelasticity, *Commun. Partial Diff. Eqs.* **13** (1988), 309–322. MR914816 (89h:35063)
- [13] O. A. Oleinik, Discontinuous solutions of non-linear differential equations. *Usp. Mat. Nauk* **12** (1957), 3–73; *AMS Translations, Ser. II*, **26**, 95–172 (in English). MR0094541 (20:1055)
- [14] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc., Providence, RI, 1934. MR1451142 (98a:01023)
- [15] M. Renardy, W. Hrusa and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Longman, New York, 1987. MR919738 (89b:35134)
- [16] W. Shen, A. Tveito, and R. Winther, On the zero relaxation limit for a system modeling the motions of a viscoelastic solid, *SIAM J. Math. Anal.* **30** (1999), 1115–1135. MR1709789 (2000f:74032)
- [17] A. I. Vol'pert, *BV space and quasilinear equations (Russian)*, *Mat. Sb. (N.S.)*, **73(115)**, 1967, 255–302. MR0216338 (35:7172)

- [18] W.-A. Yong, A difference scheme for a stiff system of conservation laws, Proc. Roy. Soc. Edinburgh, **128A** (1998), 1403–1414.
- [19] C. Christoforou, Systems of conservation laws with fading memory, J. Hyper. Diff. Eqs. **4** (2007), 435–478.

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