ON SCH AND THE APPROACHABILITY PROPERTY

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Abstract. We construct a model of \(^{\sim}\text{SCH} + \neg \text{AP}^+\) (Very Good Scale). This answers questions of Cummings, Foreman, Magidor and Woodin.

1. Introduction

Notions of Very Good Scale, \(\kappa\) (VG\(\kappa\)), Weak square \(\kappa\) (\(\square^*_\kappa\)) and the Approachability Property, \(\kappa\) (AP\(\kappa\)), for a singular \(\kappa\), play a central role in Singular Cardinals Combinatorics. They were extensively studied by Shelah [9, 10, 11] and by Cummings, Foreman and Magidor [2].

All of these properties break down above a supercompact cardinal as was shown by S. Shelah in [9]. By R. Solovay [12], the Singular Cardinal Hypothesis (SCH) holds above strong compact cardinals. Also by Ben-David and Magidor [1] the Prikry forcing adds \(\square^*_\kappa\). Hence it is natural to ask about interconnections between SCH and the above principles. Cummings, Foreman and Magidor [2] asked if \(\text{VG}^*_\kappa\) implies \(\square^*_\kappa\). Woodin previously asked if it is possible to have \(^{\sim}\text{SCH}_\kappa + \neg \square^*_\kappa\). In [4] the positive answer to the second question was claimed. The second author found a gap in the argument and was able to show that the forcing used there (extender based forcing with long extenders) actually adds a \(\square^*_\kappa\)-sequence.

Our goal here will be to give a negative answer to the first question and a positive answer to the second. Thus we prove the following:

**Theorem 1.1.** Suppose \(\kappa\) is a supercompact cardinal. Then there is a generic extension in which \(\kappa\) is a strong limit singular cardinal of cofinality \(\omega\) so that

(a) \(2^\kappa > \kappa^+\);
(b) \(^{\sim}\text{AP}^*_\kappa\) (and hence \(\neg \square^*_\kappa\));
(c) \(\text{VG}^*_\kappa\).

Using standard methods we can make \(\kappa\) into \(\aleph_{\omega^2}\). Namely the following holds:

**Theorem 1.2.** Suppose \(\kappa\) is a supercompact cardinal. Then there is a generic extension in which \(\kappa = \aleph_{\omega^2}\) is a strong limit cardinal so that
2. THE MAIN CONSTRUCTION

Let us first recall some basic definitions:

**Definition 2.1.** (S. Shelah [9]) A sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ is called an $AP_\kappa$-sequence iff

(a) $\lim(\alpha) \rightarrow C_\alpha$ is a club in $\alpha$ and o.t.$(C_\alpha) = cf(\alpha)$.
(b) There is a club subset $D$ of $\kappa^+$ such that

$$\forall \alpha \in D \forall \beta < \alpha \exists \gamma < \alpha C_\alpha \cap \beta = C_\gamma.$$ 

It is not hard to see that $\square^+_\kappa \rightarrow AP_\kappa$.

**Definition 2.2.** (a) Let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of regular cardinals such that $\bigcup_{n<\omega} \kappa_n = \kappa$. A sequence $\langle f_\alpha \mid \alpha < \kappa^+ \rangle \subseteq \prod_{n<\omega} \kappa_n$ is called a very good scale on $\prod_{n<\omega} \kappa_n$ iff

(i) $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a scale on $\prod_{n<\omega} \kappa_n$, i.e., for every $f \in \prod_{n<\omega} \kappa_n$ there exists $\beta < \kappa^+$ and $n < \omega$ such that $f(m) < f_\beta(m)$ for every $m > n$ and for every $\alpha \leq \beta < \kappa^+$, $f_\alpha(m) < f_\beta(m)$ for almost every $m$;
(ii) for every $\beta < \kappa^+$ such that $\omega < cf(\beta)$ there exists a club $C$ of $\beta$ and $n < \omega$ such that $f_\gamma(m) < f_\gamma(m)$ for every $\gamma_1 < \gamma_2 \in C$ and $m > n$.

(b) $VG_S$ holds iff there exists a sequence $\langle \kappa_n \mid n < \omega \rangle$ and $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ such that $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ is a very good scale on $\prod_{n<\omega} \kappa_n$.

**Definition 2.3.** (S. Shelah [9]) Let $\kappa$ be an uncountable cardinal such that $cf(\kappa) = \omega$, and $d : [\kappa^+]^2 \rightarrow \omega$.

(a) $d$ is called normal if $\forall \beta \forall n < \omega \mid \{ \alpha < \beta \mid d(\alpha, \beta) \leq n \} \leq \kappa$.
(b) $d$ is called subadditive if $\forall \alpha < \beta < \gamma < \kappa^+, d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.
(c) $S_0(d) = \{ \alpha < \kappa^+ \mid \exists A, B \subseteq \alpha$ unbounded in $\alpha$ such that $\forall \beta \in B \exists n_\beta \in \omega \forall \alpha \in A \cap \beta \ d(\alpha, \beta) \leq n_\beta$. 

The next Lemma, which was stated in Shelah [9], shows that such a function always exists. Let us give the proof for the benefit of the reader.

**Lemma 2.4.** (S. Shelah [9]) There exists a normal subadditive function $d : [\kappa^+]^2 \rightarrow \omega$ for every uncountable cardinal $\kappa$ such that $cf(\kappa) = \omega$.

**Proof.** Fix an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of regular cardinals cofinal in $\kappa$. For every $d : [\kappa^+]^2 \rightarrow \omega$, let $A(\beta, n)$ and $(A(\beta, \leq n))$ denote the set of all $\gamma < \beta$ such that $d(\gamma, \beta) = n$ and $d(\gamma, \beta) \leq n$ respectively. We are going to define the function $d \upharpoonright_{\gamma \times \gamma}$ by induction on $\gamma$ such that for every $\beta \geq \kappa$ the size of $A(\beta, n)$ is at most $\kappa_n$. For every $\gamma < \beta < \kappa$, we define $d(\gamma, \beta)$ to be the least $n$ such that $\gamma < \kappa_n$. Assume that $d \upharpoonright_{\gamma \times \gamma}$ is defined. If $\gamma = \eta + 1$ is a successor, then let $d(\alpha, \gamma) = d(\alpha, \eta)$ for every $\alpha < \eta$ and $d(\eta, \gamma) = 0$. It is simple to see that $d \upharpoonright_{\gamma \times \gamma}$ is normal and subadditive. Assume now that $\gamma$ is a limit ordinal. Let $\langle B_i \mid i < \omega \rangle$ be a $\subseteq$-increasing sequence such that $\bigcup_{i<\omega} B_i = \gamma$ and $|B_i| = \kappa_i$. We define the sets $A(\gamma, n)$ by induction on $n$ as follows: By the induction hypothesis we can find $A(\gamma, 0)$ such that $B_0 \subseteq A(\gamma, 0)$.
and for every $\alpha \in A(\gamma, 0)$ the set $A(\alpha, 0)$ is contained in $A(\gamma, 0)$. Assume that $A(\gamma, n - 1)$ is defined. Set

$$X_n = \bigcup_{i \leq n} \{ A(\alpha, n) \mid \alpha \in \bigcup_{i < n} A(\gamma, i) \}.$$ 

Note that by the induction hypothesis $|X_n| \leq \kappa_n$. By another application of the induction hypothesis, it is possible to find $Y_n \supseteq X_n \cup B_n$ of size $\kappa_n$ such that $A(\alpha \leq n) \subseteq Y_n$ for every $\alpha \in Y_n$. Let $A(\gamma, n) = Y_n - \bigcup_{i < n} A(\gamma, i)$. Note that the size of $A(\gamma, n)$ is $\kappa_n$. Now define

$$d(\alpha, \gamma) = n \text{ iff } \alpha \in A(\gamma, n).$$

Let us show that the function $d \upharpoonright_{\gamma \times \gamma}$ is subadditive: Let $\beta < \alpha < \gamma$. Set $n = d(\alpha, \gamma)$ and $k = d(\beta, \alpha)$. We consider two cases:

**Case 1:** $n \geq k$. But then by our construction, $\beta \in \bigcup_{i \leq n} A(\gamma, i)$ and so $d(\beta, \gamma) \leq n$.

**Case 2:** $n < k$. But then $\beta \in X_k$ and so $\beta \in Y_k$ and $d(\beta, \gamma) \leq k$.

This finishes the proof of the lemma. □

**Fact 2.5.** (S. Shelah [9]) Suppose that $\kappa$ is a strong limit cardinal of cofinality $\omega$ and $d, d' : [\kappa^+]^2 \rightarrow \omega$ are two normal functions. Then $S_0(d) \equiv S_0(d') \text{ (mod } D_{\kappa^+})$ (where $D_{\kappa^+}$ is the club filter).

**Fact 2.6.** (S. Shelah [9]) Let $\kappa$ be a singular strong limit cardinal of cofinality $\omega$. The statement $AP_{\kappa}$ is equivalent to the existence of a normal function $d : [\kappa^+]^2 \rightarrow \omega$ such that $S_0(d)$ contains a club.

$S_0(d)$ is in fact the set of all approachable points and $AP_{\kappa}$ means that modulo the club filter every point less than $\kappa^+$ is approachable.

Let us now prove Theorem 1.1. We start with a model $V$ of $ZFC + GCH$ such that $V \models \text{“} \kappa \text{ is supercompact”}$ . Iterate first in Backward Easton fashion the Cohen forcing $C(\alpha, \alpha^{\omega+2})$ for each inaccessible $\alpha \leq \kappa$, where $C(\alpha, \alpha^{\omega+2})$ is defined as the poset consisting of functions $f$ such that $Dom(f)$ is a subset of $\alpha^{\omega+2}$ of size less than $\alpha$ and for every $\beta \in Dom(f)$, $f(\beta)$ is a partial function from $\alpha$ to $\alpha$ of size less than $\alpha$.

Let $P_{\kappa}$ denote the iteration below $\kappa$ and $P_{\alpha} = P_{<\kappa} \ast C(\kappa, \kappa^{\omega+2})$. Note that the forcing $P_{\kappa}$ preserves the cofinality of the ordinals. Let $G$ be a generic subset of $P_{\kappa}$. Denote $G_{<\kappa} = P_{<\kappa} \cap G$. Let for each $\alpha < \kappa^{\omega+2}$, $F_{\alpha}$ denote the $\alpha$-th generic function from $\kappa$ to $2$ in $G$, i.e. $\{ f(\alpha) \mid f \in G \}$.

Fix in $V$ a normal ultrafilter $U$ over $P_{\kappa}(\kappa^{\omega+2})$. Let $j : V \rightarrow M \simeq Ult(V, U)$ be the corresponding elementary embedding. Then $crit(j) = \kappa$ and $\kappa^{\omega+2} \subseteq M \subseteq L[G]$.

By standard arguments (see [5]) $j$ extends in $V[G]$ to an elementary embedding $j^* : V[G] \rightarrow M[G^*]$, where $G^* \cap P_{\kappa} = G_\kappa$ and $G^*$ above $\kappa$ is constructed in $V[G]$ using closure of the forcing and the fact that the number of dense sets we need to meet is small. Also, over $j(\kappa)$, we need to start with the condition $\{ (j(\alpha), F_{\alpha}) \mid \alpha < \kappa^{\omega+2} \}$ in order to satisfy $j^* G \subseteq G^*$. This means that for each $\alpha < \kappa^{\omega+2}$ the function $F_{j(\alpha)}$ (i.e. the one $G^*$ defines to be $j(\alpha)$-th function from $j(\kappa)$ to $j(\kappa)$) should extend $F_{\alpha}$.

Note that above $\kappa$ we are free in choosing values of $F_{j(\alpha)}$. Let us require $F_{j(\alpha)}(\kappa) = \alpha$ for each $\alpha < \kappa^{\omega+2}$ and then continue to build $G^*$.

Let $U^* = \{ X \subseteq P_{\kappa}(\kappa^{\omega+2}) \mid j^* \kappa^{\omega+2} \in j^*(X) \}$. Then $U^* \supseteq U$ and it is a normal ultrafilter over $P_{\kappa}(\kappa^{\omega+2})$ in $V[G]$. 

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Definition 2.9. For every $\ell$ (1) \( F_\ell(P \cap \kappa) < F_\rho(P \cap \kappa) \in U^* \).
(2) For each $f \in \prod\{(\delta_+^{+1}) | \delta < \kappa, \delta \text{ is an inaccessible}\}$ there is $\xi < \kappa^{+1}$ such that 
\[ \{ P \in P_\kappa(\kappa^{+1}) | f(P \cap \kappa) = F_\xi(P \cap \kappa) \} \in U^*. \]

Proof. (1) In $M[G^+]$, we have $j^*(F_\xi)(\kappa) = \xi < j^*(F_\rho)(\kappa) = \rho$. Hence the conclusion follows from the definition of $U^*$.
(2) Again, in $M[G^+]$, we have $j^*(f)(\kappa) < \kappa^{+1}$. Let $\xi = j^*(f)(\kappa)$. It is simple to see that $\xi$ satisfies the desired property.

For every $n \in \omega$ let $U_n$ be the projection of $U^*$ on $P_\kappa(\kappa^n)$, i.e., $X \in U_n$ iff $\{ P \in P_\kappa(\kappa^{n+2}) | P \cap \kappa^n \in X \} \in U^*$. Clearly $U_n$ is a normal ultrafilter on $P_\kappa(\kappa^{n+1})$.

Let $a, b \in P_\kappa(\kappa^n)$ and $b \cap \kappa \in \kappa$. Set
\[ a \preceq b \iff (a \subseteq b) \wedge \text{o.t.}(p(a)) < b \cap \kappa. \]

Lemma 2.8. \( \Box \)
(a) $\forall a \in P_\kappa(\kappa^n) \{ b \in P_\kappa(\kappa^n) | a \preceq b \} \in U_n$.
(b) $\{ a \in P_\kappa(\kappa^n) | a \cap \kappa \text{ is inaccessible and } a \cap \kappa \in \kappa \} \in U_n$.
(c) Let $\vec{X} = \langle X_n \mid a \in P_\kappa(\kappa^n) \rangle$ be a sequence of sets from $U_n$. Then $\Delta \vec{X} = \{ b \in P_\kappa(\kappa^n) | \forall a \in P_\kappa(\kappa^n) a \preceq b \rightarrow b \in X_a \in U_n \}$. ($\Delta \vec{X}$ is called the diagonal intersection of $\vec{X}$.)

We now define a version of the diagonal supercompact Prikry forcing.

Definition 2.9. \( p \in Q \) iff $p = \langle a_0^p, a_1^p, \ldots, a_{n-1}^p, X_n^p, X_{n+1}^p, \ldots \rangle$ where
(i) $\forall \ell < n a_\ell^p \in P_\kappa(\kappa^\ell)$ and $a_\ell^p \cap \kappa$ is an inaccessible cardinal;
(ii) $\forall m \geq n X_m^p \in U_m$;
(iii) $\forall m \geq n \forall b \in X_m^p \forall \ell < n a_\ell^p \preceq b$;
(iv) $\forall i < j < n a_i^p \preceq a_j^p$.

$n$ is called the length of $p$ and will be denoted by $\ell(p)$.

Definition 2.10. Let $p, q \in Q$. Then $p \leq^* q$ iff
(i) $\ell(p) = \ell(q)$;
(ii) $\forall \ell < \ell(p) a_\ell^p = a_\ell^q$;
(iii) $\forall m \geq \ell(p) X_m^p \subseteq X_m^q$.

Definition 2.11. Suppose that $p \in Q$ and $\vec{a} = \langle \vec{a}(\ell(p)), \ldots, \vec{a}(m) \rangle$ where $\vec{a}(i) \in X_i^p$ for every $\ell(p) \leq i \leq m$. We denote by $p \rightarrow (\vec{a})$ the sequence
\[ \langle a_1^p, \ldots, a_{\ell(p)-1}^p, \vec{a}(\ell(p)), \ldots, \vec{a}(m), Y_{m+1}, Y_{m+2}, \ldots \rangle, \]
where
\[ Y_n = \{ b \in X_n^p | \forall \ell(p) \leq i \leq m \vec{a}(i) \preceq b \} \]
for every $n \geq m + 1$. 

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By Lemma 2.8(a) it is easy to see that $Y_k \in U_k$, for each $k > m$ and $p \cdot \langle \vec{a} \rangle \in Q$.

**Definition 2.12.** Let $p, q \in Q$. $p \leq q$ iff there exists $\vec{a}$ such that $p \cdot \langle \vec{a} \rangle \leq^* q$.

The proof of the next two claims is quite standard, and it uses the same arguments as in the case of the ordinary diagonal Prikry forcing notion; see [3].

**Lemma 2.13.**
(a) $\langle Q, \leq \rangle$ is a Prikry type forcing notion, i.e., if $\sigma$ is a statement in the forcing language, then for every $p \in P$ there exists $p \leq^* q \in P$ such that $q$ forces $\sigma$ or $\neg \sigma$.

(b) $\langle Q, \leq^* \rangle$ is $\kappa$-closed.

**Proof.** (a) Assume for simplicity that $\ell(p) = 0$. Let $\sigma$ be a statement in the forcing language. Since any two conditions of length 0 are compatible, it is sufficient to find a condition $q$ such that $\ell(q) = 0$ and $q$ decides $\sigma$. Let $\vec{a} = \langle a_0, \ldots, a_n \rangle$ be such that $a_i \in P_\kappa(\kappa^i)$ for every $i \leq n$ and $a_i \subseteq a_{i+1}$ for every $i < n$. Define a sequence $X_{\vec{a}}$ as follows: If there exists a sequence $\vec{X} = \langle X_m \mid m \geq n+1 \rangle$ such that $\vec{a} \cdot \vec{X}$ is in $Q$ and decides $\sigma$, then let $X_{\vec{a}}$ be such a sequence. Otherwise let $X_{\vec{a}}(m) = P_\kappa(\kappa^{+m})$ for every $m \geq n+1$. Using Lemma 2.8(c), we can find $Y_n \in U_n$ such that for every $Y \subseteq$ increasing sequence $\vec{a} = \langle a_0, \ldots, a_n \rangle$ and for every $m \geq n+1$,

$$\{ b \in Y_m \mid a_n \subseteq b \} \subseteq X_{\vec{a}}(m).$$

Using Lemma 2.8 again, we can find a condition $q = \langle Y'_0, Y'_1, \ldots \rangle$ such that $Y'_i \subseteq Y_i$ with the following property: if there exists $\vec{a} \in \prod_{i \leq n} Y'_i$ such that $q \cdot \vec{a}$ decides $\sigma$, then $q \cdot \langle \vec{a} \rangle$ decides $\sigma$ for every $\vec{a} \in \prod_{i \leq n} Y'_i$ (in the same way). Now it is easy to see that $q$ decides $\sigma$ and is of length 0.

(b) This is an immediate consequence of the $\kappa$ completeness of the ultrafilters.

**Lemma 2.14.** Let $G_Q$ be $Q$ generic over $V[G]$.

(a) $\langle Q, \leq \rangle$ does not add any new bounded subsets to $\kappa$.

(b) $\forall n \ c.f^{V[G][G_Q]}(\kappa^{+n}) = \omega$ (in fact for every $\kappa \leq \delta < \kappa^{+\omega}$ such that $c.f^{V[G]}(\delta) \geq \kappa$ we have $c.f^{V[G][G_Q]}(\delta) = \omega$).

**Proof.** (a) This is a consequence of Lemma 2.13.

(b) Let $\langle a_0, a_1, \ldots \rangle$ be the generic sequence added by $G_Q$. Let $\delta < \kappa^{+\omega}$ be such that $c.f^{V[G]}(\delta) \geq k$. A simple density argument shows that the sequence $\gamma_m = \sup(a_m \cap \delta)$ is cofinal in $\delta$.

The next lemma is crucial for the construction.

**Lemma 2.15.** $Q_3$ is $\kappa^{+\omega+1}$-c.c.

**Proof.** Just note that the total number of finite sequences used in the conditions is $\kappa^{+\omega}$.

The next lemma now follows easily.

**Lemma 2.16.**
(a) $V[G][G_Q] \models \kappa$ is strong limit, $2^\kappa = \kappa^{+2} = (\kappa^{+\omega+2})^{V[G]}$ and $c.f(\kappa) = \omega$.

(b) If $\models V[G][G_Q] \models \omega < \mu = c.f(\mu) < \kappa$ and $f: \mu \rightarrow V[G]$", then there is $X \in V[G]$ unbounded in $\mu$ such that $f \upharpoonright X \in V[G]$.
Proof. (b): Let $\hat{f}$ be a $Q$ name for $f$. Let $D$ be the set of all conditions $p$ in $P$ such that for every $\subseteq$ increasing sequence $\vec{a}$ in $X^p_{(p)} \times \ldots \times X^n_{(p)}$ and for every $i < \eta$, if there exists $p \hat{\upharpoonright} (\vec{a}) \leq^* q \in P$ such that $q$ decides the value of $\hat{f}(i)$, then $p \hat{\upharpoonright} (\vec{a})$ already decides the value of $\hat{f}(i)$. Let us show that $D$ is dense. Let $p$ be a condition in $P$. Assume for simplicity that $\ell(p) = 0$. Using the fact that the ultrafilters $U_n$ are $\kappa$ closed, pick for every $\subseteq$ increasing element $\vec{a}$ from $P_{\kappa}(\kappa) \times P_{\kappa}(\kappa^+) \times \ldots \times P_{\kappa}(\kappa^{+n})$ a condition $p_{\vec{a}}$ with initial segment $\vec{a}$ such that for every $i < \eta$ if there is a direct extension of $p_{\vec{a}}$ which decides the value of $\hat{f}(i)$, then $p_{\vec{a}}$ already decides this value.

Using Lemma 2.8(c), find a condition $q$ such that $\ell(q) = 0$ and $q \hat{\upharpoonright} \vec{a} \geq^* p_{\vec{a}}$ for every $\subseteq$ increasing sequence $\vec{a}$ in $X^p_0 \times \ldots \times X^n_{(p)}$. Since every two conditions of length 0 are compatible, we can assume that $q \geq^* p$. But $q$ is in $D$ and so $D$ is dense in $P$.

Pick $p \in D \cap G_Q$ and let $p \upharpoonright_{\ell(p)} \vec{a}$ be the Prikry sequence added by $G_Q$. For every $i < \eta$ we can find $m(i) < \omega$ and $q \in G$ such that $q$ is a direct extension of $p \hat{\upharpoonright} \vec{a} \upharpoonright m(i)$ and $q$ decides the value of $\hat{f}(i)$. But then $p \hat{\upharpoonright} \vec{a} \upharpoonright m(i)$ already decides the value of $\hat{f}(i)$. Since $cf(\eta) > \omega$, we can find a stationary set $X' \subseteq \eta$ and $m$ such that $m = m(i)$ for every $i \in X'$. In $V[G]$ let

$$X = \{ i < \eta \mid p \hat{\upharpoonright} \vec{a} \upharpoonright m \text{ decides the value of } \hat{f}(i) \}.$$ 

Then $X$ is as required. \hfill \ensuremath{\square}

Definition 2.17. A submodel $N$ of $H_{\kappa^{+\omega+1}}$ is called a supercompact submodel iff

1. $|N| < \kappa$ and $N \cap \kappa$ is a cardinal less than $\kappa$;
2. $cf(\sup(N \cap \kappa^{+\omega+1}) = (N \cap \kappa)^{+\omega+1}$;
3. for every $A \subseteq \kappa^{+\omega+1}$ there exists $B \in N$ such that $A \cap N = B \cap N$.

It is simple to see that if $\kappa$ is $\kappa^{+\omega+2}$ supercompact, then the collection of all supercompact submodels is stationary. The following lemma was proved by Shelah in [9]:

Lemma 2.18 ([9]). Suppose that $\kappa$ is $\kappa^{+\omega+2}$ supercompact and $d : [\kappa^{+\omega+1}]^2 \rightarrow \omega$ is normal and subadditive. Let $S$ be the set of $\delta < \kappa^{+\omega+1}$ such that $\delta = \sup(N \cap \kappa^{+\omega+1})$ for some supercompact submodel. Then $S \subseteq \kappa^{+\omega+1} \cap cf(< \kappa)$ is stationary

and $S \subseteq \kappa^{+\omega+1} - S_0(d)$.

\hfill \ensuremath{\square}

Let $G_Q$ be a generic subset of $Q$ over $V[G]$.

Proposition 2.19. $V[G][G_Q] \models \neg AP_{\kappa}$.

Proof. The idea is to try to find a normal function $d$ such that $\kappa^+ - S_0(d)$ is stationary. The next lemma shows that it is sufficient to find any two-place function $d$ with this property.

Lemma 2.20. Let $\kappa$ be a cardinal such that $cf(\kappa) = \omega$. If there is $d : [\kappa^+]^2 \rightarrow \omega$ such that $\kappa^+ - S_0(d)$ is stationary, then there is a normal $\vec{d}$ such that $\kappa^+ - S_0(\vec{d})$ is stationary.
Proof. Let $d_0 : [κ+]^2 \to ω$ be any normal function. Set $\overline{d} = d + d_0$. We need to show that $\overline{d}$ is normal and that $κ^+ - S_0(\overline{d})$ is stationary.

(i) $\overline{d}$ is normal: pick $β < κ^+$. Since $\overline{d}(α, β) ≥ d_0(α, β)$, we see that \( \{ α < β \mid \overline{d}(α, β) ≤ n \} \subseteq \{ α < β \mid d_0(α, β) ≤ n \} \) and the conclusion follows from the normality of $d_0$.

(ii) $κ^+ - S_0(\overline{d})$ is stationary: for every $β ∈ S_0(\overline{d})$, there are $A, B ⊆ β$ unbounded in $β$ which satisfy Definition 2.3(c). Since $∀ α < β d(α, β) ≤ d_0(α, β)$ we get $β ∈ S_0(d)$. We proved that $S_0(\overline{d}) ⊆ S_0(d)$ or equivalently $κ^+ - S_0(d) ⊆ κ^+ - S_0(\overline{d})$. But $κ^+ - S_0(d)$ is stationary and so $κ^+ - S_0(\overline{d})$ is also stationary.

□

Work in $V[G]$ and pick any normal subadditive function $d : [κ^{ω+1}]^2 \to ω$. Set $S = κ^+ - (S_0(d))^{V[G]}$. Since $κ$ is $κ^{ω+2}$ supercompact, we can apply Lemma 2.18 and conclude that $S$ is stationary. In $V[G, G_Q]$, $d$ is a function from $[κ]^2$ to $ω$, but $d$ is no longer normal. Let us prove that $V[G, G_Q] \models S ⊆ κ^+ - S_0(d)$. Otherwise there exists $δ ∈ S \cap S_0(d)$. Pick $A, B ∈ V[G, G_Q]$ unbounded in $δ$ such that

$$∀ β ∈ B \exists n_β \forall α < β α ∈ A → d(α, β) ≤ n_β.$$  

Since $ω < cf^{V[G, G_Q]}(δ) < κ$, we can use Lemma 2.16(b) to find $A, B ∈ V[G]$ unbounded in $δ$ such that $A ⊆ A$ and $B ⊆ B$. We have that for every $β$ in $B$ there exists $n_β$ such that $d(α, β) ≤ n_β$ for every $α < β$ in $A$. Thus $V[G] \models δ ∈ S_0(d)$. This contradicts Lemma 2.18. By Lemma 2.18 and the fact that $Q$ is $κ^{ω+1}$- c.c., we get that $S$ is stationary in $V[G, G_Q]$, and therefore $κ^+ - S_0(d)$ is stationary. By Fact 2.5 and Lemma 2.20 we get $V[G, G_Q] \models \neg AP_κ$ as required. □


Proof. Let $⟨ P_n \mid n < ω ⟩$ be the supercompact Prikry sequence defined from $G_Q$, i.e., for each $m < ω$, there is $p ∈ G_Q$ such that

$$⟨ P_n \mid n < m ⟩ = \{ a_0^n, \ldots, a_{m-1}^n \} .$$

Let $κ_n = P_n \cap κ$ for each $n < ω$. Then $⟨ κ_n \mid n < ω ⟩$ is an increasing sequence of inaccessible cardinals cofinal in $κ$. Consider $\prod_{n < ω} κ_n^{ω+1}$. For each $α < (κ^{ω+1})^V = κ^+$ and $n < ω$ let $t_α(n) = F_n(κ_n)$ if $F_n(κ_n) < κ_α^{ω+1}$ and $t_α(n) = 0$ otherwise. Clearly $\{ t_α \mid α < κ^+ \} ⊆ \prod_{n < ω} κ_n^{ω+1}$. We show below that it is a scale and a very good one.

Claim 2.22. For each $α < β < κ^+$ we have $t_α(n) < t_β(n)$ for all but finitely many $n$’s.

Proof. Note that the set $Y = \{ P ∈ P_n(κ^{ω+2}) \mid F_α(P \cap κ) < F_β(P \cap κ) < (P \cap κ)^{ω+1} \} ⊆ U^*$. Hence for each $n < ω$ the projection $Y_n$ of $Y$ to $P_n(κ^{ω+1})$ belongs to $U_n$, i.e., the set $Y_n = \{ P \cap κ^{ω+1} \mid P ∈ Y \} ∈ U_n$. By a simple density argument, we can find $q ∈ G_Q$ such that $X_q^n ⊆ Y_n$ for every $n ≥ ℓ(q)$. But by the choice of $Y_n$, $q$ forces that $t_α(n) < t_β(n) < κ_α^{ω+1}$ for every $n ≥ ℓ(q)$ as required. □

Claim 2.23. For each $t ∈ \prod_{n < ω} κ_n^{ω+1}$ there exists $α$ such that $t_α(n) > t(n)$ for all but finitely many $n$’s.

Proof. Let $i$ be a name for $t$ and assume that $\models i ∈ \prod_{n < ω} κ_n^{ω+1}$. Let us show that for every $q$ there is $q ≤^* p$ and $α < κ^{ω+1}$ such that

$$\models t_α(n) > t(n) \text{ for almost every } n,$$

(*)
Assume for simplicity that ℓ(q) = 0. Let ă be as in Definition 2.11. Since q−(ă) forces that t(m) < ă(m) ∩ κ+ω+1 < κ, we can use the Prikry condition and the fact that ≤* is κ closed to find r ≥* q−(ă), which determines the value of ℓ(m).

Using the same argument as in the proof of the Prikry property, we can find p′ ≥* q such that for every ă as in Definition 2.11 there exists βă such that p′−(ă) forces that ℓ(m) = βă. Let h(ă) = βă. Note that for each n we have

\[ j^*(h)(κ), j^*(κ^+), ..., j^*(κ^n)) ≤ α_n < κ^{+ω+1}. \]

Let α = sup{α_n | n < ω} + 1. By the construction of F_α, we know that j^*(F_α)(κ) = α, and so using (**), we can shrink the sets of measure one of p′ to form a condition p so that for every ă, βă < F_α(ă(m) ∩ κ). It is simple to see that α and p satisfy (*).

Claim 2.24. \(|t_\alpha | \alpha < κ^+\) is a very good scale.

Proof. Let α < κ+ be of uncountable cofinality below κ. Then (cfα)^V[G,G_α] = (cfα)^V[G] = (cfα)^V. Then pick a club C ⊆ α in V with o.t.(p(C)) = cfα. Now by the choice of U* we have

\[ A = \{ P ∈ P_κ(κ^{+ω+2}) | \forall γ, β ∈ C(γ < β → F_γ(P ∩ κ) < F_β(P ∩ κ)) \} \in U^* \]

since j^*(F_γ)(κ) = γ < β = j^*(F_β)(κ) in M[G^*] for each γ < β < (κ^{+ω+1})^V and |C| = cfα < κ.

Let A_n be the projection of A to P_κ(κ+n). The set of q such that X^q_n ⊆ A_n for every n ≥ ℓ(q) is dense in Q and so we can find such a condition q in G_κ. Now it is simple to see that q forces that t_α(m) < t_β(m) for every m ≥ ℓ(q), and we are done.

Remark 2.25. (a) The same argument shows that \(|t_\alpha | \alpha < κ^{++} = (κ^{+ω+2})^V\)

is a very good scale in \(Π_{κ<κ} κ^{+ω+2}\).

(b) It is possible instead of using the explicit construction producing the scale just to start with an indestructible under κ-directed closed forcing supercompact cardinal κ. Then set 2^κ = κ^{+ω+2}. Any functions H_α such that \([H_α]_κ = α (α < κ^{+ω+2})\) with \(V\) being the projection of a supercompact measure from \(P_κ(κ^{+ω+2})\) to κ can be used instead of the F_α's.

(c) Cummings and Foreman have shown in an unpublished work that in V^Q there is a scale on \(Π_{κ<κ} κ^{κ+ω+1}\) which is not good. This gives an alternative argument for the failure of AP_κ in V^Q.

Our next task will be to push everything down to \(κ_{ω^2}\). The argument is quite standard, so let us only concentrate on the main points.

Let j : V → M be a κ^{+ω+1} supercompact embedding. We would like to find an extension j^* of j to V[G] such that all the ordinals α < j(κ) will be of the form j^*(g)(κ) for some g : κ → κ.

Work in V[G]. Since κ^{ω+1} M[G] ⊆ M[G] and the number of antichains of j(P_{<κ})/G in M[G] is κ^{ω+2}, we can find a generic subset H of j(P_{<κ})/G over M[G]. Set M^* = M[G*H] and let \(|x_α | α < κ^{ω+2}\) be an enumeration of j(κ).

Lemma 2.26. There exists a generic subset K of \(C := (C(j(κ)), j(κ^{ω+1}))^{M[G*H]}\)

with the following properties:

(a) \(j'(G_α) ⊆ K\);
(b) \(j(F_α)(κ) = x_α\), where F_α is the α-th Cohen function.
Proof. Let \( \langle A_i \mid i < \kappa^{+\omega+2} \rangle \) be an enumeration of the antichains of \( C \) in \( M^* \).
Since \( C \) is \( \kappa^{\omega+1} \) closed in \( V[G] \), we can find a \( C \) generic subset \( K^* \) over \( M^* \).
For each \( \alpha < j(j^{+\omega+1}) \), set \( K^* \mid \alpha = \{ p \mid p \in K^* \} \). Set \( F = \bigcup_j j''(G_\alpha) \).
Note that \( F \subseteq j''(\kappa^{+\omega+2}) \times \kappa \times \kappa \). For each \( \alpha < j(j^{+\omega+1}) \), we let \( K \mid \alpha \) be the set of all conditions \( p \) such that for every \( \delta < \kappa^{+\omega+2} \), if \( j(\delta) < \alpha \), then \( p(j(\delta)) \supseteq j(F(\delta)) = F(\delta) \) and \( p(j(\delta))(\kappa) = x_\delta \).
Note that since \( \sup(j''(\kappa^{+\omega+2})) = j(\kappa^{+\omega+2}) \),
we need to change only \( \kappa^{+\omega+1} \) many coordinates and so \( p \) is in \( M^* \).
Since \( K^* \mid \alpha \) is \( C \mid \alpha := \langle C(j(j, k), \alpha) \rangle \) \( M^* \) generic over \( M^* \),
and the number of changes is small (that is, \( \kappa^{+\omega+1} < j(j) \)),
we conclude that \( K \mid \alpha \) is also \( \langle C(j(j), k), \alpha \rangle \) \( M^* \) generic over \( M^* \).
Let \( K = \bigcup_{\alpha < j(j^{+\omega+1})} K \mid \alpha \).
Since every antichain in \( C \) is an antichain of \( C \mid \alpha \) for some \( \alpha < j(j^{+\omega+2}) \), we get that \( K \) is \( C \) generic over \( M^* \).
Also by our construction, \( K \) satisfies (a) and (b) and we are done.
\( \square \)

Let \( j^* : V[G] \rightarrow M^*[K] \) be the extension of \( j \) to \( V[G] \). Let \( U_n^\omega \) be the \( \kappa^\omega \) ultrafilter derived from \( j^* \), i.e.,
\( X \in U_n^\omega \) iff \( j''(\kappa^\omega) \in j^*(X) \).

Let \( i_n^* : V[G] \rightarrow Ult(V[G], U_n^\omega) \cong N_n \) and \( k_n : N_n \rightarrow M^*[K] \). By standard arguments we can find an \( M^*[K] \) generic subset \( H^* \) of \( Col(\kappa^{+\omega+2}, j(k)) \). Now by our construction, the range of \( k_n \) contains \( \{ j^*(F_\alpha)(\kappa) \mid \alpha < \kappa^{+\omega+2} \} \cup \{ k_n(i_n(\kappa)) \} = j(\kappa) + 1 \) and so \( crit(k_n) > i_n(\kappa) \).
But since \( (\kappa^{+\omega+2}, i_n(\kappa)) )^{N_n} \) satisfies \( i_n(\kappa) \)-c.c.,
the filter generated by \( k_n^{-1}(H^*) \) is \( (\kappa^{+\omega+2}, i_n(\kappa)) )^{N_n} \) generic over \( N_n \). Denote this filter by \( H_n \).

Now we are ready to define a new forcing \( Q \).

**Definition 2.27.** \( p \in Q \) iff
\( p = \langle a_0^p, a_0^p, a_1^p, f_1^p, \ldots, a_n^p, f_n^p, X_0^p, X_n^p, F_n^p, X_{n+1}^p, F_{n+1}^p, \ldots \rangle \)
so that the following holds:

1. \( \langle a_0^p, a_0^p, \ldots, a_n^p, X_0^p, X_n^p, \ldots \rangle \) is as in Definition 2.9 with the \( U_n^\omega \)'s replacing the \( U_n^\omega \)'s.
2. \( \forall \ell < n - 1 \ f_\ell^p \in Col((a_\ell^p \cap \kappa)^{+\omega+2}, a_{\ell+1}^p \cap \kappa) \).
3. \( f_n^p \in Col((a_{n-1}^p \cap \kappa)^{+\omega+2}, \kappa) \).
4. \( \forall \ell \geq n \ F_n \) is a function on \( X_\ell^p \) such that
   \( a) \ F_n(P) \in Col((P \cap \kappa)^{+\omega+2}) \).
   \( b) \ j_n^p(F_n(j''n^\omega)) \in H_n \).

All the previous claims remain valid here. Only in Lemma 2.16(2) do we restrict ourselves to \( \mu \)'s of the form \( \kappa^{+\omega+1} \) or \( \kappa^{+\omega+2} \) for the Prikry sequence \( \langle \kappa_n \mid n < \omega \rangle \).

Let us conclude with two questions.

**Question 1.** Is it consistent that \( \aleph_\omega \) is a strong limit, \( 2^{\aleph_\omega} > \aleph_{\omega+1} \) and \( \Box \aleph_\omega \) (or \( \sim AP_{\aleph_\omega} \))?

**Question 2.** Is it consistent that \( GCH \) holds below \( \kappa \), \( 2^\kappa > \kappa^+ \) and \( \Box^* \kappa \) (or \( \sim AP_\kappa \)) for a singular cardinal \( \kappa \)?

**Question 3** (Cummings). Is it consistent that there is a very good scale on every increasing sequence \( \langle \kappa_n \mid n < \omega \rangle \) of regular cardinals such that \( \bigcup_{n<\omega} \kappa_n = \kappa \) and \( \sim AP_\kappa \)?
References

1. S. Ben-David and M. Magidor, The weak □* is really weaker than the full square, J. Symbolic Logic 51 (1986) 1029–1033. MR865928 (88a:03117)


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