THE GLOBAL ATTRACTIVITY OF THE RATIONAL DIFFERENCE EQUATION $y_n = A + \left( \frac{y_{n-k}}{y_{n-m}} \right)^p$

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Abstract. This paper studies the behavior of positive solutions of the recursive equation

$$y_n = A + \left( \frac{y_{n-k}}{y_{n-m}} \right)^p, \quad n = 0, 1, 2, \ldots,$$

with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. We prove that if $\gcd(k, m) = 1$, and $p \leq \min\{1, (A + 1)/2\}$, then $y_n$ tends to $A + 1$. This complements several results in the recent literature, including the main result in K. S. Berenhaut, J. D. Foley and S. Stević, The global attractivity of the rational difference equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$, Proc. Amer. Math. Soc., 135 (2007) 1133–1140.

1. Introduction

This paper studies the behavior of positive solutions of the recursive equation

$$y_n = A + \left( \frac{y_{n-k}}{y_{n-m}} \right)^p, \quad n = 0, 1, 2, \ldots,$$

with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$.

The case $k = m$ is trivial, so from now on we will assume that $k \neq m$.

Note that if $g = \gcd(k, m) > 1$, then the set $\{y_i\}$ can be separated into $g$ different equations of the form

$$y_n^{(j)} = A + \left( \frac{y_n^{(j)} - \frac{k}{g}}{y_n^{(j)} - \frac{m}{g}} \right)^p,$$

where $j \in \{1, 2, \ldots, g\}$. Hence, we may assume that $\gcd(k, m) = 1$.

The study of properties of rational and nonlinear difference equations has been an area of intense interest in recent years; cf. [11–25] and the references therein.

There is a relatively long history in studying equation (1). For example, for $p = 1$, the case $k = 2, m = 1$ was studied in [2] by Amleh et al., the case $k \in \mathbb{N}$, $m = 1$ was studied by DeVault et al. in [11], and the case $A > 1, k = 1, m \in \mathbb{N}$ was studied by Stević in [20]. The investigation of global stability and periodicity of...
positive solutions of equation (1), for the case \( p = A = 1, k, m \in \mathbb{N} \) was completed by results in [3] and [15]; see also [17] and [21].

The study of the case \( p > 1 \) was suggested in [14], where the authors noted that some results from [2] for the case \( p = 1, k = 2, m = 1 \), can be translated to the case \( p > 1, k = 2, m = 1 \). The first results for the case \( p < 1 \) were given in [23].

The existence of monotone solutions, for the case \( p > 0 \) and \( A > 1 \) was shown in [5] by developing the technique from [6, 7, 8, 9, 10, 24] and [25]. Equations in papers [4] and [12] were investigated by transforming them into some special cases of equation (1).

The linearized equation associated with equation (1) for the case \( k = 2 \) and \( m = 1 \) is

\[
(A + 1)z_n + pz_{n-1} - pz_{n-2} = 0,
\]

and its characteristic roots are

\[
\lambda_1 = -\frac{p + \sqrt{p^2 + 4p(A + 1)}}{2(A + 1)} \quad \text{and} \quad \lambda_2 = -\frac{p - \sqrt{p^2 + 4p(A + 1)}}{2(A + 1)}.
\]

By some simple calculation we obtain

\[
|\lambda_1| = \frac{2p}{p + \sqrt{p^2 + 4p(A + 1)}} < 1,
\]

for every \( p, A > 0 \).

On the other hand, we have that

\[
|\lambda_2| < 1 \iff 2p < A + 1.
\]

Hence, when \( 2p < A + 1 \) equation (1) for the case \( k = 2 \) and \( m = 1 \) is locally asymptotically stable by the Linearized Stability Theorem.

Motivated by this local stability result, in [22] Stević has posed the following conjecture.

**Conjecture 1.** If \( k = 2, m = 1 \) and \( p, A \in (0, 1) \) are such that \( p < (A + 1)/2 \), then every positive solution of equation (1) converges to the unique equilibrium \( A + 1 \).

Among other results, here we confirm the conjecture by proving that the following holds true for every \( k, m \in \mathbb{N}, 0 < A < 1 \) and \( 0 < p \leq (A + 1)/2 \).

**Theorem 1.** Suppose that \( m, k \geq 1 \), and that \( p, A \) are positive numbers satisfying \( 0 < A < 1 \) and \( 0 < p \leq (A + 1)/2 \). If the sequence \( \{y_i\} \) satisfies (1) with \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty) \) where \( s = \max\{m, k\} \), then \( \{y_i\} \) converges to the unique equilibrium \( A + 1 \).

**Remark 1** (The case \( k \) even and \( m \) odd). Note that the general characteristic equation associated with the linearized equation for equation (1) is

\[
(A + 1)\lambda^s + p\lambda^{s-m} - p\lambda^{s-k} = 0,
\]

and for \( k \) even and \( m \) odd, equation (3) has a real root \( \lambda_0 < -1 \), when \( p > (A + 1)/2 \).

To see this, suppose that \( p > (A + 1)/2 \), and set

\[
f(\lambda) = A + 1 + \frac{p}{\lambda^m} - \frac{p}{\lambda^k}.
\]

Now, note that \( f(-1) = A + 1 - 2p < 0 \), and for \( \lambda < -1 \),

\[
f(\lambda) = A + 1 - \frac{p}{|\lambda|^m} - \frac{p}{|\lambda|^k} > A + 1 - \frac{2p}{|\lambda|^{\min\{m, k\}}} > 0,
\]
for sufficiently large $|\lambda|$.

Hence, by the continuity of the function $f$ on the interval $(-\infty, -1)$ it follows that $f(\lambda) = 0$ for some $\lambda \in (-\infty, -1)$, as required.

Thus, by the Linearized Stability Theorem, the positive equilibrium $\bar{y} = A + 1$ of equation (1) is not stable, in this case. This fact in conjunction with Theorem 1 gives a full characterization of stability for the case $k$ even and $m$ odd, for $A, p \in (0, 1)$.

The paper proceeds as follows. In Section 2, we introduce some preliminary lemmas and notation. Section 3 is devoted to global stability, where among other results we give a proof of Theorem 1.

2. Preliminaries and notation

In this section, we introduce some preliminary lemmas and notation. First, consider the simple transformed sequence \{z_i\} defined by $z_n = y_n - A$, for $n \geq -s$. Then, equation (1) becomes

$$z_n = \left( \frac{A + z_{n-k}}{A + z_{n-m}} \right)^p,$$

for $n \geq 0$.

Now, define \{z^*_i\} by

$$z^*_i = \begin{cases} z_i, & \text{if } z_i \geq 1, \\
\frac{1}{z_i}, & \text{otherwise}. \end{cases}$$

The following elementary lemma will be useful.

**Lemma 1.** If $x > 1$ and $0 < A < 1$, then

$$\left( \frac{A + x}{A + 1/x} \right)^{\frac{1}{x}} \leq x,$$

with the inequality if and only if $x = 1$, and if $x \geq 1$ and $A > 1$, then the reverse inequality to inequality (8) holds.

**Proof.** Assume first that $A \in (0, 1)$. Then the inequality in (8) is equivalent to

$$g_A(x) \overset{def}{=} (A + 1) \ln \left( \frac{A + x}{Ax + 1} \right) - (1 - A) \ln x \leq 0.$$  

Note that

$$\lim_{x \to +\infty} g_A(x) = -\infty \text{ and } g_A(1) = 0.$$

By some simple calculations we obtain that

$$g'_A(x) = -\frac{A(x - 1)^2(1 - A)}{(Ax + 1)^2} < 0,$$

when $x \neq 1$, since $A \in (0, 1)$.

Hence, $g_A(x)$ is decreasing, and thus by (11) is negative on the interval $(1, \infty)$.

Now, assume that $A > 1$. Then $\lim_{x \to +\infty} g_A(x) = +\infty$ and $g_A(1) = 0$. On the other hand, by (11) it follows that $g'_A(x) > 0$, from which the desired inequality follows. \qed

**Remark 2.** Note that if $A = 1$ the inequality in (8) becomes equality.
Next we prove a contraction lemma which will be helpful in showing convergence of solutions in the transformed space obtained through (7).

**Lemma 2.** Suppose \( \{ z_i \} \) satisfies (8) with \( p \leq (A + 1)/2 \) and \( A \in (0,1] \). Then,

\[
1 \leq z_n^* \leq \max \{ z_{n-k}^*, z_{n-m}^* \},
\]

for all \( n \geq s \).

**Proof.** Suppose that \( z_{n-k}^* > z_{n-m}^* \) and set \( x = \max \{ z_{n-k}^*, z_{n-m}^* \} \). Note that if \( z_{n-k}^* \geq 1 \), then \( 1 \leq z_{n-k}^* \leq x \) and consequently

\[
1/x \leq z_{n-k}^* \leq x,
\]

and if \( z_{n-k}^* < 1 \), then \( 1/z_{n-k}^* = z_{n-k}^* \leq x \) from which (13) also holds. Similarly, we have that

\[
1/x \leq z_{n-m}^* \leq x.
\]

Then, from (8), (13) and (14), for \( n \geq s \), we have that

\[
z_n^* = z_n = \left( \frac{A + z_{n-k}^*}{A + z_{n-m}^*} \right)^p \leq \left( \frac{A + z_{n-k}^*}{A + z_{n-m}^*} \right) \frac{A + 1}{A + 1} \leq x,
\]

where the final inequality in (13) follows from Lemma 1. Similarly, if \( z_{n-k}^* \leq z_{n-m}^* \), then

\[
z_n^* = \frac{1}{z_n} = \left( \frac{A + z_{n-m}^*}{A + z_{n-k}^*} \right)^p \leq \left( \frac{A + z_{n-m}^*}{A + z_{n-k}^*} \right) \frac{A + 1}{A + 1} \leq x,
\]

and the lemma is proved. \( \square \)

Now, set

\[
D_n = \max_{n-s \leq i \leq n-1} \{ z_i^* \},
\]

for \( n \geq s \).

The following result is a simple consequence of Lemma 2 and (17).

**Lemma 3.** The sequence \( \{ D_i \} \) is monotonically nonincreasing in \( i \), for \( i \geq s \).

Since \( D_i \geq 1 \) for \( i \geq s \), Lemma 3 implies that, as \( i \) tends to infinity, the sequence \( \{ D_i \} \) converges to some limit, say \( D \), where \( D \geq 1 \).

Next, we have the following lemma concerning boundedness of solutions to equation (11).

**Lemma 4.** If \( p \in (0,1) \), then every positive solution of equation (11) is bounded.

**Proof.** First, note that each \( n \in \mathbb{N}_0 \) can be written in the form \( lk + i \), for some \( l \in \mathbb{N}_0 \) and \( i \in \{0,1,\ldots,k-1\} \). Let \( l_0 = l_0(i) \) be the smallest element of \( \mathbb{N}_0 \) such that \( l_0 k + i \geq m \). From (11) and since \( y_n > A \) for every \( n \geq 0 \), we have that

\[
y_{lk+i} = A + \frac{y_{l(1)k+i}}{y_{l(i-1)k+i}} < A + \frac{y_{l(1)k+i}}{A^p},
\]

for every \( l \in \mathbb{N}_0 \) and \( i \in \{0,1,\ldots,k-1\} \) such that \( lk + i \geq m \). Let \( (u_{l(i)}^{(i)})_{l \in \mathbb{N}} \) be the solution of the difference equation

\[
u_{l(i)}^{(i)} = A + \frac{u_{l-1(i)}}{A^p}, \quad u_{l(i)}^{(i)} = y_{l(l_0-1)+i},
\]
By (18) and induction we see that \( y_{(l-1)k+i} \leq u_l^{(i)} \), \( l \geq l_0 \). Hence it is enough to prove that the sequences \( (u_l^{(i)})_{l \geq l_0}, \ i \in \{0, 1, \ldots, k-1\} \), are bounded.

Since the function
\[
f(x) = A + \frac{x^p}{Ap}, \quad x \in (0, \infty),
\]
is increasing and concave for \( p \in (0, 1) \) it follows that there is a unique fixed point \( \bar{x} \) of the equation \( f(x) = x \) and that the function \( f \) satisfies the condition
\[
(f(x) - x)(x - \bar{x}) < 0, \quad x \in (0, \infty).
\]

Using this fact it is easy to see that if \( u_l^{(i)} \in (0, \bar{x}] \) the sequence is nondecreasing and bounded above by \( \bar{x} \) and if \( u_l^{(i)} \geq \bar{x} \), it is nonincreasing and bounded below by \( \bar{x} \). Hence for every \( u_l^{(i)} \in (0, \infty) \), each of the sequences \( u_l^{(i)}, \ i \in \{0, 1, \ldots, k-1\} \), is bounded, from which the result follows.

\[\square\]

3. Convergence of Solutions to Equation (1)

In this section, we study the global attractivity of the positive solutions of equation (1). First, we give a proof of Theorem 1.

Proof of Theorem 1. Note that it suffices to show that the transformed sequence \( \{z^*_i\} \) converges to 1.

By the definition in (17), the values of \( D_i \) are taken on by entries in the sequence \( \{z^*_j\} \), and as well, by Lemma 2, \( z^*_i \in [1, D_i] \) for \( i \geq s \). Now, for any \( \epsilon > 0 \), we can find an \( N \) such that
\[
z^*_N \in [D, D + \epsilon],
\]
and for \( i \geq N - s \),
\[
z^*_i \in [1, D + \epsilon].
\]

Note that, similar to (18) and (14), (20) implies that
\[
\frac{1}{D + \epsilon} \leq z_{N-m}, z_{N-k} \leq D + \epsilon.
\]

We will show that \( D = 1 \), and from this, (7), (17) and the definition of \( D \), the result follows.

Now, suppose \( D > 1 \), and note that (19) implies that \( z_N \neq 1 \).

First, consider the case \( z_N > 1 \). Then, from (19), we have that
\[
z_N = z^*_N \in [D, D + \epsilon].
\]

Solving for \( z_{n-k} \) in (6), and employing (22) and (21), gives
\[
D + \epsilon \geq z_{n-k} = z_N^{1/p} (A + z_{N-m}) - A \\
\geq D^{1/p} \left( A + \frac{1}{D + \epsilon} \right) - A \\
\geq D^{\frac{1}{p}} \left( A + \frac{1}{D + \epsilon} \right) - A.
\]

This implies that
\[
\left( \frac{A + D + \epsilon}{A + \frac{1}{D + \epsilon}} \right) \geq D^{\frac{1}{p}}.
\]
Assume now that \( z_N < 1 \). Then, 
\[
\frac{1}{z_N} = z_N' \in [D, D + \epsilon].
\] (25)

From (9), and employing (21) and (25), it follows that 
\[
D + \epsilon \geq z_{N-m} = \left( z_N' \right)^{1/p} (A + z_{N-k}) - A \\
\geq D^{1/p} \left( A + \frac{1}{D + \epsilon} \right) - A \\
\geq D^{\frac{1}{p+1}} \left( A + \frac{1}{D + \epsilon} \right) - A.
\] (26)

From (26) we have that (24) holds in this case, as well. Since \( \epsilon > 0 \) was arbitrary and \( D > 1 \), by Lemma 1 we arrive at a contradiction, which implies that \( D = 1 \), and the theorem follows.

\[ \square \]

Remark 3. Note that the above argument breaks down when \( p > (A + 1)/2 \). In particular, we have that for \( p > (A + 1)/2 \),
\[
\left( \frac{A + x}{A + 1/x} \right)^p > x,
\]
for \( x = 1 + \epsilon \), for sufficiently small \( \epsilon > 0 \). To see this, similar to (9), set
\[
h_{A,p}(x) = p\left( \ln(A + x) - \ln(Ax + 1) \right) - (1 - p)\ln x,
\]
and note that the condition \( h_{A,p}(x) > 0 \) is equivalent to (27). Now, \( h_{A,p}(1) = 0 \) and
\[
h'_{A,p}(1) = \frac{2p - (A + 1)}{(A + 1)} > 0.
\]

Hence for sufficiently small \( \epsilon > 0 \), \( h_{A,p}(1 + \epsilon) > 0 \).

The next theorem is devoted to the case \( p \in (0, 1) \) and \( A \geq 1 \). It is simpler than Theorem 1 and is essentially a consequence of the boundedness result in Lemma 1.

**Theorem 2.** Suppose that \( m, k \geq 1 \), \( p \in (0, 1) \) and \( A \geq 1 \). If the sequence \( \{y_i\} \) satisfies (11) with \( y_{s-s}, y_{s-s+1}, \ldots, y_{s-1} \in (0, \infty) \) where \( s = \max\{m, k\} \), then, \( \{y_i\} \) converges to the unique equilibrium \( A + 1 \).

**Proof.** By Lemma 1 every solution \( \{y_n\} \) of equation (11) is bounded which implies that there are finite lim inf \( y_n = \lambda \) and lim sup \( y_n = \Lambda \). Assume to the contrary that \( \lambda < \Lambda \). Taking the lim inf and lim sup in (11) it follows that
\[
A + \frac{\lambda^p}{\Lambda^p} \leq \lambda < \Lambda \leq A + \frac{\Lambda^p}{\lambda^p}.
\]
From this and since \( p \in (0, 1) \), it follows that
\[
A\Lambda^p + \lambda^p \leq \Lambda^p \lambda < \Lambda\lambda^p \leq A\lambda^p + \lambda^p;
\]
i.e.,
\[
(A - 1)\Lambda^p < (A - 1)\lambda^p.
\]
Since \( A \geq 1 \), this is impossible. Therefore we have that \( \lambda = \Lambda \), which implies the result. \[ \square \]
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