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GORENSTEIN RINGS AND IRREDUCIBLE PARAMETER IDEALS

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Dedicated to Professor Shiro Goto on the occasion of his sixtieth birthday

Abstract. Given a Noetherian local ring (R, m) it is shown that there exists an integer ℓ such that R is Gorenstein if and only if some system of parameters contained in m^ℓ generates an irreducible ideal. We obtain as a corollary that R is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal.

1. Introduction

It is well-known that a commutative Noetherian local ring (R, m) is Gorenstein if and only if R is Cohen-Macaulay and some ideal generated by a system of parameters (s.o.p.) is irreducible. Perhaps less widely known is a result of Northcott and Rees which states that if every ideal generated by an s.o.p. (henceforth called a *parameter ideal*) is irreducible, then R is Cohen-Macaulay [NR, Theorem 1]. Thus, R is Gorenstein if and only if every parameter ideal is irreducible. There are, however, easy examples of non-Gorenstein rings possessing irreducible parameter ideals: $(y)R$ is irreducible in the local ring $R = \mathbb{Q}[[x, y]]/(x^2, xy)$, while $(y^n)R$ is reducible for all $n \geq 2$. In 1982, S. Goto showed that if there exists an s.o.p. $x_1, \ldots, x_d$ for R such that $(x_1^n, \ldots, x_d^n)$ is irreducible for all sufficiently large n, then R is Gorenstein [G, Proposition 3.4]. Rephrasing Goto's result, if R is not Gorenstein then for every s.o.p. $x_1, \ldots, x_d$ of R there exists an integer n (depending on the s.o.p.) such that $(x_1^n, \ldots, x_d^n)$ is reducible. In light of this, given a non-Gorenstein local ring R it is natural to ask whether there exists a *uniform* exponent n such that $(x_1^n, \ldots, x_d^n)$ is reducible for all s.o.p.'s $x_1, \ldots, x_d$ of R. Our main result (Theorem 2.7) gives an affirmative answer to this question:

Theorem. *Let (R, m) be a Noetherian local ring. Then there exists an integer ℓ such that R is Gorenstein if and only if some parameter ideal contained in m^ℓ is irreducible.*

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As a consequence, a local ring R is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal. The integer ℓ identified in this theorem may be taken to be the least integer δ such that the canonical map

$$\operatorname{Ext}^d_R(R/m^\delta, R) \to \varinjlim \operatorname{Ext}^d_R(R/m^n, R) \cong H^d_m(R)$$

is surjective after applying the functor $\operatorname{Hom}_R(R/m, -)$, where $d = \dim R$.

We note that the theorem above has previously been established when R is quasi-Buchsbaum or when R has finite local cohomologies and $H^i_m(R) \neq 0$ for at most two positive values of i. These results are either explicitly stated or obtained as easy consequences of [GSa1, Corollary 3.13], [GSa2, Theorem 4.2], [LR, Corollary 2.2], and [R, Corollary 3.4]

A condition weaker than what is studied here was investigated by Hochster: R is called *approximately Gorenstein* if every power of m contains an irreducible m-primary ideal. While approximately Gorenstein rings must have positive depth, they need not be Cohen-Macaulay. In fact, every complete Noetherian domain is approximately Gorenstein [Ho, Theorem 1.6].

2. Main results

As general references for terminology and well-known results, we refer the reader to [Mat] or [BH]. Throughout, R denotes a Noetherian local ring with maximal ideal m and M is a finitely generated R-module. The *socle* of M is defined to be $(0 :_M m) = \{x \in M \mid mx = 0\}$, and is denoted by $\operatorname{Soc}_R M$, or simply $\operatorname{Soc} M$ if there is no confusion about the ring. It is clear that $\operatorname{Soc}(-)$ is a left exact covariant functor in a natural way. We identify this functor with $\operatorname{Hom}_R(R/m, -)$.

In the sequel we adopt the following notation: For a sequence of elements $\mathbf{x} = x_1, \ldots, x_r$ of R and $t \in \mathbb{N}$ we let $\mathbf{x}^t$ denote the sequence $x_1^t, \ldots, x_r^t$. The Koszul chain complex on R with respect to $\mathbf{x}$ is denoted by $K(\mathbf{x})$. For an R-module M, the ith Koszul cohomology of M with respect to $\mathbf{x}$, denoted by $\mathrm{H}^i(\mathbf{x}; M)$, is the ith cohomology of $\operatorname{Hom}_R(K(\mathbf{x}), M)$. For $1 \leq s \leq t$ there exist canonical chain maps $\phi^t_s \colon K(\mathbf{x}^t) \to K(\mathbf{x}^s)$ which induce natural homomorphisms

$$\mathrm{H}^i(\mathbf{x}^s; M) \to \mathrm{H}^i(\mathbf{x}^t; M)$$

for all i. By [Gr, Theorem 2.8], we have $\varinjlim \mathrm{H}^i(\mathbf{x}^n; M) \cong \mathrm{H}^i_{(\mathbf{x})}(M)$, the ith local cohomology of M with support in $(\mathbf{x})$.

Following [Hu], we define the *limit closure* of a sequence $\mathbf{x}$ on a module M as follows:

Definition 2.1. Let $\mathbf{x} = x_1, \ldots, x_r \in R$ and let M be an R-module. Define

$$\{\mathbf{x}\}^{\lim}_M := \bigcup_{n \geq 0} \left(\left(x_1^{n+1}, \ldots, x_r^{n+1}\right) M :_M x_1^n \cdots x_r^n \right).$$

Remark 2.2. For a sequence $\mathbf{x} = x_1, \ldots, x_r$ of R consider the direct system $\{M/(\mathbf{x}^n)M\}_{n \geq 1}$ given by the maps

$$M/(\mathbf{x}^s)M \xrightarrow{(x_1 \cdots x_r)^{t-s}} M/(\mathbf{x}^t)M$$

for $1 \leq s \leq t$. Then the kernel of the canonical map

$$M/(\mathbf{x})M \to \varinjlim M/(\mathbf{x}^n)M \cong \mathrm{H}^r_{(\mathbf{x})}(M)$$

is $\{x_1, \ldots, x_r\}_M^{\lim}/(x_1, \ldots, x_r)M$.

The following proposition is known in a more general setting [St, Theorem 5.2.3], but we include a brief proof for the reader's convenience.

Proposition 2.3. *Let $x_1, \ldots, x_r \in m$ and let M be a finitely generated R-module. Assume that $\{x_1, \ldots, x_r\}_M^{\lim} = (x_1, \ldots, x_r)M$. Then $x_1, \ldots, x_r$ is a regular sequence on M.*

Proof. We proceed by induction on r. In the case $r = 1$, let $x = x_1$. Suppose $\{x\}_M^{\lim} = (x)M$ and $x\alpha = 0$ for some $\alpha \in M$. We claim that $\alpha \in (x^k)M$ for all $k \geq 0$. This is clearly true for $k = 0$, so suppose $\alpha = x^k\beta$ for some $k \geq 0$ and $\beta \in M$. Then $x^{k+1}\beta = 0$, and thus $\beta \in \{x\}_M^{\lim} = (x)M$. Hence, $\alpha \in \left(x^{k+1}\right) M$. As x is in the Jacobson radical and M is finitely generated, $\bigcap_k (x^k)M = 0$ by Krull's Intersection Theorem. Hence, $\alpha = 0$ and x is a non-zero-divisor on M.

Suppose now that $r > 1$. To complete the proof, we will show the following:

(1) $\{x_1, \ldots, x_{r-1}\}_M^{\lim} = (x_1, \ldots, x_{r-1})M$.
(2) x_r is a non-zero-divisor on $M/(x_1, \ldots, x_{r-1})M$.

Item (1) will allow us to use the inductive hypothesis to conclude that $x_1, \ldots, x_{r-1}$ is a regular sequence on M.

To prove (1), let $\alpha \in \{x_1, \ldots, x_{r-1}\}_M^{\lim}$. We claim that for all $k \geq 0$, $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^k)M$. Again by Krull's Intersection Theorem, this will imply that $\alpha \in (x_1, \ldots, x_{r-1})M$. The case $k = 0$ is clear, so suppose $\alpha = \omega + x_r^k\beta$ where $\omega \in (x_1, \ldots, x_{r-1})M$ and $\beta \in M$. Thus, $x_r^k\beta \in \{x_1, \ldots, x_{r-1}\}_M^{\lim}$. Hence, there exists $t \geq 0$ such that

$$(x_1 \cdots x_{r-1})^t x_r^k \beta \in \left(x_1^{t+1}, \ldots, x_{r-1}^{t+1}\right) M.$$

Multiplying by $(x_1 \cdots x_{r-1})^k x_r^t$, we obtain

$$(x_1 \cdots x_r)^{t+k}\beta \in \left(x_1^{t+k+1}, \ldots, x_{r-1}^{t+k+1}\right) M \subseteq \left(x_1^{t+k+1}, \ldots, x_r^{t+k+1}\right) M.$$

Hence, $\beta \in \{x_1, \ldots, x_r\}_M^{\lim} = (x_1, \ldots, x_r)M$. Thus, $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^{k+1})M$.

The proof of (2) is similar: Suppose $x_r\alpha \in (x_1, \ldots, x_{r-1})M$ for some $\alpha \in M$. We claim that $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^k)M$ for all $k \geq 0$. Suppose $\alpha = \omega + x_r^k\beta$ where $\omega \in (x_1, \ldots, x_{r-1})M$ and $\beta \in M$. Then $x_r\alpha = x_r\omega + x_r^{k+1}\beta$. Hence, $x_r^{k+1}\beta \in (x_1, \ldots, x_{r-1})M$. Multiplying by $(x_1 \cdots x_{r-1})^{k+1}$, we obtain that

$$(x_1 \cdots x_r)^{k+1}\beta \in (x_1^{k+2}, \ldots, x_{r-1}^{k+2})M \subseteq (x_1^{k+2}, \ldots, x_r^{k+2})M.$$

Hence, $\beta \in \{x_1, \ldots, x_r\}_M^{\lim} = (x_1, \ldots, x_r)M$ and $\alpha \in (x_1, \ldots, x_{r-1})\, M + \left(x_r^{k+1}\right) M$. □

Definition 2.4. Let (R, m) be a local ring, let M be a finitely generated R-module, and let $i \geq 0$. We will define the invariant $\ell_i(M)$. By applying $\operatorname{Ext}_R^i(-, M)$ to the system of surjections

$$\cdots \to R/m^3 \to R/m^2 \to R/m$$

we obtain a direct system whose limit is $\varinjlim \operatorname{Ext}_R^i(R/m^n, M) \cong \mathrm{H}_m^i(M)$. Since direct limits commute with $\operatorname{Hom}_R(R/m, -)$,

$$\varinjlim \operatorname{Soc} \operatorname{Ext}_R^i(R/m^n, M) \cong \operatorname{Soc} \varinjlim \operatorname{Ext}_R^i(R/m^n, M) \cong \operatorname{Soc} \mathrm{H}_m^i(M).$$

Since $\mathrm{H}^i_m(M)$ is Artinian, $\operatorname{Soc}\mathrm{H}^i_m(M)$ is finitely generated, and so there exists a non-negative integer δ such that the map $\operatorname{Soc}\operatorname{Ext}^i_R(R/m^t, M) \to \operatorname{Soc}\mathrm{H}^i_m(M)$ is surjective for all $t \geq \delta$. We define $\ell_i(M)$ to be the least integer δ with this property.

Let M be a finitely generated R-module. We say an ideal $q = (x_1, \ldots, x_d)$ is a *parameter ideal* for M if $d = \dim M$ and $\dim M/qM = 0$. The following proposition is essentially [GSa1, Lemma 3.12].

Proposition 2.5 ([GSa1, Lemma 3.12])**.** *Let (R, m) be a Noetherian local ring, let M be a finitely generated R-module of dimension d, and let $i \geq 0$. For all systems $x_1, \ldots, x_d = \mathbf{x}$ of parameters for M contained in $m^{\ell_i(M)}$, the map*

$$\operatorname{Soc}\mathrm{H}^i(\mathbf{x}; M) \to \operatorname{Soc}\mathrm{H}^i_m(M)$$

induced by the canonical map $\mathrm{H}^i(\mathbf{x}; M) \to \varinjlim \mathrm{H}^i(\mathbf{x}^n; M)$ is surjective.

Remark 2.6. There are two differences between the above result and [GSa1, Lemma 3.12]. The first difference is that in [GSa1, Lemma 3.12], instead of defining each integer $\ell_i(M)$ independently, the integer $\max_{i=1}^d \ell_i(M)$ is used so that one power of the maximal ideal works to make all of the canonical maps surjective on the socles. In this paper, we only need Proposition 2.5 for the dth local cohomology module.

The second difference is that the result [GSa1, Lemma 3.12] is stated for systems of parameters of the ring R instead of the module M. However, this is remedied in [GSa2, Corollary 2.15] by passing to the ring $R/\operatorname{Ann}_R M$.

The main result now follows readily from the two propositions above:

Theorem 2.7. *Let (R, m) be a Noetherian local ring, let M be a finitely generated R-module of dimension d, and let $\ell = \ell_d(M)$. Then M is Cohen-Macaulay with Cohen-Macaulay type $r(M) = 1$ if and only if some parameter ideal q for M contained in m^ℓ has the property that qM is irreducible.*

Proof. It suffices to show that if there exists a system of parameters $\mathbf{x} = x_1, \ldots, x_d$ for M contained in m^ℓ such that $(\mathbf{x})M$ is irreducible, then M is Cohen-Macaulay. Let ϕ denote the canonical homomorphism from $H^d(\mathbf{x}; M) \cong M/(\mathbf{x})M$ to

$$\varinjlim H^d(\mathbf{x}^n; M) \cong \mathrm{H}^d_m(M).$$

By Remark 2.2 we have an exact sequence

$$0 \longrightarrow \frac{\{\mathbf{x}\}^{\lim}_M}{(\mathbf{x})M} \longrightarrow \frac{M}{(\mathbf{x})M} \xrightarrow{\phi} \mathrm{H}^d_m(M).$$

Applying the socle functor and using Proposition 2.5 we obtain the exact sequence

$$0 \longrightarrow \operatorname{Soc}\frac{\{\mathbf{x}\}^{\lim}_M}{(\mathbf{x})M} \longrightarrow \operatorname{Soc}\frac{M}{(\mathbf{x})M} \longrightarrow \operatorname{Soc}\mathrm{H}^d_m(M) \longrightarrow 0.$$

Since $\mathrm{H}^d_m(M)$ is a non-zero Artinian module, it has a non-zero socle. Since $(\mathbf{x})M$ is irreducible, $M/(\mathbf{x})M$ has a one-dimensional socle. Hence, $\operatorname{Soc}(\{\mathbf{x}\}^{\lim}_M/(\mathbf{x})M) = 0$, which implies $\{\mathbf{x}\}^{\lim}_M = (\mathbf{x})M$. By Proposition 2.3, we see that $\mathbf{x}$ is M-regular, M is Cohen-Macaulay and $r(M) = \dim_{R/m} \operatorname{Soc} H^d_m(M) = 1$. □

Corollary 2.8. *Let (R, m) be a Noetherian local ring. Then R is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal.*

Proof. Take $M = R$ in Theorem 2.7. □

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