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COMPACT QUANTUM GROUP ACTIONS ON C*-ALGEBRAS AND INVARIANT DERIVATIONS

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Abstract. We define the notion of invariant derivation of a C*-algebra under a compact quantum group action and prove that in certain conditions, such derivations are generators of one parameter automorphism groups.

1. Introduction

The investigation of unbounded derivations started with the seminal work of Sakai in [9] and was motivated by problems of mathematical physics and in particular the problem of the construction of dynamics in statistical mechanics.

The principal interest of symmetric derivations and the main motivation for their analysis arise from the fact that they occur as the generators of one parameter groups of automorphisms. In particular, unbounded derivations commuting with group actions and their generator property have been considered by several authors (see [5] and the references therein). These papers were motivated by the problem of invariance with respect to symmetry groups in statistical mechanics. The subject is also inspired by Connes' work on non-commutative geometry.

The recent developments in the theory of Hopf algebras and quantum groups, as a new symmetry between quantum observables and quantum states, provide a new impetus for future research if symmetric derivations are invariant with respect to compact quantum group actions. Our results contained in Theorem 3.8 and Theorem 3.10 show that, under certain conditions, symmetric, unbounded derivations which are invariant with respect to compact quantum group actions are generators.

Let $G = (A, \Delta)$ be a compact quantum group (see [12]) and δ be an action of G on a C*-algebra B. This paper is organized as follows: In Section 2 we collect some preliminary results about the spectral subspaces of such actions. In Section 3 we define the notion of invariant derivation under a compact quantum group action and prove that such derivations are generators of one-parameter automorphism groups if their domain contains all the spectral subspaces (Theorem 3.8) or, if $B = \mathcal{K}(H)$, if the domain contains the fixed point algebra B^δ (Theorem 3.10). Our results extend and improve on results obtained in [5], [6], [7] for the case of groups.

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2. Compact quantum group actions on C*-algebras

Let $G = (A, \Delta)$ be a compact quantum group, i.e. a unital C*-algebra A and $\Delta : A \to A \otimes A$ a $*$ - homomorphism such that

(i) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, where ι is the identity map

and

(ii) $\overline{\Delta(A)(1 \otimes A)} = A \otimes A$ and $\overline{\Delta(A)(A \otimes 1)} = A \otimes A$.

Let $\widehat{G}$ denote the dual of G, i.e. the set of all unitary equivalence classes of irreducible representations of G ([11], [12]). For each $\alpha \in \widehat{G}$, denote by u^α a representative of each class. Let $\{u^\alpha_{ij}\}_{i,j=\overline{1,d_\alpha}} \subset A$ be the matrix elements of u^α.

Set $\chi_\alpha = \sum_{i=1}^{d_\alpha} u^\alpha_{ii}$. Then χ_α is called the character of α. By ([11], [12]) there is a unique invertible operator $F_\alpha \in B(H_\alpha)$, where H_α is the finite dimensional Hilbert space of the representation u^α that intertwines u^α with its double contragradient representation $(u^\alpha)^{cc}$ such that $tr(F_\alpha) = tr(F_\alpha^{-1})$. Set $M_\alpha = tr(F_\alpha)$.

F_α can be represented as a matrix $F_\alpha = [f_1(u^\alpha_{ij})]$ where f_1 is a linear functional on the dense $*$-subalgebra $\mathcal{A} \subset A$ spanned (linearly) by $\{u^\alpha_{ij}\}_{\alpha \in \widehat{G}, i,j}$.

If $a \in A$ (or $a \in \mathcal{A}$) and ξ is a linear functional (on A or $\mathcal{A}$), we denote ([11], [12])

$$a * \xi = (\xi \otimes \iota)(\Delta(a)),$$
$$\xi * a = (\iota \otimes \xi)(\Delta(a)).$$

Both $a * \xi$ and $\xi * a$ are elements of A (respectively $\mathcal{A}$).

If h is the Haar measure on A ([11]), set

$$h_\alpha = M_\alpha h \cdot (\chi_\alpha * f_1)^*,$$

where $(h \cdot a)(b) = h(ab)$.

Now let B be a C*-algebra and $\delta : B \to B \otimes A$ be a $*$- homomorphism of B into the multiplier algebra of the minimal tensor product $B \otimes A$. If:

(a) $(\iota \otimes \Delta)\delta = (\delta \otimes \iota)\delta$ and

(b) $\overline{\delta(B)(1 \otimes A)} = B \otimes A$, then δ is called an action of G on B, or a coaction of A on B.

2.1. *Remark.* The condition (b) above implies that $\delta \in Mor(B, B \otimes A)$ in the sense of [12], Introduction. Then, by the discussion in [12], δ can be uniquely extended to a $*$-homomorphism $\delta'' : \mathcal{M}(B) \to \mathcal{M}(B \otimes A)$.

Let $P_\alpha : B \to B$ be the linear map $P_\alpha(x) = (\iota \otimes h_\alpha)(\delta(x))$.

In particular, if $\alpha = \iota$ (the trivial one-dimensional representation), $P_\iota = (\iota \otimes h) \circ \delta$ is a conditional expectation from B to $B^\delta = \{x \in B \mid \delta(x) = x \otimes 1_A\}$ which will be called the fixed point algebra of the action.

2.2. **Definition.**

(1) For every $\alpha \in \widehat{G}$, denote
$$B_\alpha = \{P_\alpha(x) \mid x \in B\}.$$
The subspace $B_\alpha \subseteq B$ is called the spectral subspace corresponding to α.

(2) For $\alpha \in \widehat{G}$, $i, j = 1, ..., d_\alpha$, we denote
$$c^\alpha_{ij} = M_\alpha (u^\alpha_{ij} * f_1)^*$$
and
$$P^\alpha_{ij}(x) = (id \otimes h \cdot c^\alpha_{ij})(\delta(x)),\ x \in B.$$

We collect some properties of these objects in the following Lemma (see [3], [8], or easy calculations).

2.3. **Lemma.**

(i) $P^{\alpha}_{ij}P^{\beta}_{kl} = \delta_{il}\delta_{\alpha\beta}P^{\alpha}_{kj}$, *where* δ_{il}, $\delta_{\alpha\beta}$ *denote the Kronecker symbols.*
(ii) $P^{\alpha}P^{\alpha}_{ij} = P^{\alpha}_{ij}$.
(iii) *If* $x \in B_{\alpha}$, *then* $x = \sum_{i=1}^{d_{\alpha}} P^{\alpha}_{ii}(x)$.
(iv) *The algebraic direct sum* $\sum_{\alpha \in \widehat{G}} B_{\alpha}$ *is a dense* $*$*-subalgebra of* B.
(v) $\delta(P^{\alpha}_{ij}(x)) = \sum_{l=1}^{d_{\alpha}} P^{\alpha}_{il}(x) \otimes u^{\alpha}_{lj}$, $\forall x \in B$.

For $\alpha \in \widehat{G}$, we can consider a matricial spectral subspace:

$$B_2(\alpha) = \{[P^{\alpha}_{ij}(x)] \mid x \in B\}.$$

Then $B_2(\alpha) \subset B \otimes B(H_{\alpha})$.

By Lemma 2.3 (iii) the map $x \mapsto [P^{\alpha}_{ij}(x)]$ is a linear isomorphism between B_{α} and $B_2(\alpha)$.

2.4. **Definition.** If $\delta : B \to \mathcal{M}(B \otimes A)$ and $\theta : C \to \mathcal{M}(C \otimes A)$ are actions of G on B and C, denote by $\delta \textcircled{T} \theta$ the tensor product of the two actions, i.e.

$$(\delta \textcircled{T} \theta)(b \otimes c) = \delta(b)_{13}\theta(c)_{23}.$$

We used the leg-numbering notation (see for instance [1]). Straightforward calculations show that this is a coaction of A on $B \otimes C$.

In particular, if ι denotes the trivial action of G on $B(H_{\alpha})$, $\iota(x) = x \otimes 1$, $\forall x \in B(H_{\alpha})$, then $\tilde{\delta} = \delta \otimes \iota$ is an action of G on $B \otimes B(H_{\alpha})$.

2.5. *Remark.* For $x \in B_{\alpha}$, let $X = [P_{ij}(x)] \in B_2(\alpha)$. Then:

$$\tilde{\delta}(X) = (X \otimes 1_A)(1_B \otimes u^{\alpha}).$$

This follows from Lemma 2.3 (v).

2.6. **Lemma.** *B^{δ} is a C*-subalgebra of B and $B^{\delta} \neq (0)$.*

Proof. It is immediate that $B^{\delta}$is a C*-subalgebra.

We prove next that $B^{\delta} \neq (0)$. By Lemma 2.3 (iv) there is $\alpha \in \widehat{G}$ such that $B_{\alpha} \neq (0)$. Let $x \in B_{\alpha}$, $x \neq 0$. Let

$$X = [X_{ij}] \in B_2(\alpha)$$

as in Remark 2.5. Then $X \neq 0$. By Remark 2.5, we have

$$\tilde{\delta}(X) = (X \otimes 1_A)(1 \otimes u^{\alpha}).$$

Hence

$$\tilde{\delta}(XX^*) = XX^* \otimes 1_A.$$

Therefore each entry of XX^* belongs to B^{δ}. Since $X \neq 0$, it follows that $B^{\delta} \neq (0)$. $\square$

2.7. **Lemma.** *If $(e_{\lambda})_{\lambda}$ is an approximate identity of B^{δ}, then $(e_{\lambda})_{\lambda}$ is an approximate identity of B.*

Proof. Let $x \in B_\alpha$, for some $\alpha \in \widehat{G}$ and $X = [X_{ij}] \in B_2(\alpha)$ as in Remark 2.5. Then, as in the proof of the previous lemma, $XX^* \in (B \otimes B(H_\alpha))^{\tilde{\delta}}$.

For each λ, let $E_\lambda = [e_{ij}^\lambda]$ with $e_{ii}^\lambda = e_\lambda$ for every i and $e_{ii}^\lambda = 0$ if $i \neq j$.

Since $(e_\lambda)_\lambda$ is an approximate identity of B^δ and $XX^* \in (B \otimes B(H_\alpha))^{\tilde{\delta}}$, we obviously have:

$$\begin{aligned}\lim_\lambda \|E_\lambda X - X\| &= \lim_\lambda \|(E_\lambda X - X)(E_\lambda X - X)^*\|^{\frac{1}{2}} \\ &= \lim_\lambda \|E_\lambda XX^* E_\lambda - E_\lambda XX^* - XX^* E_\lambda + XX^*\|^{\frac{1}{2}} = 0.\end{aligned}$$

Since, by Lemma 2.3 (iii), $x = \sum X_{ii}$, it follows that $\lim_\lambda \|e_\lambda x - x\| = 0$. As $\overline{\sum_{\alpha \in \widehat{G}} B_\alpha} = B$ (Lemma 2.3 (iv)), the proof is complete. □

Since $B^\delta \neq (0)$, then the minimum dimension that B^δ can have is 1.

2.8. **Definition.** The action δ is called ergodic if $\dim B^\delta = 1$.

2.9. *Remark.* If δ is ergodic, then B has a unit. Indeed, by Lemma 2.6, B^δ is a one-dimensional C*-subalgebra of B and thus it has a unit. By Lemma 2.7 this is a unit of B also.

Interesting examples and results on ergodic actions can be found in [2], [3], [10].

3. Unbounded derivations commuting with quantum group actions

Let $d : D(d) \to B$ be a closed, densely defined $*$-derivation. Here $D(d)$ is a dense $*$-subalgebra of B.

As it is easy to show, $d \otimes i_A : D(d) \otimes A \to B \otimes A$ is a closable linear operator. Denote by $\overline{d \otimes i_A}$ its closure.

3.1. **Definition.** We say that d is δ-invariant (or that d commutes with δ) if for every $x \in D(d)$, it follows that:

$$a)\ \delta(x) \in D(\overline{d \otimes i_A}),$$
$$b)\ \delta(d(x)) = (\overline{d \otimes i_A})(\delta(x)),\ \forall x \in D(d).$$

In order to give an equivalent formulation of the concept of invariant derivation, we need the following:

3.2. **Definition.** If $\delta_1 : B_1 \to \mathcal{M}(B_1 \otimes A)$ and $\delta_2 : B_2 \to \mathcal{M}(B_2 \otimes A)$ are actions of $G = (A, \Delta)$ on B_1, respectively B_2, define the direct sum action of G on $B_1 \oplus B_2$ by

$$\delta_1 \oplus \delta_2 = (i_1 \otimes i_A) \circ \delta_1 \circ p_1 + (i_2 \otimes i_A) \circ \delta_2 \circ p_2$$

where $i_k : B_k \hookrightarrow B_1 \oplus B_2$ is the inclusion (k=1,2), $p_k : B_1 \oplus B_2 \to B_k$ is the projection (k=1,2) and $i_A : A \to A$ is the identity map.

Notice that $i_k \otimes i_A$ can be extended as maps from $\mathcal{M}(B_k \otimes A)$ to $\mathcal{M}((B_1 \oplus B_2) \otimes A)$.

If $d : D(d) \to B$ is a closed, densely defined, $*$-derivation, denote by Γ_d its graph, i.e. :

$$\Gamma_d = \{x \oplus d(x) \mid x \in D(d)\} \subseteq B \oplus B.$$

3.3. **Proposition.** *d is a δ-invariant derivation if and only if $(\delta \oplus \delta)(\Gamma_d) \subseteq \overline{\Gamma_d \odot A}$, where $\overline{\Gamma_d \odot A}$ is the closure of the algebraic tensor product $\Gamma_d \odot A$.*

Proof. Straightforward from definitions. □

3.4. **Lemma.** *If d is a δ-invariant derivation, then $P_\alpha(D(d)) \subseteq D(d)$ and $P_\alpha(D(d))$ are dense in B_α.*

Proof. Let $b \in D(d)$. Since d is δ-invariant,

$$(\delta \oplus \delta)(b \oplus d(b)) \in \overline{\Gamma_d \odot A}.$$

Therefore

$$(i \otimes h_\alpha)((\delta \oplus \delta)(b \oplus d(b))) \in \Gamma_d$$

But this means in particular that $P_\alpha(b) \in D(d)$. Then, since P_α are continuous maps, we get

$$B_\alpha = P_\alpha(B) = P_\alpha(\overline{D(d)}) \subseteq \overline{P_\alpha(D(d))} \subseteq B_\alpha.$$

Hence $P_\alpha(D(d))$ are dense in B_α. $\square$

3.5. **Assumption.** For the next results we will assume the following:
Either
a) Both δ and h are faithful, in which case, the conditional expectation $P_\iota : B \to B^\delta$, $P_\iota = (\iota \otimes h) \circ \delta$, is faithful,
or
b) B is a simple C*-algebra.

3.6. *Remark.* Either one of a) and b) implies that for every $x \in B$, $\|x\| = \sup_{\tilde\varphi} \|\pi_{\tilde\varphi}(x)\|$ where $\tilde\varphi$ runs through $\{\tilde\varphi = \varphi \circ P_\iota \mid \varphi \text{ is a state of } B^\delta\}$ and $\pi_{\tilde\varphi}$ is the GNS representation of B with respect to the state $\tilde\varphi$.

3.7. *Remark.* Assume that Assumption 3.5 holds. If $d : \sum_{\alpha \in \widehat{G}} B_\alpha \to B$ is a not necessarily closed, $*$-derivation that commutes with δ (i.e. $\delta(d(x)) = (d \otimes i)(\delta(x))$ for every $x \in \sum B_\alpha$), then d is closable and its closure is δ-invariant.

Proof. Since d commutes with δ, it follows in particular that $d \circ P_\iota = P_\iota \circ d$. Since $B_\iota = B^\delta \subset D(d)$, the restriction of d to B^δ is a bounded derivation. By our Assumption 3.5, the conditions of [5], Proposition 1.4.11, are satisfied and the conclusion follows. $\square$

Next we give our main results. For a comprehensive theory of generators we refer to [4] and [5].

3.8. **Theorem.** *Let (B, G, δ) be a quantum dynamical system satisfying Assumption 3.5. Let $d : \sum B_\alpha \to B$ be a densely defined $*$-derivation that commutes with δ in the sense of Remark 3.7. Then d is closable and its closure is the generator of a one parameter group of δ-invariant automorphisms of B.*

Proof. Notice first that if $x \in B_\alpha$ we have:

$$\begin{aligned} d(x) &= d((i \otimes h_\alpha)(\delta(x))) = (d \otimes h_\alpha)(\delta(x)) \\ &= (i \otimes h_\alpha)((\overline{d \otimes i})(\delta(x))) = (i \otimes h_\alpha)(\delta(d(x))). \end{aligned}$$

Hence $d(x) \in B_\alpha$. By Remark 3.7 d is closable and its closure, also denoted d, is δ-invariant.

Since B_α, $\alpha \in \widehat{G}$, are closed subspaces of B which are d-invariant, they consist of analytic elements for d, i.e. for every $x \in B_\alpha$, the series $\sum \frac{z^n}{n!} d^n(x)$ is absolutely convergent for every $z \in \mathbb{C}$. Therefore, by Lemma 2.3 (iv), d has a dense set of analytic elements.

We will check next that $\|(1+\alpha d)(x)\| \geqslant \|x\|$, for every $\alpha \in \mathbb{R}$, $x \in D(d)$ and then apply [4], Theorem 3.2.50, to conclude that d is a generator.

Let φ be a state of B^δ and $\widetilde{\varphi} = \varphi \circ P_\iota$ be the corresponding state of B. Let $(\pi_{\widetilde{\varphi}}, H_{\widetilde{\varphi}}, \xi_{\widetilde{\varphi}})$ be the associated GNS representation of B. Let $\pi = \bigoplus_\varphi \pi_{\widetilde{\varphi}}$, where φ is a state of B^δ. Then, by Assumption 3.5, π is a faithful representation of B on $H_\pi = \bigoplus_\varphi H_{\pi_{\widetilde{\varphi}}}$. We will identify B with $\pi(B) \subset B(H_\pi)$.

Since the restriction of d to B^δ is a bounded derivation, by Sakai's Theorem, its extension to the weak closure $\overline{B^\delta}^w$ is an inner derivation. Let $h_0 \in \overline{B^\delta}^w$ be a selfadjoint element such that $d|_{B^\delta} = ad(ih_0)|_{B^\delta}$. On the other hand, since, in particular, $B^\delta \subset D(d)$, by the von Neumann algebra version of [5], Proposition 1.4.11, it follows that d is σ-weakly closable on $\overline{B}^w$. Let $\overline{d}$ denote the σ-weak closure of d.

It is clear that $h_0 \in D(\overline{d})$ and $\overline{d}(h_0) = 0$. Therefore $\overline{d} \circ ad(ih_0) = ad(ih_0) \circ \overline{d}$.

Set $d_1 = \overline{d} - ad(ih_0)$. Then d_1 satisfies the conditions of [5], Corollary 1.5.6, and as in the proof of that corollary, there is a skew symmetric operator S on $\sum_\varphi D(d_1)\xi_{\widetilde{\varphi}}$ such that:

$$\begin{aligned} &1) \quad S(x\xi_{\widetilde{\varphi}}) = d_1(x)\xi_{\widetilde{\varphi}},\ \forall \varphi \text{ state on } B^\delta,\ \forall x \in D(d_1),\\ &2) \quad d_1(x) = Sx - xS,\ \forall x \in D(d_1). \end{aligned}$$

Therefore, $\overline{d} = ad(S + ih_0)$. Since $\sum_\alpha B_\alpha$ is a set of analytic elements of d, h_0 is bounded and $Sh_0 = h_0 S$, it follows that $\sum_\alpha B_\alpha \xi_{\widetilde{\varphi}}$ are analytic elements of $S + ih_0$ for every φ and therefore the skew symmetric operator $S + ih_0$ has a dense set of analytic elements. Indeed, if $x \in B_\alpha$, then

$$\begin{aligned} \|(S+ih_0)^n x\xi_{\widetilde{\varphi}}\| &= \|\sum_{k=1}^n \binom{n}{k} i^k (Adh_0)^k S^{n-k} x\xi_{\widetilde{\varphi}}\| \\ &= \|\sum_{k=1}^n \binom{n}{k} i^k (Adh_0)^k d_1^{n-k}(x)\xi_{\widetilde{\varphi}}\| \\ &\leqslant (2\|h_0\| + \rho)^n \|x\| \end{aligned}$$

where $\rho = \|d_1|_{B_\alpha}\| < \infty$. Therefore $S + ih_0 = iK$, where K is essentially selfadjoint.

Applying [4], Corollary 3.2.56, it follows that $\|(1+\alpha\overline{d})(x)\| \geqslant \|x\|$ for every $x \in D(\overline{d})$ and the proof is complete. □

3.9. **Corollary.** *If* (B, G, δ) *is an ergodic system satisfying Assumption* 3.5 *and* $d : D(d) \to B$ *is a* δ*-invariant* $*$*-derivation, then* d *is a generator.*

Proof. By [3], Theorem 17, the spectral subspaces B_α are finite dimensional. By Lemma 3.4, $P_\alpha(D(d)) \subseteq B_\alpha \cap D(d)$ and $P_\alpha(D(d))$ is dense in B_α. Therefore $P_\alpha(D(d)) = B_\alpha \subset D(d)$ and the conditions of the Theorem 3.8 are satisfied. □

Our next result refers to quantum group actions on the C*-algebra of compact operators $K(H) \subset B(H)$. This result is an extension of [6], Theorem 4.1. The condition we use is also slightly weaker than the tangential condition used in [6]. We do not require that $K(H)^\delta \subseteq ker(d)$.

3.10. **Theorem.** *Let δ be an action of $G = (A, \Delta)$ on $K(H)$. If $d : D(d) \to K(H)$ is a closed densely defined $*$-derivation such that:*

$$a)\ d \text{ is } \delta - invariant, \text{ and}$$
$$b)\ K(H)^{\delta} \subseteq D(d),$$

then δ is a generator.

Proof. We will check the conditions A1), B2) and C1) of [4], Theorem 3.2.50.

Condition A1) is satisfied by hypothesis. We will show next that d has a dense set of analytic elements (condition B2)).

Since $K(H)^{\delta} \subseteq D(d)$ and d is δ-invariant, it follows that d is a (bounded) derivation on $K(H)^{\delta}$. Since every bounded derivation is inner in the σ-closure, there exists a self-adjoint operator $h_0 \in \overline{K(H)^{\delta}}^{\sigma}$ such that $d|_{K(H)^{\delta}} = ad(ih_0)|_{K(H)^{\delta}}$.

Notice that, by [4], Corollary 3.2.27, d is σ-weakly closable on $B(H)$ and $\overline{d}(h_0) = 0$, where $\overline{d}$ denotes the σ-weak closure of d.

Consider the derivation $d_0 = ad(ih_0)$ on $B(H)$. Then $d_0(D(d)) \subseteq D(d)$. Indeed, let $x \in D(d)$ and $(e_\lambda)_\lambda$ be an approximate identity of $K(H)^{\delta}$. By Lemma 2.7, (e_λ) is an approximate identity of $K(H)$.

Then

$$x = norm - \lim_{\lambda}(e_\lambda x)$$

and

$$h_0 x = norm - \lim_{\lambda}(h_0 e_\lambda x).$$

This last equality follows since $h_0 e_\lambda \in K(H)$, $\sigma - lim h_0 e_\lambda = h_0$ and $x \in K(H)$.

By Remark 2.1, $\delta(h_0 e_\lambda) = \delta''(h_0)\delta(e_\lambda) = h_0 e_\lambda \otimes 1$ and hence $h_0 x \in K(H)^{\delta}$. Therefore $h_0 x$ (and $x h_0$) are elements of $D(d)$, so $d_0(D(d)) \subseteq D(d)$.

Moreover, since $\delta''(h_0) = h_0 \otimes 1$ it follows that d_0 is δ-invariant.

It is obvious that d commutes with d_0. Since d is closed and d_0 bounded, it follows that $d_1 = d - d_0$ is a closed, densely defined (on $D(d)$) δ-invariant derivation. Clearly $d_1|_{K(H)^{\delta}} = 0$.

Let $(e_\lambda)_\lambda$ be an approximate identity of $K(H)^{\delta}$ consisting of projections. Then, by Lemma 2.7 $(e_\lambda)_\lambda$ is an approximate identity of $K(H)$.

Since e_λ are finite dimensional, $e_\lambda K(H) e_\lambda$ are finite dimensional C*-subalgebras of $K(H)$. Further, since $d_1|_{K(H)^{\delta}} = 0$, it follows that $d_1(e_\lambda K(H) e_\lambda) \subseteq e_\lambda K(H) e_\lambda$, $\forall \lambda \in \Lambda$. Therefore $\bigcup_{\lambda \in \Lambda} e_\lambda K(H) e_\lambda$ is a dense set of analytic elements of d_1. Since d_0 commutes with d_1 and d_0 is bounded, it follows that d has a dense set of analytic elements. Indeed, if $x \in e_\lambda \mathcal{K}(H) e_\lambda$ we have

$$\begin{aligned} \|d^n(x)\| &= \|(d_0 + d_1)^n(x)\| \\ &= \|\sum_{k=1}^{n} \binom{n}{k} d_0^k d_1^{n-k}(x)\| \leqslant (2\|h_0\| + \rho)^n \|x\| \end{aligned}$$

where $\rho = \|d_1|_{e_\lambda \mathcal{K}(H) e_\lambda}\| < \infty$ (since e_λ is finite dimensional and $d_1(e_\lambda \mathcal{K}(H) e_\lambda) \subseteq e_\lambda \mathcal{K}(H) e_\lambda$).

By [5], Example 1.6.4, and its proof, there is a skew symmetric operator S such that $d_1 \subseteq ad(S)$ and S has a dense set of analytic elements. Therefore S is essentially skew adjoint.

Since h_0 is a bounded selfadjoint operator, we have that $S + ih_0$ is essentially skew adjoint.

Since $d \subseteq ad(S + ih_0)$, by [5], Corollary 1.5.6, it follows that condition C1) of [4], Theorem 3.2.50, is satisfied and we are done. $\square$

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