DEGREE ONE MAPS
BETWEEN HYPERBOLIC SMALL 3-MANIFOLDS

MICHEL BOILEAU AND SHICHENG WANG

(Communicated by Daniel Ruberman)

ABSTRACT. We construct the first examples of degree one maps between non-homeomorphic closed hyperbolic small 3-manifolds.

1. INTRODUCTION

All terminologies not defined in this paper are standard; see [Ja], [Ro] and [Th].

A compact 3-manifold $M$ is small if it is orientable, irreducible and if any incompressible surface in $M$ is parallel to $\partial M$. A knot $k$ in a 3-manifold $M$ is small if its exterior $M - \text{int} \mathcal{N}(k)$, denoted by $E(k)$, is a small 3-manifold.

The main theme of this work is to study the existence of degree one maps between closed hyperbolic small 3-manifolds. All known and expected closed small 3-manifolds are either Seifert fibered or hyperbolic. The existence of degree one maps between closed small Seifert manifolds has been studied (see [HWZ] and the references there).

For closed hyperbolic small 3-manifolds, even some general results for degree one maps between them have been established (see [RW] and [BW2]), but to our knowledge, there are no known examples of such degree one maps. Indeed the authors of [RW] wondered how to find such degree one maps when they were working on [RW], which is the first motivation of the present work.

Note that there are many ways of producing degree one maps between closed hyperbolic 3-manifolds (cf. [BW1], [Ka], [Ru]), but none of them insure that both hyperbolic manifolds are small. The following theorem provides the first examples of degree one maps between non-homeomorphic closed small hyperbolic 3-manifolds.

**Theorem 1.1.** There are infinitely many pairs of non-homeomorphic, closed, small hyperbolic 3-manifolds $(M, N)$ such that there is a degree one map $f : M \to N$.

**Remark 1.2.** The construction of degree one maps in Theorem 1.1 is based on Thurston’s hyperbolic Dehn surgery theorem [Th] and Proposition 2.2, which provides non-trivial proper degree one maps between the exteriors of hyperbolic small knots in the 3-sphere. These are also, to our knowledge, the first such examples. Our examples of degree one maps in Proposition 2.2 are constructed from the exteriors of some $n$-bridge knots, with $n > 2$, to the exteriors of some 2-bridge knots.
It would be interesting to find a degree one map between the exteriors of two hyperbolic 2-bridge knots, since such a degree one map will produce degree one maps between closed 3-manifolds obtained by Dehn surgeries on those knots with the same coefficients, and for most given coefficients those two closed 3-manifolds are small and hyperbolic.

In general we wonder if there is a degree one map \( f : E(k_1) \to E(k_2) \) for \( k_1 \) and \( k_2 \) two knots in \( S^3 \), is the bridge number of \( k_1 \) not smaller than that of \( k_2 \) [W]?

2. Examples of degree one maps between small hyperbolic 3-manifolds

This section is devoted to the proof of Theorem 1.1. Our construction uses tangle sum in the sense of Conway.

A tangle \( T = (B^3, a_1 \cup a_2) \) is a properly embedded disjoint pair of arcs \( (a_1 \cup a_2, \partial a_1 \cup \partial a_2) \hookrightarrow (B^3, \partial B^3) \).

Such a tangle is irreducible if there is no 2-sphere \( S \subset B^3 \) meeting transversely an arc \( a_i \) in two points such that the intersection of the 3-ball \( V \) bounded by \( S \) in \( B^3 \) and \( a_i \) is a knotted arc in \( V \).

We denote by \( T_0 = (B^3, b_1 \cup b_2) \) the trivial tangle. It is formed by two unknotted arcs separated by a properly embedded disk in \( B^3 \) disjoint from them.

Our construction of non-trivial degree one maps between small closed hyperbolic 3-manifolds relies on Thurston’ hyperbolic Dehn surgery theorem and the following two propositions:

**Proposition 2.1.** There exists an irreducible, non-trivial tangle \( T = (B^3, a_1 \cup a_2) \) with the following properties:

1. The 2-fold covering of \( B^3 \) branched along \( a_1 \cup a_2 \) is the exterior \( E \) of a small hyperbolic knot in \( S^3 \).
2. There is a proper degree one map \( f : T = (B^3, a_1 \cup a_2) \to T_0 = (B^3, b_1 \cup b_2) \) onto the trivial tangle \( T_0 \) such that
   - the restriction \( f|_{\partial B^3 : \partial B^3} \to \partial B^3 \) is a homeomorphism,
   - for \( i \in \{1, 2\} \), \( f^{-1}(b_i) = a_i \) and the restriction \( f| : a_i \to b_i \) is a homeomorphism.

By using a Conway sum of the tangle \( T \) with rational tangles, we deduce from Proposition 2.1 the following result:

**Proposition 2.2.** There are infinitely many hyperbolic small knots in \( S^3 \) with bridge number \( \geq 3 \) such that their exteriors admit a proper degree one map to the exterior of a hyperbolic 2-bridge knot.

The three subsections below are devoted to the proofs of Proposition 2.1, Proposition 2.2 and Theorem 1.1, respectively.

2.1. Construction of the tangle \( T \) and proof of Proposition 2.1. We consider the non-alternating knot \( \tilde{k} \) with eight crossings, the knot \( 8_{21} \) in Rolfsen’s book tabulation [Ro]. It is the Montesinos knot \( M(1; 1/2, 2/3, 2/3) \) with 3-branches in the notation of [BoZ]. By Oertel’s work [O1] it is a small hyperbolic knot. It is also a fibred knot with fibre a surface of genus 2; see for example [Ga].

---

1 After the acceptance of this paper, T. Ohtsuki, R. Riley and M. Sakuma [ORS] informed us that they have constructed degree one maps between 2-bridge knot complements.
Moreover, it is a strongly invertible knot; i.e., there is a smooth involution \( \tau \) of the pair \((S^3, \hat{k})\) such that \( \text{Fix}(\tau) \), the fixed point set of \( \tau \), is an unknotted circle and meets \( \hat{k} \) in exactly two points (Figure 1).

After an isotopy of the fibration of the exterior \( E = E(\hat{k}) \), we can assume that the restriction of \( \tau \) on \( E \) is fibre preserving. Hence there are two fibres \( F_1 \) and \( F_2 \) invariant by \( \tau \) in \( E \). Moreover \( \text{Fix}(\tau) \cap E = \tilde{a}_1 \cup \tilde{a}_2 \), where \( \tilde{a}_i \) is a properly embedded arc in \( F_i \) which separates \( F_i \) into two symmetric parts, \( i = 1, 2 \) (Figure 1).

The orbifold quotient \( E/\tau \) has an underlying space \(|E/\tau| \) homeomorphic to \( B^3 \) and the ramification locus formed by the union of two properly embedded disjoint arcs \( a_1 \cup a_2 \) in \( E/\tau \). That gives naturally a tangle \( T = (|E/\tau|, a_1 \cup a_2) \), which by construction verifies property (1) of Proposition 2.1.

To verify property (2), we construct a proper degree one map \( \tilde{f} : E \to S^1 \times D^2 \), which is equivariant with respect to the involution \( \tau \) on \( E \) and to the involution \( \tau_0 \) obtained on \( S^1 \times D^2 \) by extending the involution \( \tau|_{\partial E} \) to the solid torus.

We identify \( \partial E \) with \( S^1 \times \partial D^2 \) by choosing a preferred meridian-longitude coordinate system \((\mu, \lambda)\) on \( \partial E \) and by identifying the meridian \( \mu \) with \( S^1 \times \{\ast\} \) and the longitude \( \lambda \), which is the boundary of a fibre of the fibration of \( E \), with \( \{\ast\} \times \partial D^2 \).

The involution \( \tau_0 \) preserves two meridian disks \( D_1 \) and \( D_2 \), and \( \text{Fix}(\tau_0) \cap D_i = \tilde{b}_i \) is a properly embedded arc in \( D_i \) for \( i \in \{1, 2\} \). In particular the orbifold quotient \( S^1 \times D^2/\tau_0 \) has \( B^3 \) as its underlying space and the union of the two disjoint properly embedded arcs \( b_1 \cup b_2 \) as its ramification locus. That gives a trivial tangle \( T_0 = (|S^1 \times D^2/\tau_0|, b_1 \cup b_2) \).

The construction of the equivariant degree one map \( \tilde{f} \) is done in three steps:

**Step 1.** By the choice of \( \tau_0 \) on \( S^1 \times D^2 \) and the identification of \( \partial E \) with \( S^1 \times \partial D^2 \), we can take \( \tilde{f} : \partial E \to \partial(S^1 \times D^2) \) to be the identity.

**Step 2.** We extend \( \tilde{f} \) equivariantly to the two fibres \( F_1 \) and \( F_2 \), so that for \( i \in \{1, 2\} \), \( \tilde{f}(F_i) = D_i \), \( \tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i \), and \( \tilde{f}|_{\tilde{a}_i} : \tilde{a}_i \to \tilde{b}_i \) is a homeomorphism.

Since the properly embedded arc \( \tilde{a}_i \subset F_i \) is separating, we have \( F_i = F'_i \cup \tau(F'_i) \) for \( i \in \{1, 2\} \), where \( F'_i \) is a genus 1 surface. In the same way, we have \( D_i = D'_i \cup \tau(D'_i) \) for \( i \in \{1, 2\} \), where \( D'_i \) is a disk.

We consider the pinch \( p_i : F'_i \to D'_i \) which is the identity on a collar neighborhood of \( \partial F'_i \), for \( i \in \{1, 2\} \). Then we extend \( \tilde{f} \) equivariantly on \( F_i \) by taking \( \tilde{f}|_{F'_i} = p_i \) and \( \tilde{f}|_{\tau(F'_i)} = \tau_0 \circ p_i \circ \tau \), for \( i \in \{1, 2\} \).
At this point, for $i \in \{1, 2\}$, $\tilde{f}(F_i) = D_i$, $\tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i$ and $\tilde{f} : \tilde{a}_i \to \tilde{b}_i$ is a homeomorphism.

**Step 3.** We extend $\tilde{f}$ equivariantly to the remaining part of $E$.

By cutting $E$ along the two fibres $F_1 \cup F_2$, we get $E = F \times [1, 2] \cup F_1 \cup F_2 \tau(F \times [1, 2])$. In the same way $S^1 \times D^2 = D^2 \times [1, 2] \cup D_1 \cup D_2 \tau_0(D^2 \times [1, 2])$.

From Steps 1 and 2 we have defined a degree one map $\tilde{f} : \partial(F \times [1, 2]) \to \partial(D^2 \times [1, 2])$. Since $D^2 \times [1, 2]$ is a 3-ball, we can extend this map to a degree one map $\tilde{f} : F \times [1, 2] \to D^2 \times [1, 2]$.

Since $\tilde{f} : F_1 \cup F_2 \cup \partial E \to D_1 \cup D_2 \cup \partial(S^1 \times D^2)$ is already equivariant with respect to $\tau$ and $\tau_0$, we can finally define the map $\tau_0 \circ \tilde{f} \circ \tau : \tau(F \times [1, 2]) \to \tau_0(D^2 \times [1, 2])$ to get the desired proper degree one map $\tilde{f} : E \to S^1 \times D^2$ with the following properties:

(a) $\tau_0 \circ \tilde{f} = \tilde{f} \circ \tau$.

(b) The restriction $\tilde{f}|_{\partial E} : \partial E \to S^1 \times \partial D^2$ is the identity with respect to the chosen parametrization of $\partial E$.

(c) For $i \in \{1, 2\}$, $\tilde{f}^{-1}(\tilde{b}_i) = \tilde{a}_i$ and the restriction $\tilde{f}|_{\tilde{a}_i} : \tilde{a}_i \to \tilde{b}_i$ is a homeomorphism.

Now this equivariant proper degree one map $\tilde{f}$ covers through the involutions $\tau$ and $\tau_0$ a proper degree one map: $f : T = (|E/\tau|, a_1 \cup a_2) \to T_0 = (|S^1 \times D^2/\tau_0|, b_1 \cup b_2)$, between the tangle $T$ and the trivial tangle $T_0$. This degree one map verifies the properties of (2) of Proposition 2.1 because of properties (b) and (c) of $\tilde{f}$. That finishes the proof of Proposition 2.1.

### 2.2. Proof of Proposition 2.2

In the parametrization of $\partial E$ by the preferred meridian-longitude pair $(\tilde{\mu}, \tilde{\lambda})$, any simple closed curve on $\partial E$ is determined by a unique slope $(p, q)$, where $p \geq 0$ and $q \in \mathbb{Z}$ are coprime. We denote by $E(p, q)$ the closed orientable 3-manifold obtained from $E$ by Dehn filling $\partial E$ along the slope $(p, q)$.

The involution $\tau$ on the knot exterior $E$ can be extended to the glued solid torus to get an involution still called $\tau$ on the closed 3-manifold $E(p, q)$. According to Montesinos's construction [Mo], the quotient of $E(p, q)$ by $\tau$ is $S^3$ and the branching locus $k(p, q)$ is a knot or a link with two components, according to whether $p$ is odd or even.

This knot or link $k(p, q)$ is obtained by a Conway sum of the tangle $T$ with the rational tangle $R(p, q)$ of type $(p, q)$ (see Figure 2), where the outside tangle $T$ is drawn by using Montesinos’s technique.
Now, the degree one map $f : T \to T_0$ can be extended trivially by a homeomorphism to a degree one map of pairs $g : (S^3, k(p, q)) \to (S^3, b(p, q))$, such that

- $b(p, q)$ is the 2-bridge knot or link, obtained by a Conway sum of the trivial tangle $T_0$ with the rational tangle $R(p, q)$,
- $g^{-1}(b(p, q)) = k(p, q)$ and the restriction $\bar{g} : k(p, q) \to b(p, q)$ is a homeomorphism.

This last property follows immediately from the properties of (2) of $f$ in Proposition 2.1.

By considering the restriction of $g$ to the exterior of $k(p, q)$, Proposition 2.2 follows now from the following lemma:

**Lemma 2.3.** For $p = 2p' + 1, p' > 1$ and $q \neq np \pm 1, n \in \mathbb{Z}$, $k(p, q)$ and $b(p, q)$ are small hyperbolic knots in $S^3$. Moreover $k(p, q)$ has bridge number $\geq 3$.

**Proof.** By the classification of 2-bridge knots or links (cf. [BuZ]), $b(p, q)$ is a hyperbolic knot iff $p = 2p' + 1, p' > 1$ and $q \neq np \pm 1, n \in \mathbb{Z}$. By [HT] it is a small knot.

By Oertel [O2] (see also [Dun1]) the boundary slopes of the knot 8_{21} are the following integral slopes:

$$\{(12, -1), (6, -1), (2, -1), (0, 1), (1, 1)\}.$$}

In particular all odd $p > 1$ are not in this list. So for the slopes $(p, q)$ given in Lemma 2.3, the closed 3-manifolds $E(p, q)$ are small.

Since $E(p, q)$ is the 2-fold branched covering of the knot $k(p, q)$, it follows from the equivariant Dehn lemma that $k(p, q)$ is a small knot in $S^3$ (cf. [GL]). Hence it is either a hyperbolic or a torus knot.

It cannot be a torus knot since its exterior admits a proper degree one map onto the exterior of a hyperbolic 2-bridge knot. This would contradict the fact that the simplicial volume of a torus knot exterior vanishes, while it is always non-zero for a hyperbolic knot exterior.

The knot $k(p, q)$ has bridge number $\geq 3$. Otherwise its 2-fold branched covering would be a lens space and by the cyclic surgery theorem [CGLS] $q$ would be equal to $\pm 1$, since $\hat{k}$ is a hyperbolic knot. This contradicts our choice for $q$.

That finishes the proof of Lemma 2.3 and hence of Proposition 2.2. \qed

### 2.3. Degree one map between closed small hyperbolic 3-manifolds and proof of Theorem 1.1.

In Proposition 2.2 we constructed a small hyperbolic knot $k_1 \subset S^3$ with bridge number $\geq 3$ and a hyperbolic 2-bridge knot $k_2 \subset S^3$ such that there is a degree one map $g : (S^3, k_1) \to (S^3, k_2)$ such that $g^{-1}(k_2) = k_1$ and such that the restriction $\bar{g} : k_1 \to k_2$ is a homeomorphism. Let $E_1$ and $E_2$ be the exteriors of $k_1$ and $k_2$, respectively. As before we choose for $i \in \{1, 2\}$ a trivialization of $\partial E_i$ by a preferred meridian-longitude pair $(\mu_i, \lambda_i)$. Then (after possibly some isotopy on the boundary) $g$ induces a proper degree one map $h : E_1 \to E_2$ such that

- the restriction $h| : \partial E_1 \to \partial E_2$ is a homeomorphism,
- $h(\mu_1) = \mu_2$ and $h(\lambda_1) = \lambda_2$.

For any slope $(r, s)$ on $\partial E_i, i = 1, 2$, this degree one map $h$ extends trivially by a homeomorphism to a degree one map $h_{r, s} : E_1(r, s) \to E_2(r, s)$. Now Theorem 1.1 is a consequence of the following lemma:
Lemma 2.4. For almost all slopes \((r, s)\) (i.e. except finitely many), the two closed orientable 3-manifolds \(E_1(r, s)\) and \(E_2(r, s)\) are small, hyperbolic and not homeomorphic.

**Proof.** By [Hat] there are only finitely many slopes \((p, q)\) that can be boundary slopes on either \(\partial E_1\) or \(\partial E_2\). Since \(k_1\) and \(k_2\) are small knots in \(S^3\), if \((r, s)\) avoids this finite set of slopes, then \(E_1(r, s)\) and \(E_2(r, s)\) are small, closed 3-manifolds.

Let \(v_i = \text{vol}(E_i)\) be the hyperbolic volume of \(E_i, i \in \{1, 2\}\). Since \(k_1\) is not a 2-bridge knot, \(E_1\) is not homeomorphic to \(E_2\), because knots are determined by their complement in \(S^3\). Since there is a proper degree one \(h : E_1 \to E_2\), Gromov-Thurston’s strict rigidity theorem ([Th, Chap. 6], [Dun2]) implies that \(v_1 > v_2\).

By Thurston’s hyperbolic Dehn surgery theorem ([Th, Chap. 4]; see also [BP, Appendix B]) and Schlafli’s formula (cf. [Mi]), there is a constant \(c > 0\) (depending only on \(k_1\) and \(k_2\)) such that for \(r^2 + s^2 \geq c^2\) the following happens:

- both \(E_1(r, s)\) and \(E_2(r, s)\) are hyperbolic,
- \(v_1 > \text{vol}(E_1(r, s)) > v_2 > \text{vol}(E_2(r, s))\).

Therefore if \((r, s)\) avoids the finite set of slopes \(r^2 + s^2 < c^2\), then \(E_1(r, s)\) and \(E_2(r, s)\) are both hyperbolic and not mutually homeomorphic.

This finishes the proof of Lemma 2.4 and of Theorem 1.1. □

**Remark 2.5.** With further effort, one can show that the bridge number of the knots \(k(p, q)\) in Proposition 2.2 is at most 4 and the Heegaard genus of the 3-manifolds \(E_1(r, s)\) in Lemma 2.3 is at most 3.

**ACKNOWLEDGMENTS**

We wish to thank A. Kawauchi and A. Reid for helpful conversations.

**REFERENCES**


Laboratoire Émile Picard, CNRS UMR 5580, Université Paul Sabatier, 118 Route de Narbonne, F-31062 Toulouse Cedex 4, France
E-mail address: boileau@picard.ups-tlse.fr

Department of Mathematics, LAMA, Peking University, Beijing 100871, People’s Republic of China
E-mail address: wangsc@math.pku.edu.cn