EXTREMAL PSEUDOCOMPACT ABELIAN GROUPS
ARE COMPACT METRIZABLE

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Abstract. Every pseudocompact Abelian group of uncountable weight has both a proper dense pseudocompact subgroup and a strictly finer pseudocompact group topology.

1. Introduction

All topological groups here are assumed to satisfy the Hausdorff separation axiom. A pseudocompact group $G$ is said to be $r$-extremal [resp. $s$-extremal] if $G$ admits no strictly finer pseudocompact group topology [resp. $G$ has no proper dense pseudocompact subgroup]. Early formulations of these notions appeared in [6], [7]. From the fact that a pseudocompact space of countable weight is compact and metrizable it follows readily (as in [7, 2.3]) that every pseudocompact group of countable weight is both $r$-extremal and $s$-extremal. It is natural to ask whether there are extremal pseudocompact groups of uncountable weight. This question has generated much attention during the last two decades. See [1], [12] and [10] for more information. An affirmative answer was given in [7] for zero-dimensional Abelian groups. In [2] it was shown that no pseudocompact Abelian group of cardinality greater than $c$ is $s$-extremal. For partial answers in the class of connected groups, see for example [3], [12] and [1].

The aim of this paper is to answer the question for Abelian groups.

Theorem 1.1. A pseudocompact Abelian group of uncountable weight is neither $r$-extremal nor $s$-extremal.

We keep this presentation short by invoking several essential results established in the literature. We plan in [5] to present a polished, complete and self-contained proof of Theorem 1.1.

We announced our results at the annual meeting of the American Mathematical Society in January, 2006 [4].

2. Preliminaries

In this section we fix notation and we cite the results we need from the literature.

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The symbol \( wX \) denotes the weight of a topological space \( X \). A subspace of a space \( X \) is \( G_δ \)-dense in \( X \) if it meets every nonempty \( G_δ \)-subset of \( X \). If \( X \) is a set and \( \kappa \) a cardinal number, then \( |X|^\leq \kappa \) denotes \( \{ A \subseteq X : |A| \leq \kappa \} \).

For Abelian groups we use additive notation. Let \( G \) be an Abelian group. If \( A \subseteq G \), then \( \langle A \rangle \) denotes the subgroup of \( G \) generated by \( A \). A subset \( X \) of \( G \) is called independent if for every \( x \in X \) we have \( \langle \{x\} \rangle \cap \langle X \setminus \{x\} \rangle = \{0\} \). If \( A \) is a subgroup of \( G \), then a subset \( X \) of \( G \) is said to be independent over \( A \) if it is independent and \( \langle X \rangle \cap A = \{0\} \). The cardinality of a maximal independent set of elements of infinite order is called the torsion-free rank of \( G \), here denoted \( r_0(G) \). It is known that \( r_0(G) \) is an invariant of \( G \), i.e., all such maximal independent subsets of \( G \) have the same cardinality. It is clear that if \( h : G \to H \) is a surjective homomorphism, then \( r_0(H) \leq r_0(G) \). See [11] pp. 85-86 for additional details.

The torsion subgroup of an Abelian group \( G \) is denoted by \( tG \).

If \( G \) is a (not necessarily Abelian) totally bounded group, then \( \overline{G} \) denotes its (compact) Weil completion. It was shown in [5] that a topological group \( G \) is pseudocompact if and only if it is \( G_δ \)-dense in \( G \). Hence a dense subgroup of a pseudocompact group \( G \) is pseudocompact if and only if it is \( G_δ \)-dense in \( G \).

Let \( G \) be a topological group. Then

\[
\Lambda(G) = \{ N \subseteq G : N \text{ is a closed, normal, } G_δ\text{-subgroup of } G \}.
\]

Now we collect some information needed later in our proof of the main result.

**Theorem 2.1.** Let \( G \) be a pseudocompact group such that \( wG > \omega \), and let \( N \in \Lambda(G) \). Then

(a) [9, 3.3] \( G/N \) is compact and metrizable,
(b) [11, 6.2] \( N \) is pseudocompact, and
(c) [9, 2.7] \( wN = wG \).

**Lemma 2.2** ([3, 2.13(b),(c)]). Let \( G \) be a pseudocompact group and let \( G = \bigcup_{n<\omega} A_n \), where each \( A_n \) is a subgroup of \( G \). Then there exist \( N \in \Lambda(G) \) and \( n < \omega \) such that \( A_n \cap N \) is \( G_δ \)-dense in \( N \).

**Theorem 2.3** ([11, 4.4], [12, 3.7.1]). Let \( G \) be a pseudocompact Abelian group. If \( G \) contains a proper, dense pseudocompact subgroup \( H \) such that \( G/H \) can be mapped homomorphically onto some nondegenerate compact group, then \( G \) is not \( r \)-extremal.

**Theorem 2.4** ([11, 5.7], [12, 6.4.2]). Let \( G \) be a pseudocompact Abelian group of uncountable weight. If there exists \( N \in \Lambda(G) \) such that no connected \( M \in \Lambda(G) \) is contained in \( N \), then \( G \) is neither \( r \)-extremal nor \( s \)-extremal.

**Theorem 2.5** ([2, 4.5], [11, 5.10], [12, 7.3]). Let \( G \) be a pseudocompact Abelian group of uncountable weight such that \( r_0(G) > c \). Then \( G \) is neither \( r \)-extremal nor \( s \)-extremal.

### 3. Lemmas

In this section we collect some simple results to be used later. The technique used in the proof of Lemma 3.1 is well-known, and was used in many earlier results. See e.g., [3, 2, 12, 1]. For the benefit of the reader we provide the (simple) details. (Note added September 15, 2006. The referee has pointed out that a proof of Lemma 3.1 is also available in the preprint [10].)
Lemma 3.1. Let $G$ be a pseudocompact Abelian group, and let $A$ be a $G_δ$-dense subgroup of some $N \in \Lambda(G)$ such that $r_0(N/A) \geq \kappa$. Then $G$ contains a $G_δ$-dense subgroup $H$ such that $r_0(G/H) \geq \kappa$.

Proof. The conditions imply that there is a subset $X$ of $N \setminus A$ of elements of infinite order such that $|X| = \kappa$ and $X$ is independent over $A$. Split $X$ into two disjoint sets $X_0$ and $X_1$, each of cardinality $\kappa$.

By Theorem 2.1(a), the number of cosets of $N$ in $G$ is at most $\kappa$. (In fact, either $|G/N| < \omega$ or $|G/N| = \kappa$.) Let $\{a_\alpha + N : \alpha < \lambda\}$ be a faithful enumeration of $G/N$. We assume without loss of generality that $a_0 = 0$. By recursion on $\alpha < \lambda$ we will choose $x_\alpha \in X_0 \cup \{0\}$ such that

$$\langle\langle x_\alpha \rangle\rangle \subseteq \langle\langle a_\beta + x_\beta : \beta \leq \alpha \rangle\rangle + A = \{0\}.$$ 

Let $x_0 = 0$. Let $\alpha < \lambda$ and suppose that $x_\beta$ has been defined for all $\beta < \alpha$. Put $B_\alpha = \langle\langle a_\beta + x_\beta : \beta < \alpha \rangle\rangle$. Then $|B_\alpha| < \kappa$ and $\langle\langle X_1 \rangle\rangle \cap (B_\alpha + A) = \{0\}$. Suppose that for every $x \in X_0$ we have that

$$\langle\langle X_1 \rangle\rangle \cap (\langle\langle B_\alpha \cup \{a_\alpha + x\} \rangle\rangle + A) \neq \{0\}.$$ 

Then for every $x \in X_0$ there exist $b_x \in B_\alpha$, $n_x \in \mathbb{Z}$, $p_x \in A$ and $q_x \in \langle\langle X_1 \rangle\rangle \setminus \{0\}$ such that

$$(\dagger) \quad q_x = b_x + n_x(a_\alpha + x) + p_x.$$ 

Note (since $q_x \notin B_\alpha + A$) that no $n_x$ is equal to 0. Since $|X_0| = \kappa$, there are distinct $x, y \in X_0$, $n \in \mathbb{Z} \setminus \{0\}$ and $b \in B_\alpha$ such that $n = n_x = n_y$ and $b = b_x = b_y$. But then by subtracting the equation $(\dagger)$ for $x$ and $y$, we get

$$n(x - y) = q_x - q_y + p_y - p_x \in \langle\langle X_1 \rangle\rangle + A,$$

which contradicts the independence of $X$ over $A$. This completes the transfinite recursion.

Now put $B = \bigcup_{\alpha < \lambda} B_\alpha$. Then $\langle\langle X_1 \rangle\rangle \cap (B + A) = \{0\}$, hence $r_0(G/(B + A)) \geq |X_1| = \kappa$. It is clear that $B + A$ is $G_δ$-dense in $G$. □

Lemma 3.2. Let $\kappa$ be an infinite cardinal. Suppose that $A$ is a family of subsets of $2^\kappa$ with the following properties:

1. If $B \subseteq [A]^{<\kappa}$, then $\bigcap B \in A$, and
2. each element of $A$ has cardinality $2^\kappa$.

Then there is a countably infinite family $B$ of subsets of $2^\kappa$ such that

(i) $B$ is pairwise disjoint, and
(ii) if $A \in A$ and $B \in B$, then $|A \cap B| = 2^\kappa$.

Proof. We give $2^\kappa$ the standard Tychonov product topology. Let $\mathcal{V}$ be the collection of all nonempty clopen subsets $V$ of $2^\kappa$ for which there is an element $A(V) \in A$ such that $|V \cap A(V)| < 2^\kappa$. Clearly, $|\mathcal{V}| \leq \kappa$. Let $\mathcal{D} = \{A(V) : V \in \mathcal{V}\}$, $Y = \bigcap \mathcal{D}$, and $\tilde{V} = \bigcup \mathcal{V}$. Now $|V \cap Y| \leq |V \cap A(V)| < 2^\kappa$ for every $V \in \mathcal{V}$, so

$$|\tilde{V} \cap Y| < 2^\kappa$$

since $2^\kappa$ has cofinality at least $\kappa^+$. Then $|Y| = 2^\kappa$ by (1) and (2), hence $|Y \setminus \tilde{V}| = 2^\kappa$. There is consequently a countably infinite pairwise disjoint family $B$ of clopen subsets of $2^\kappa$ such that $B \cap (Y \setminus \tilde{V}) \neq \emptyset$ for every $B \in B$. To see that $B$ is as
required, pick arbitrary \( B \in \mathcal{B} \) and \( A \in \mathcal{A} \). If \( |B \cap A| < 2^\kappa \), then \( B \in \mathcal{V} \) and hence \( B \subseteq \widetilde{V} \), which contradicts the fact that \( B \cap (Y \setminus \widetilde{V}) \neq \emptyset \). \( \square \)

**Remark 3.3.** The inclusion \( \bigcup B \subseteq 2^\kappa \) is necessarily proper (since \( 2^\kappa \) is compact). Replacing any one element of \( \mathcal{B} \) by the complement in \( 2^\kappa \) of the union of the remaining elements, we may hence assume without loss of generality that \( \mathcal{B} \) is a partition.

4. **Proof of Theorem 1.1**

We now present the proof of our main result. By Theorem 2.5, it suffices to consider groups \( G \) of torsion-free rank at most \( \kappa \). Furthermore, by Theorem 2.4 we may assume that every \( N \in \Lambda(G) \) contains a connected \( M \in \Lambda(G) \). Henceforth, let \( G \) be a pseudocompact Abelian group of uncountable weight satisfying those two conditions.

**Lemma 4.1.** If \( H \) is a nontrivial connected subgroup of \( G \), then \( r_0(H) = \kappa \).

**Proof.** It is clear that \( r_0(H) \leq \kappa \). Let \( 0 \neq x \in H \) and let \( h \) be a continuous homomorphism from \( H \) to \( T \) such that \( h(x) \neq h(0) \). Then \( h(H) = T \) since \( H \) (and hence \( h(H) \)) is connected. It follows that \( \kappa \geq r_0(H) \geq r_0(T) = \kappa \), as asserted. \( \square \)

Since \( G \in \Lambda(G) \), there is a connected \( C \in \Lambda(G) \). Hence \( r_0(C) = \kappa \) by Theorem 2.1(b),(c) and Lemma 4.1.

Let \( \overline{C} \) be the closure of \( C \) in \( G \). Then \( \overline{C} \) is a compact, connected group, hence is divisible [13 Theorem 24.25]. By [13 Theorem A.14] or [11 Theorem 23.1], there is a cardinal number \( \lambda \) such that \( \overline{C} \) is (algebraically) isomorphic to

\[
\bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha \oplus t\overline{C},
\]

where each \( \mathbb{Q}_\alpha \) is a copy of the group of rational numbers \( \mathbb{Q} \). Then \( \overline{C}/t\overline{C} = \bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha \). Let \( \pi : \overline{C} \to \bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha \) be the natural homomorphism. For \( x \in \overline{C} \) let \( S(x) = \{ \alpha < \lambda : \pi(x)_\alpha \neq 0_\alpha \} \), and for \( E \subseteq \overline{C} \) let \( S(E) = \bigcup_{x \in E} S(x) \).

**Lemma 4.2.** If \( N \in \Lambda(C) \), then \( |S(\pi(N))| = \kappa \).

**Proof.** Since \( N \in \Lambda(G) \), we may assume without loss of generality that \( N \) is connected. Moreover, \( N \) is nontrivial by Theorem 2.1(c). So \( r_0(N) = \kappa \) by Lemma 4.1. That \( |S(\pi(N))| = \kappa \) is then clear. \( \square \)

Writing \( S = S(\pi(C)) \), we have \( |S| = \kappa \). Hence \( \lambda \geq \kappa \), and

\[
C \subseteq \bigoplus_{\alpha \in S} \mathbb{Q}_\alpha \oplus \bigoplus_{\alpha \notin S} \{0_\alpha\} \oplus t\overline{C}.
\]

For every \( \beta \in S \), let \( \rho_\beta : \bigoplus_{\alpha \in S} \mathbb{Q}_\alpha \to \mathbb{Q}_\beta \) be the projection.

For every nonempty \( A \subseteq S \), put

\[
C(G(A)) = C \cap \left( \bigoplus_{\alpha \in A} \mathbb{Q}_\alpha \oplus \bigoplus_{\alpha \notin A} \{0_\alpha\} \oplus t\overline{C} \right),
\]

and let

\[
\mathcal{A} = \{ A \subseteq S : \text{there is } N \in \Lambda(C) \text{ such that } N \subseteq G(A) \}.
\]

**Lemma 4.3.** \( \mathcal{A} \) is closed under countable intersections, and every \( A \in \mathcal{A} \) has size \( \kappa \).
Proof. That $A$ is closed under countable intersections is clear, since if $B$ is any family of subsets of $S$, then
\[ \bigcap_{B \in \mathcal{B}} G(B) = G(\bigcap \mathcal{B}) \]
and $\Lambda(G)$ is closed under countable intersections.

Now take an arbitrary $A \in A$. We want to prove that $|A| = c$. Take $N \in \Lambda(C)$ such that $N \subseteq G(A)$. Then $\pi(N) \subseteq A$, so $c = |S| \geq |A| \geq |\pi(N)| = c$ by Lemma 4.2.

By Lemma 3.2 and Remark 3.3, there consequently is a (faithfully indexed) partition $\mathcal{B} = \{B_n : n < \omega\}$ of $S$ such that $|B_n \cap A| = c$ for each $B_n \in \mathcal{B}$, $A \in A$.

For every $n < \omega$, let
\[ V_n = G\left( \bigcup_{i \leq n} B_i \right). \]
Then $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots$, and $C = \bigcup_{n < \omega} V_n$. By Theorem 2.1(b) and Lemma 2.2, there exist $N \in \Lambda(C)$ and $m < \omega$ such that $H := V_m \cap N$ is $G_\delta$-dense in $N$. We may assume without loss of generality that $m = 0$, i.e., that $V_0 = V = G(B_0)$.

Lemma 4.4. $r_0(N/H) \geq c$.

Proof. We will prove that there is a subset $X$ of $N$ of cardinality $c$ such that

1. each $x \in X$ has infinite order,
2. $X$ is independent,
3. $\langle X \rangle \cap H = \{0\}$

(hence $\langle X \rangle$ is isomorphic to $\bigoplus_{\alpha < c} \mathbb{Z}_\alpha$, where each $\mathbb{Z}_\alpha$ is a copy of the group of integers $\mathbb{Z}$). Choose $x_0 \in N \setminus G(B_0)$ and define $W_0 = B_0$. Let $0 < \alpha < c$ and suppose that $x_\beta$ and $W_\beta$ have been defined for all $\beta < \alpha$. Then, set
\[ W_\alpha = B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta), \]
and observe that
\[ |B_1 \cap W_\alpha| = \left| B_1 \cap \bigcup_{\beta < \alpha} S(x_\beta) \right| < c. \]

Hence $W_\alpha \not\subseteq A$, since $B_1$ meets every element of $A$ in a set of size $c$, which means that $N \not\subseteq G(W_\alpha)$; let $x_\alpha$ be any point in $N \setminus G(W_\alpha)$. This completes the transfinite construction.

We claim that $X = \{x_\alpha : \alpha < c\}$ satisfies (1), (2) and (3). To prove this, let $\alpha < c$, and let $n \in \mathbb{Z} \setminus \{0\}$ be arbitrary. By construction we have
\[ x_\alpha \not\in G\left( B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta) \right), \]
so $S(x_\alpha) \not\subseteq B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta)$; let $\gamma \in S(x_\alpha)$ witness that relation. Then $\rho_\gamma(\pi(x_\alpha)) \neq 0$ and $\rho_\gamma(\pi(x_\beta + h)) = 0$ for every $\beta < \alpha$ and $h \in H$. Then clearly $x_\alpha$ has infinite order, and
\[ nx_\alpha \not\in \langle \{x_\beta : \beta < \alpha\} \rangle + H, \]
as required. \qed
Now we complete the proof of Theorem 1.1. From Lemmas 3.1 and 4.4 we conclude that $G$ has a proper dense pseudocompact subgroup $H$ such that $r_0(G/H) \geq c$. This proves that $G$ is not $s$-extremal. To see that $G$ is not $r$-extremal, note first that $r_0(G/H) \geq c$, so $G/H$ contains a subgroup isomorphic to $\bigoplus_{\alpha < \kappa} \mathbb{Z}_\alpha$. The latter can be mapped homomorphically onto $T$, and since homomorphisms into a divisible group always extend by [13, Theorem A.7], we are done by Theorem 2.3.

References