SUBMANIFOLDS OF REAL ALGEBRAIC VARIETIES

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ABSTRACT. By the Nash-Tognoli theorem, each compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic set, called an algebraic model of $M$. We construct algebraic models $X$ of $M$ with controlled behavior of the group of cohomology classes represented by algebraic subsets of $X$.

1. INTRODUCTION

A compact smooth (of class $C^\infty$) manifold is said to be a boundary if it is diffeomorphic to the boundary of a compact smooth manifold with boundary. By convention, the empty manifold is a boundary. The goal of the present paper is to demonstrate that boundaries play a surprisingly interesting role in real algebraic geometry (cf. [4, Theorem 1.4] for the first result of this kind).

All real algebraic varieties in this paper are assumed to be affine (that is, isomorphic to an algebraic subset of $\mathbb{R}^n$ for some $n$). For background material on real algebraic geometry the reader may consult [2]. Every real algebraic variety carries also the Euclidean topology, induced by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a compact nonsingular real algebraic variety $X$, we denote by $H^\text{alg}_d(X, \mathbb{Z}/2)$ the subgroup of $H_d(X, \mathbb{Z}/2)$ of homology classes represented by $d$-dimensional algebraic subsets of $X$; cf. [1, 2, 3, 5]. For technical reasons it is more convenient to work with cohomology groups. We set

$$H^\text{c,alg}_d(X, \mathbb{Z}/2) := D_X^{-1}(H^\text{alg}_d(X, \mathbb{Z}/2)),$$

where $c + d = \dim X$ and $D_X : H^c(X, \mathbb{Z}/2) \to H_d(X, \mathbb{Z}/2)$ is the Poincaré duality isomorphism. The groups $H^\text{c,alg}_d(-, \mathbb{Z}/2)$ are of fundamental interest in real algebraic geometry and will also be in the center of our attention here. Their basic properties and applications are surveyed in [3].

Let $M$ be a compact smooth manifold of dimension $m$. Denote by $[M]$ the fundamental class of $M$ in $H_m(M, \mathbb{Z}/2)$. For any smooth $d$-dimensional submanifold $N$ of $M$ (submanifolds are always assumed to be closed subsets), we let $[N]_M$ denote the homology class in $H_d(M, \mathbb{Z}/2)$ represented by $N$. We set $[N]_M = 0$ if $N$ is empty.

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Given a finite collection $\mathcal{F}$ of smooth submanifolds of $M$, we define

$$G_k(\mathcal{F}) := \{ u \in H^k(M, \mathbb{Z}/2) \mid \langle u, [N]_M \rangle = 0 \text{ for all } N \in \mathcal{F} \text{ with } \dim N = k \}.$$ 

Here, as usual, $\langle \cdot, \cdot \rangle$ stands for scalar product (Kronecker index). If there is no $k$-dimensional submanifold in $\mathcal{F}$, then $G_k(\mathcal{F}) := H^k(M, \mathbb{Z}/2)$. Clearly, $G_k(\mathcal{F})$ is a subgroup of $H^k(M, \mathbb{Z}/2)$.

Recall that $\mathcal{F}$ is said to be in general position if for each point $x$ in $M$, the collection $\mathcal{F}_x = \{ N \in \mathcal{F} \mid x \in N \}$ is either empty or otherwise codim$(\bigcap_{N \in \mathcal{F}_x} T_x N) = \sum_{N \in \mathcal{F}_x} \text{codim} T_x N$, where $T_x N$ is regarded as a subspace of the tangent space $T_x M$.

If $\mathcal{F}$ is in general position and $N_1, \ldots, N_s$ are in $\mathcal{F}$, then $N_1 \cap \ldots \cap N_s$ is a smooth submanifold of $M$.

**Definition 1.1.** A finite collection $\mathcal{F}$ of smooth submanifolds of $M$ is said to be admissible if:

(i) $\dim N < \dim M$ for all $N$ in $\mathcal{F}$;
(ii) $\mathcal{F}$ is in general position;
(iii) if $N_1, \ldots, N_s$ are in $\mathcal{F}$, then $N_1 \cap \ldots \cap N_s$ is in $\mathcal{F}$;
(iv) each submanifold in $\mathcal{F}$ has trivial normal vector bundle.

If $M'$ is another smooth manifold and if $\varphi : M' \to M$ is a smooth diffeomorphism, then

$$\varphi^* \mathcal{F} := \{ \varphi^{-1}(N) \mid N \in \mathcal{F} \}$$

is a collection of smooth submanifolds of $M'$. The collection $\varphi^* \mathcal{F}$ is admissible, provided $\mathcal{F}$ is admissible.

**Theorem 1.2.** Let $X$ be a compact nonsingular real algebraic variety and let $\mathcal{F}$ be an admissible collection of smooth submanifolds of $X$. If

$$H^k_{\text{alg}}(X, \mathbb{Z}/2) \subseteq G_k(\mathcal{F})$$

for some nonnegative integer $k$, then each $k$-dimensional manifold in $\mathcal{F}$ is a boundary.

Theorem 1.2 is easy to prove. Its purpose is to provide motivation for our main result, Theorem 1.3, below. By Tognoli’s theorem [14] (cf. also [2, Theorem 14.1.10] and, for a weaker but influential result, [12]), each compact smooth manifold $M$ is diffeomorphic to a nonsingular real algebraic variety, called an algebraic model of $M$. Constructing algebraic models satisfying some additional desirable conditions is an interesting and active area of research; cf. [1, 2, 3, 8, 9].

**Theorem 1.3.** Let $M$ be a compact smooth manifold and let $\mathcal{F}$ be an admissible collection of smooth submanifolds of $M$. If each manifold in $\mathcal{F}$ is a boundary, then there exist an irreducible algebraic model $X$ of $M$ and a smooth diffeomorphism $\varphi : X \to M$ such that

$$H^k_{\text{alg}}(X, \mathbb{Z}/2) \subseteq G^k(\varphi^* \mathcal{F})$$

for all nonnegative integers $k$.

Theorem 1.3 is useful for constructing examples such that $H^k_{\text{alg}}(-, \mathbb{Z}/2) \neq H^k(-, \mathbb{Z}/2)$.
Example 1.4. The $n$-fold product $T^n = S^1 \times \cdots \times S^1$ of the unit circle $S^1$ has an algebraic model $X$ with $H^k_{alg}(X, \mathbb{Z}/2) \neq H^k(X, \mathbb{Z}/2)$ for $1 \leq k \leq n - 1$. Indeed, suppose $n \geq 2$, since for $n = 1$ the assertion is void. Choose $n - 1$ distinct points $p_1, \ldots, p_{n-1}$ in $S^1$ and set

$$N_i = \{(x_1, \ldots, x_n) \in T^n \mid x_1 = \cdots = x_i = p_i\}$$

for $1 \leq i \leq n - 1$. Then $\mathcal{F} := \{N_1, \ldots, N_{n-1}\}$ is an admissible collection of smooth submanifolds of $T^n$. Moreover, $\text{codim}_{T^n} N = i$ and $N_i$ is a boundary for $1 \leq i \leq n - 1$. Hence the assertion follows from Theorem 1.3.

Theorems 1.2 and 1.3 will be proved in the next section.

2. Proofs

As usual, the $i$th Stiefel-Whitney class of a smooth manifold $M$ will be denoted by $w_i(M)$. Given a smooth submanifold $N$ of $M$, we set


Henceforth, both the homology class $[N]_M$ and cohomology class $[N]^M$ will be used. Basic properties of the products $\langle \cdot, \cdot \rangle, \cap, \cup$ familiar from algebraic topology [6] will be used without further explanation.

Lemma 2.1. Let $M$ be a compact smooth manifold and let $N$ be a $k$-dimensional smooth submanifold of $M$. If the normal vector bundle of $N$ in $M$ is trivial, then the following conditions are equivalent:

(a) $N$ is a boundary,

(b) $\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M), [N]_M \rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$,

(c) $\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup [N]^M, [M] \rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$.

Proof. Let $e : N \hookrightarrow M$ be the inclusion map. Then


Setting $w = w_{i_1}(M) \cup \ldots \cup w_{i_r}(M)$, we have

$$\langle w \cup [N]^M, [M] \rangle = \langle w, [N]^M \cap [M] \rangle = \langle w, [N]_M \rangle = \langle w, e_*([N]) \rangle = \langle e^*(w), [N] \rangle = \langle e^*(w_{i_1}(M)) \cup \ldots \cup e^*(w_{i_r}(M)), [N] \rangle = \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N] \rangle,$$

where the last equality holds since the triviality of the normal vector bundle of $N$ implies that $e^*(w_i(M)) = w_i(N)$ for all $i \geq 0$. The proof is complete since, by [13, Théorèmes IV.3, IV.10], $N$ is a boundary if and only if $\langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N), [N] \rangle = 0$ for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$. □

Proof of Theorem 1.2. Every cup product $w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)$ is in $H^k_{alg}(X, \mathbb{Z}/2)$, where $k = i_1 + \cdots + i_r$; cf. [1, 2, 3, 5]. By assumption, for any $k$-dimensional submanifold $N$ in $\mathcal{F}$, we have

$$\langle w_{i_1}(X) \cup \ldots \cup w_{i_r}(X), [N]_X \rangle = 0,$$

which completes the proof in view of Lemma 2.1. □
The proof of Theorem 1.3 requires further preparations. Given a smooth manifold $P$, let $\mathfrak{N}_s(P)$ denote the unoriented bordism group of $P$; cf. [7]. The following fundamental result will play a crucial role.

**Theorem 2.2.** Let $P$ be a smooth manifold. Two smooth maps $f : M \to P$ and $g : N \to P$, where $M$ and $N$ are compact smooth manifolds of dimension $d$, represent the same bordism class in $\mathfrak{N}_s(P)$ if and only if for every nonnegative integer $q$ and every cohomology class $v$ in $H^q(P,\mathbb{Z}/2)$, one has

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(v), [M] \rangle = \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N) \cup g^*(v), [N] \rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = d - q$.

Reference for the proof. [7, (17.3)].

If $W$ is a nonsingular real algebraic variety, then a bordism class in $\mathfrak{N}_s(W)$ is said to be algebraic if it can be represented by a regular map $h : Y \to W$, where $Y$ is a compact nonsingular real algebraic variety.

We will also make use of a certain construction from real algebraic geometry. Let $X$ be a compact nonsingular real algebraic variety. Define $\text{Alg}^l(X)$ to be the subset of $H^l(X,\mathbb{Z}/2)$ that consists of all elements $v$ for which there exist a compact irreducible nonsingular real algebraic variety $T$ (depending on $v$), two points $t_0$ and $t_1$ in $T$, and a cohomology class $z$ in $H^{l+1}(X \times T,\mathbb{Z}/2)$ such that

$$v = i_t^*(z) - i_{t_0}^*(z).$$

Here given $t$ in $T$, we let $i_t : X \to X \times T$ denote the map defined by $i_t(x) = (x, t)$ for all $x$ in $X$. For properties and an alternative definition of $\text{Alg}^l(-)$, the reader may refer to [10, 11]. Below we will need the following facts. The set $\text{Alg}^l(X)$ is a subgroup of $H^l_{\text{alg}}(X,\mathbb{Z}/2)$, which has the expected functorial property: if $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphism $f^* : H^l(Y,\mathbb{Z}/2) \to H^l(X,\mathbb{Z}/2)$ satisfies

$$f^*(\text{Alg}^l(Y)) \subseteq \text{Alg}^l(X);$$

cf. [1, p. 114].

**Proposition 2.3.** Let $X$ be a compact nonsingular real algebraic variety. If $u$ is in $H^k_{\text{alg}}(X,\mathbb{Z}/2)$ and $v$ is in $\text{Alg}^l(X)$, where $k + l = \dim X$, then $\langle u \cup v, [X] \rangle = 0$.

References for the proof. [10, Theorem 2.1] (cf. also [11, Theorem 4.4]).

**Example 2.4.** Given a positive integer $l$ let $B^l$ be an irreducible nonsingular real algebraic variety with precisely two connected components $B_0^l$ and $B_1^l$, each diffeomorphic to the unit $l$-sphere $S^l$ (for example $B^l = \{(x_0, \ldots, x_l) \in \mathbb{R}^{l+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \cdots + x_l^2 = 0\}$). Let $B = B^l_1 \times \cdots \times B^{l_r}$ and $B_0 = B^l_0 \times \cdots \times B_0^{l_r}$. Then

$$H^q(B_0,\mathbb{Z}/2) = \delta^*(H^q(B,\mathbb{Z}/2)) = \delta^*(\text{Alg}^q(B))$$

for all $q \geq 0$, where $\delta : B_0 \hookrightarrow B$ is the inclusion map. This assertion is a minor generalization of [11, Example 4.5].

For convenience we introduce the following notation. For any finite collection $\mathcal{F}$ of smooth submanifolds of a compact smooth manifold $M$, denote by $A^l(\mathcal{F})$ the subgroup of $H^l(M,\mathbb{Z}/2)$ generated by $\{[N]^M \mid N \in \mathcal{F}, \text{codim}_MN = l\}$. If $k + l = \dim M$, then

$$G^k(\mathcal{F}) = \{u \in H^k(M,\mathbb{Z}/2) \mid \langle u \cup v, [M] \rangle = 0 \text{ for all } v \in A^l(\mathcal{F})\}.$$
This follows immediately from the equality $\langle u \cup v, [M] \rangle = \langle u, D_M(v) \rangle$.

Proof of Theorem 1.3. Observe that $G^0(F)$ contains the subgroup of $H^0(M, \mathbb{Z}/2)$ generated by 1. On the other hand, if $Y$ is a compact irreducible nonsingular real algebraic variety, then $H^0_{\text{alg}}(Y, \mathbb{Z}/2)$ is the subgroup of $H^0(Y, \mathbb{Z}/2)$ generated by 1. Hence, without loss of generality, we can enlarge the collection $F$ by adding to it finitely many 0-dimensional submanifolds of $M$, each consisting of an even number of points. Thus we can assume that $F$ has the following property: if $M_1$ and $M_2$ are distinct connected components of $M$, then $\{x_1, x_2\}$ is in $F$ for some points $x_1$ in $M_1$ and $x_2$ in $M_2$.

Let $F = \{N_1, \ldots, N_s\}$ and $l_i = \text{codim}_M N_i$ for $1 \leq i \leq s$. Note that $l_i \geq 1$. We will use notation introduced in Example 2.4. There is a smooth map $f_i : M \rightarrow B^{l_i}$ such that $f_i(M) \subseteq B^{l_i}_0$ and $f_i^*(H^0(B^{l_i}, \mathbb{Z}/2))$ is the subgroup of $H^0(M, \mathbb{Z}/2)$ generated by $[N_i]^M$. This assertion follows from a well-known fact that if the normal vector bundle of a smooth submanifold $N$ of $M$ is trivial and $\text{codim}_M N = l \geq 1$, then one can find a smooth map $h : M \rightarrow S^l$ with $h^*(\lambda) = [N]^M$, where $\lambda$ is the unique generator of $H^0(S^l, \mathbb{Z}/2) \cong \mathbb{Z}/2$; cf. [13, Théorème II.2]. Setting $f = (f_1, \ldots, f_s) : M \rightarrow B = B^{l_1} \times \cdots \times B^{l_s}$, we have

$$f^*(H^q(B, \mathbb{Z}/2)) = A^q(F) \quad \text{for all } q \geq 1. \quad (1)$$

Let $c : M \rightarrow B$ be a constant map, whose single value is a point in $B_0$. Since $M$ has an algebraic model, the bordism class of $c$ in $\mathcal{R}_s(B)$ is algebraic. We claim that for every nonnegative integer $q$ and every cohomology class $b$ in $H^q(B, \mathbb{Z}/2)$,

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(b), [M] \rangle = \langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup c^*(b), [M] \rangle$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = \text{dim } M - q$. The claim is obvious if $q = 0$. For $q \geq 1$ condition (2) is equivalent to

$$\langle w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup f^*(b), [M] \rangle = 0.$$ 

The last equality holds in view of (1) and Lemma 2.1, and hence (2) is proved. It follows from (2) and Theorem 2.2 that $f$ and $c$ represent the same bordism class in $\mathcal{R}_s(B)$. In particular, the bordism class of $f$ in $\mathcal{R}_s(B)$ is algebraic. By [1, Theorem 2.8.4], there exist an algebraic model $X$ of $M$, a smooth diffeomorphism $\varphi : X \rightarrow M$, and a regular map $g : X \rightarrow B$ such that $g$ is homotopic to $f \circ \varphi$. It remains to show that $X$ and $\varphi$ have the properties stated in the theorem. This can be done as follows. Since $g$ is homotopic to $f \circ \varphi$, we get $\varphi^* \circ f^* = (f \circ \varphi)^* = g^*$ in cohomology, and hence for all $q \geq 0$,

$$\varphi^*(f^*(H^q(B, \mathbb{Z}/2))) = g^*(H^q(B, \mathbb{Z}/2)) = g^*(\text{Alg}^q(B)), \quad (3)$$

where the last equality follows from the inclusion $g(X) \subseteq B_0$ and Example 2.4. We also have

$$g^*(\text{Alg}^q(B)) \subseteq \text{Alg}^q(X), \quad (4)$$

the map $g : X \rightarrow B$ being regular. Conditions (1), (3), and (4) combined yield

$$\varphi^*(A^q(F)) \subseteq \text{Alg}^q(X) \quad \text{for all } q \geq 1. \quad (5)$$

Let $k$ and $l$ be integers satisfying $k \geq 0$, $l \geq 0$, and $k + l = m$, where $m = \text{dim } M = \text{dim } X$. Since $\varphi^*(A^l(F)) = A^l(\varphi^* F)$ and

$$G^k(\varphi^* F) = \{ u \in H^k(X, \mathbb{Z}/2) \mid \langle u \cup v, [X] \rangle = 0 \text{ for all } v \in A^l(\varphi^* F) \},$$

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it follows from (5) and Proposition 2.3 that
\[ H^k_{\text{alg}}(X, \mathbb{Z}/2) \subseteq G^k(\varphi^*\mathcal{F}) \]
for \( 0 \leq k \leq m - 1 \). In fact (6) holds for all \( k \geq 0 \), the equality \( G^k(\varphi^*\mathcal{F}) = H^k(X, \mathbb{Z}/2) \) being automatically satisfied if \( k \geq m \).

In order to complete the proof, we need to demonstrate that \( X \) is irreducible. This is obvious if \( M \) is connected. Suppose then that \( M \) is disconnected and \( X \) is reducible. Let \( X_1 \) and \( X_2 \) be distinct irreducible components of \( X \). There are points \( p_1 \) in \( X_1 \) and \( p_2 \) in \( X_2 \) such that \( P = \{p_1, p_2\} \) is in \( \varphi^*\mathcal{F} \) (see the beginning of the proof). In particular, \( [P]^X \) belongs to \( A^m(\varphi^*\mathcal{F}) = \varphi^*(A^m(\mathcal{F})) \), which in view of (5) implies that \( [P]^X \) is in \( \text{Alg}^m(X) \). If \( e : X_1 \hookrightarrow X \) is the inclusion map, then \( e^*(\text{Alg}^m(X)) \subseteq \text{Alg}^m(X_1) \), and hence \( e^*([P]^X) = \{[p_1]\}^X_1 \) is in \( \text{Alg}^m(X_1) \). However, \( \langle \{[p_1]\}^X_1, [X_1] \rangle = 1 \), which contradicts Proposition 2.3. Thus \( X \) is irreducible, as required. \( \square \)

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