PENCILS AND INFINITE DIHEDRAL COVERS OF $\mathbb{P}^2$

ENRIQUE ARTAL BARTOLO, JOSÉ IGNACIO COGOLLUDO, AND HIRO-O TOKUNAGA

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Abstract. In this work we study the connection between the existence of finite dihedral covers of the projective plane ramified along an algebraic curve $C$, infinite dihedral covers, and pencils of curves containing $C$.

Introduction

Let us consider a reduced plane curve $C \subset \mathbb{P}^2$. The third author has extensively studied algebraic conditions for the existence of dihedral covers of $\mathbb{P}^2$ ramified along $C$. In this paper, $C$ will be supposed to have two irreducible components $C_1$ and $C_2$ with the purpose of studying the existence of $D_{2n}$-covers of $\mathbb{P}^2$ branched at $2C_1 + nC_2$, for $n$ odd (see the comments before Theorem 1 for the notation). Such covers are related to epimorphisms $\pi_1(\mathbb{P}^2 \setminus C) \to D_{2n}$ sending a meridian of $C_1$ (resp. $C_2$) to a conjugate of $\sigma$ (resp. $\tau$); see subsection 1.2. Our goal is to derive the existence of $(\mathbb{Z}/2 \ast \mathbb{Z}/2)$-covers that factorize through such finite dihedral covers. This will be related to the existence of pencils of curves containing $C$ and the existence of infinite dihedral covers of $\mathbb{P}^2$. We will impose some restrictions on the curves $C$; some of them are necessary conditions for the existence of the above $D_{2n}$-covers and others will be set for the sake of simplicity.

(i) $\deg C_1$ is even: this is a necessary condition for the existence of the intermediate double cover ramified along $C_1$; see subsection 1.2.
(ii) $C_1$ has at most simple singularities: this condition will simplify some proofs.
(iii) $C_2 \cap \text{Sing}(C_1) = \emptyset$.
(iv) For each local branch $\varphi$ of $C_2$ at $P \in C_1 \cap C_2$, $(\varphi \cdot C_1)_P$ is even: this is also a necessary condition for the reducibility of the preimage of $C_2$ by the double cover ramified on $C_1$; see Proposition 1.3.

Let us introduce the general setting of this work. Let $X$ and $Y$ be normal projective varieties. Let $\pi : X \to Y$ be a finite surjective morphism. Under these conditions, the rational function field $\mathbb{C}(X)$ of $X$ is regarded as a field extension of $\mathbb{C}(Y)$, the function field of $Y$. We call $X$ a $D_{2n}$-cover of $Y$ if the field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois and its Galois group is isomorphic to the dihedral group $D_{2n}$ of order $2n$.

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The branch locus of \( \pi : X \to Y \), denoted by \( \Delta(X/Y) \) or \( \Delta_\pi \), is the subset of \( Y \) given by
\[
\Delta_\pi := \{ y \in Y \mid \pi \text{ is not a local isomorphism at } y \}.
\]
It is well known that \( \Delta_\pi \) is an algebraic subset of codimension 1 if \( Y \) is smooth; see [11]. Suppose that \( Y \) is smooth and let \( \Delta_\pi = B_1 + \cdots + B_r \) be its irreducible decomposition. We say \( \pi : X \to Y \) is branched at \( e_1B_1 + \cdots + e_rB_r \) if the ramification index along \( B_i \) is \( e_i \).

Let us state our main results:

**Theorem 1.** If \( D_{2n} \)-covers of \( \mathbb{P}^2 \) branched at \( 2C_1 + nC_2 \) exist for enough odd numbers \( n \in \mathbb{N} \), then they exist for any \( n \in \mathbb{N} \). Moreover, if \( F_i \) denote defining equations of \( C_i \), \( i = 1, 2 \), then there exist homogeneous polynomials \( G_1 \) and \( G_2 \) such that \( F_2 = G_1^2 - G_2^2 F_1 \).

**Corollary 2.** Under the hypothesis of Theorem 1 there exists an epimorphism from \( \pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2)) \) onto the infinite dihedral group \( \mathbb{Z}_2 * \mathbb{Z}_2 \).

**Remark 3.** It is possible to be more precise in the statement of Theorem 1 in terms of the curves \( C_1, C_2 \). Consider the standard resolution of the double cover of \( \mathbb{P}^2 \) ramified along \( C_1 \). By Proposition 1.5 the preimage of \( C_2 \) under this cover decomposes as \( C_2^+ \cup C_2^- \) into two irreducible components. As shown in equation (2), divisibility properties of \( C_2^+ - C_2^- \) are required for Theorem 1 to hold. For instance, let \( \nu \) be the self-intersection of \( C_2^+ - C_2^- \) and assume that \( \nu \neq 0 \); then the existence of a single \( D_{2n} \)-cover of \( \mathbb{P}^2 \) branched at \( 2C_1 + nC_2 \), where \( n^2 \) does not divide \( \nu \), is enough for Theorem 1 to hold.

1. Preliminaries

1.1. Topology of a double cover of \( \mathbb{P}^2 \).

Let \( B \) be a reduced plane curve of even degree \( d \). Assume that singularities of \( B \) are all simple. Let \( \delta : Z \to \mathbb{P}^2 \) be a double cover branched at \( B \) and let \( \mu : \tilde{Z} \to Z \) be the canonical resolution; see [6].

**Lemma 1.1.** \( \tilde{Z} \) is simply connected.

**Proof.** By using results on the simultaneous resolution ([8], [4]) we know that if \( (S,0) \subset (\mathbb{C}^3,0) \) is a double simple singularity, then the total space of its resolution is \( (\mathbb{C}^\infty) \) diffeomorphic to its Milnor fiber; this implies that the surface \( \tilde{Z} \) obtained as the minimal resolution of the double covering of \( \mathbb{P}^2 \) ramified along a curve of even degree \( 2m \) having only simple singularities is diffeomorphic to the double covering of \( \mathbb{P}^2 \) ramified along a smooth curve of degree \( 2m \). We assume that \( B \) is smooth. In this case, \( \tilde{Z} = Z \). If \( B \) is smooth, \( \pi_1(\mathbb{P}^3 \setminus B) \cong \mathbb{Z}/d\mathbb{Z} \). Hence \( \pi_1(Z \setminus \delta^{-1}(B)) \cong \mathbb{Z}/(d/2)\mathbb{Z} \), and it is generated by a meridian around \( \delta^{-1}(B) \). In \( Z \), this lasso is homotopic to zero. Hence \( \pi_1(Z) = \{1\} \).

**Corollary 1.2.** \( \text{Pic}(\tilde{Z}) = \text{NS}(\tilde{Z}); \text{Pic}(\tilde{Z}) \) is a lattice with respect to the intersection pairing.

1.2. Dihedral covers.

To present \( D_{2n} \), we use the notation
\[
D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle,
\]
There exists a line bundle \( L \) on \( Z \) such that \( D - \sigma^* D \sim nL \), where \( \sim \) means linear equivalence.

Let us also suppose that either \( n \) is odd, or \( n \) is even and \( Y \) is simply connected. Then there exists a \( D_{2n} \)-cover \( \pi : X \rightarrow Y \) such that \( D(X/Y) = Z \) and \( \pi \) is branched at \( 2\Delta_\delta + n\delta(D) \).

**Proof.** If \( n \) is odd, our statement is a special case of [8], Remark 3.1 and a similar argument to the proof of [8] Proposition 1.1, the result follows.

**Corollary 1.4.** Suppose that \( Y \) is simply connected. If \( \sigma^*_D \sim D \), then there exists a \( D_{2n} \)-cover of \( Y \) branched at \( 2\Delta_\delta + nf(D) \) for any \( n \geq 3 \).

As for a necessary condition for the existence of \( D_{2n} \)-covers, we have the following.

**Proposition 1.5** ([7] §2). Let \( \pi : X \rightarrow Y \) be a \( D_{2n} \)-cover such that \( D(X/Y) \) is smooth. Then there exist a (possibly empty) effective divisor \( D_1 \) and a line bundle \( L \) on \( D(X/Y) \) satisfying the following conditions:

1. \( D_1 \) and \( \sigma^* D_1 \) have no common components.
2. \( D_1 - \sigma^* D_1 \sim nL \).
3. \( \Delta_\delta(D(X/Y)) = \text{Supp}(D_1 + \sigma^* D_1) \).
4. The ramification index along \( D_{1,j} \) is \( \frac{n}{\gcd(a_j, n)} \), where \( D_1 = \sum_j a_j D_{1,j} \) \((a_j > 0)\) is the irreducible decomposition.

**Corollary 1.6.** Let \( D \) be an irreducible component of \( \beta_1(\pi)(\Delta_{\beta_2(\pi)}) \). Then the divisor \( \beta_1(\pi)^* D \) is of the form \( D' + \sigma^* D' \) for some irreducible divisor on \( D(X/Y) \).

In other words, \( \beta_2(\pi) \) is not branched along any irreducible divisor \( D \) with \( D = \sigma^* D \).

2. Certain \( D_{2n} \)-Covers of Algebraic Surfaces

Let \( \Sigma_o \) be a smooth projective surface. Let \( C_1 \) and \( C_2 \) be reduced divisors on \( \Sigma_o \) such that

- \( C_1 \) has at most simple singularities;
- \( C_2 \) is irreducible;
- \( C_2 \cap \text{Sing}(C_1) = \emptyset \);
- there exists a double cover \( \delta : Z \rightarrow \Sigma_o \) branched at \( C_1 \);
- its canonical resolution \( \mu : \tilde{Z} \rightarrow Z \) is simply connected.
Proposition 2.1. If there exists a $D_{2k}$-cover $\pi_k : S_k \to \Sigma_o$ branched at $2C_1 + kC_2$ for finitely many enough odd natural numbers $k$ (see Remark 3), then there exist $D_{2n}$-covers of $\Sigma_o$ branched at $2C_1 + nC_2$ for any integer $n \geq 3$.

Proof. By our assumption, $D(S_k/\Sigma_o) = Z$ and $\beta_1(S_k) = \delta$. Let

$$
\begin{array}{ccc}
Z & \xrightarrow{\mu} & \tilde{Z} \\
\delta \downarrow & & \downarrow \delta \\
\Sigma_o & \xrightarrow{q} & \Sigma
\end{array}
$$

denote the diagram where $q$ is the composition of the minimal sequence of blow-ups such that the pull-back $\tilde{Z}$ is smooth. Let $\tilde{S}_k$ be the $\mathbb{C}(S_k)$-normalization of $\Sigma$. The variety $\tilde{S}_k$ is a $D_{2k}$-cover of $\Sigma$ and we denote the cover morphism by $\tilde{\pi}_k$. Summing up, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
S_k & \xleftarrow{} & \tilde{S}_k \\
\downarrow & & \downarrow \\
Z & \xleftarrow{\mu} & \tilde{Z} \\
\delta \downarrow & & \downarrow \delta \\
\Sigma_o & \xleftarrow{q} & \Sigma
\end{array}
$$

Note that

$$
\Delta_\delta = q^{-1}C_1 + \text{ Some irreducible components of the exceptional set of } q,
$$

$$
\Delta_{\beta_1(\tilde{\pi}_k)} = \delta^{-1}(q^{-1}C_2) + \text{ Some irreducible components of the exceptional set of } \mu,
$$

where $\bullet^{-1}$ denote proper transforms.

By Corollary 1.6 $\delta^{-1}(q^{-1}C_2)$ is of the form $C_2^+ + C_2^-$, $\sigma_\delta^*(C_2^+) = C_2^-$. Since $\pi_k$ is branched at $2C_1 + kC_2$, by Proposition 1.3 for all $k$ as in the statement there exists a line bundle $L_k$ such that

$$
(1) \quad C_2^+ - C_2^- + R_k - \sigma_\delta^*R_k \sim kL_k
$$

where $\text{Supp}(R_k \cup \sigma_\delta^*R_k)$ is contained in the exceptional set of $\mu$. The subgroup $T$ of $\text{NS}(\tilde{Z})$ generated by the irreducible components of the exceptional divisors of $\mu$ is a negative definite sublattice in $\text{NS}(\tilde{Z})$. Let us consider the relation (1) in $\text{NS}(\tilde{Z})/T$. Then we have

$$
(2) \quad C_2^+ - C_2^- \equiv kL_k \mod T.
$$

Since $\text{NS}(\tilde{Z})/T$ is a finitely generated Abelian group, the hypothesis implies that $C_2^+ - C_2^-$ is a torsion element of $\text{NS}(\tilde{Z})/T$; we can apply Remark 3 since $C_2^+$ is orthogonal to $T$. Hence there exists a certain $\ell \in \mathbb{N}$ such that $\ell(C_2^+ - C_2^-) \in T$. Put

$$
\ell(C_2^+ - C_2^-) = \sum_i c_i \Theta_i,
$$

where $\Theta_i$’s denote the irreducible components of the exceptional divisor of $\mu$. Since $C_2$ does not pass through the singularities of $C_1$ then $\Theta_i \cdot C_2^+ = 0$ for all $i$. Hence $\ell(C_2^+ - C_2^-) = 0$, and as $T$ is a free $\mathbb{Z}$-module generated by $\Theta_i$’s, then $C_2^+ = C_2^-$ in $\text{NS}(\tilde{Z})$. Since $\tilde{Z}$ is simply connected, $\text{Pic}(\tilde{Z}) = \text{NS}(\tilde{Z})$. This implies $C_2^+ - C_2^- \sim 0$. Hence by Corollary 1.3 our statement follows. \qed
3. Proof of Theorem 1

Let $\delta : \hat{Z} \to \mathbb{P}^2$ be a double cover branched at $C_1$, and let $\mu : \hat{Z} \to Z$ be its canonical resolution. Since $C_1$ has at most simple singularities, $\hat{Z}$ is simply connected by Lemma 1.1. Hence the first half of Theorem 1 follows from Proposition 2.1.

We now go on to the second half. Assume that $C_1$ and $C_2$ are given by the equations:

$$ C_1 : F_1(U, V, W) = 0, $$

$$ C_2 : F_2(U, V, W) = 0. $$

Since $C_2 \sim C_2$, there exists a rational function $\varphi \in \mathbb{C}(\hat{Z})(= \mathbb{C}(Z))$ such that

$$ (\varphi) = C_2^+ - C_2^-.$$

Put $\theta_n = \sqrt[n]{\varphi}$ ($n \geq 3$) and consider the $\mathbb{C}(Z)(\theta_n)$-normalization $S_n$ of $Z$. We denote the induced covering morphism $S_n \to Z$ by $g_n$.

**Lemma 3.1.** $S_n$ is a $D_{2n}$-cover of $\mathbb{P}^2$ branched at $2C_1 + nC_2$.

**Proof.** Since $\varphi \neq 1/\varphi$, $\mathbb{C}(Z) = \mathbb{C}(\mathbb{P}^2)(\varphi)$ and this implies that $\mathbb{C}(S_n) = \mathbb{C}(\mathbb{P}^2)(\theta_n)$ and $[\mathbb{C}(S_n) : \mathbb{C}(\mathbb{P}^2)] = 2n$. One can see that $\mathbb{C}(S_n)/\mathbb{C}(\mathbb{P}^2)$ is a $D_{2n}$-extension, as a $D_{2n}$-action over $\mathbb{C}(\mathbb{P}^2)$ is given by $\theta_n^2 = 1/\theta_n$ and $\theta_n^n = \zeta_n\theta_n$, $\zeta_n = \exp(2\pi \sqrt{-1}/n)$. Hence $\delta \circ g_n : S_n \to \mathbb{P}^2$ is a $D_{2n}$-cover. As $(\varphi) = C_2^+ - C_2^-$ and $C_2^+ \cup C_2^-$ is contained in the smooth part of $Z$, the branch locus of $g_n$ is $(C_2^+ + C_2^-)$ and the ramification index along $C_2^+ \cup C_2^-$ is $n$. Since the branch locus of $\delta$ is $C_1$, $\delta \circ g_n$ is branched at $2C_1 + nC_2$. \qed

Put $u := \varphi + 1/\varphi$. As $u$ is $\sigma$-invariant, there exists a rational function $\psi \in \mathbb{C}(\mathbb{P}^2)$ such that $\delta^* \psi = u$.

**Lemma 3.2.** The polar divisor of $\psi$ is $C_2$.

**Proof.** Let $C_\infty$ be the polar divisor of $\psi$. Since the polar divisor of $\varphi + 1/\varphi$ is $C_2^+ + C_2^-$, we have $\delta^* C_\infty = C_2^+ + C_2^-$. \qed

Let $\varpi_n : \mathbb{P}^1 \to \mathbb{P}^1$ be a $D_{2n}$-cover given by

$$ t \mapsto \frac{1}{2} \left( t^n + \frac{1}{t^n} \right) =: s, $$

where $t, s$ are inhomogeneous coordinates. Let $\Phi_n : S_n \to \mathbb{P}^1$ and $\varpi_n : \mathbb{P}^2 \to \mathbb{P}^1$ be rational maps given by $\theta_n$ and $\psi$, respectively. The rational map $\Phi_n$ is $D_{2n}$-equivariant, and we have the following commutative diagram:

$$
\begin{array}{ccc}
S_n & \xrightarrow{\Phi_n} & \mathbb{P}^1 \\
\downarrow{\delta \circ g_n} & & \downarrow{\varpi_n} \\
\mathbb{P}^2 & \xrightarrow{\varpi_n} & \mathbb{P}^1
\end{array}
$$

From this diagram, we can infer that $S_n$ is obtained as a *rational* pullback by $\varpi_n$; note that any $D_{2n}$-cover is obtained as a rational pullback as above if $n$ is odd; see [1]. Since $\varpi_n$ is branched at $2[1 : \pm 1] + n[0 : 1]$, $[a : b] \equiv [1 : s]$ being a homogenous coordinate of $\mathbb{P}^1$, we may assume that the images of $C_1$ and $C_2$ are $[1 : 1]$ and $[0 : 1]$, respectively.
Following Lemma 3.2 we can write \( \psi := F_0/F_2 \), where \( F_0 \) is a homogeneous polynomial, \( \deg F_0 = \deg F_2 \). Then the images of the curves given by \( F_0 - F_2 = 0 \) and \( F_0 + F_2 = 0 \) under \( \mathbb{P}_n \) are \([1 : 1]\) and \([1 : -1]\). This implies that the divisors given by \( F_0 - F_2 = 0 \) and \( F_0 + F_2 = 0 \) are of the form \( C_1 + 2D_1 \) and \( 2D_2 \). Hence there exist homogeneous polynomials \( G_1 \) and \( G_2 \) such that \( F_0 + F_2 = G_1^2 \) and \( F_0 - F_2 = G_2^2 F_1 \), and we deduce

\[
F_2 = \frac{G_1^2 - G_2^2 F_1}{2}.
\]

The second half of Theorem 1 follows.

4. Pencils and Fundamental Groups

Let \( C \) be a complex projective plane curve. In this section we intend to exhibit the connection between the existence of pencils of curves related to \( C \) and the fundamental group of its complement \( X_C := \mathbb{P}^2 \setminus C \) from a topological point of view. We will apply it to curves satisfying the statement of Theorem 1.

Definition 4.1. Let \( D \) be a compact algebraic curve, let \( p_1, \ldots, p_r, q_1, \ldots, q_s \in D \) be distinct points and let \( n_1, \ldots, n_r \in \mathbb{Z}_{\geq 2} \). An orbifold \( D^{p_1 \cdots p_rq_1 \cdots q_s} \) is a punctured curve \( D \setminus \{q_1, \ldots, q_s\} \) where the points \( p_i \) are weighted with the integers \( n_i, i = 1, \ldots, r \). For the sake of simplicity sometimes it will be denoted by \( D^{n_1 \cdots n_r} \).

We may think that the charts around the points \( p_i \) are obtained as the quotient of disks in \( \mathbb{C} \) by the action of the \( n_i \)-roots of unity. This justifies the following definition.

Definition 4.2. The orbifold-fundamental group \( \pi_1^{orb}(D^{p_1 \cdots p_rq_1 \cdots q_s}; \ast) \), \( \ast \in D := D \setminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\} \) is defined as the quotient of \( \pi_1(D; \ast) \) by the normal subgroup generated by \( \mu_i^{n_i}, i = 1, \ldots, r \), where \( \mu_i \) is a meridian of \( p_i \).

Examples 4.3. We will fix \( D = \mathbb{P}^1 \).

1. \( \pi_1^{orb}(\mathbb{P}^1)_{2,2,n} ; \ast \) is the dihedral group \( D_{2n} \).
2. \( \mathbb{P}_{p,q,r} := \pi_1^{orb}(\mathbb{P}^1)_{p,q,r} ; \ast \) is the corresponding triangle group.
3. \( \mathbb{P}_{n_1,\ldots,n_r} := \pi_1^{orb}(\mathbb{P}^1)_{n_1,\ldots,n_r} ; \ast \) is the free product \( \mathbb{Z}/n_1 \ast \cdots \ast \mathbb{Z}/n_r \).

Let us now fix a connected smooth projective surface \( X \), a connected smooth projective curve \( \Gamma \) and a non-constant rational map \( \tilde{\rho} : X \rightarrow \Gamma \). Let \( C \subset X \) be a compact curve such that \( \tilde{\rho} \) is well defined on \( X \setminus C \) and let \( A := \Gamma \setminus \tilde{\rho}(X \setminus C) \), which is a finite set of points. We denote by \( \rho : X \setminus C \rightarrow \Gamma \setminus A \) the restriction of \( \tilde{\rho} \), which is assumed to have connected fibers.

Let \( p \in \Gamma \setminus A \); we consider the divisor \( \rho^*(p) \), which is the restriction of \( \tilde{\rho}^*(p) \) to \( X \setminus C \). For each \( p \) we denote \( n_p \) as the gcd of the multiplicities of \( \rho^*(p) \). We consider the orbifold \( \Gamma_p := \Gamma^A \{ (p, n_p) | n_p \geq 1 \} \). Fix \( q \in \Gamma \setminus A \) such that \( n_q = 1 \) and \( \ast \in \rho^{-1}(q) \).

Proposition 4.4. The mapping \( \rho \) induces a natural epimorphism

\[
\rho_* : \pi_1(X \setminus C; \ast) \twoheadrightarrow \pi_1^{orb}(\Gamma_p; q).
\]

Proof. Let us denote \( \tilde{C} := C \cup \bigcup_{n_p > 1} \tilde{\rho}^*(p) \) and \( \Gamma_1 := \Gamma \setminus (A \cup \{(p, n_p) | n_p > 1\}) \). The rational map \( \tilde{\rho} \) induces a well-defined surjective morphism \( \rho_1 : X \setminus \tilde{C} \rightarrow \Gamma_1 \). It
is a standard fact that \( \rho \) induces an epimorphism
\[
\pi_1(X \setminus \tilde{C}; *) \twoheadrightarrow \pi_1(\Gamma_1; q).
\]
Recall that \( \pi_1(X \setminus C; *) \) is the quotient of \( \pi_1(X \setminus \tilde{C}; *) \) by the subgroup generated by the components of \( \tilde{C} \) not in \( C \). The condition on the gcd of multiplicities guarantees the following commutative diagram which gives the result:
\[
\begin{array}{ccc}
\pi_1(X \setminus \tilde{C}; *) & \twoheadrightarrow & \pi_1(\Gamma_1; q) \\
\downarrow & & \downarrow \\
\pi_1(X \setminus C; *) & \twoheadrightarrow & \pi_{\text{orb}}^1(\Gamma_\rho; q).
\end{array}
\]
Let us note that a meridian of a component of \( \tilde{C} \) not in \( C \) is sent by \( \rho \) to the power of a meridian \( \mu_i \); the power is a multiple of \( n_i \).

We say that a pencil \( \mathcal{P} := \{ F_p \} \subseteq \mathbb{P}^1 \) contains \( C \) if each irreducible component of \( C \) is contained in a member of \( \mathcal{P} \). Let \( A \subseteq \mathbb{P}^1 \) be the subset of \( p \in \mathbb{P}^1 \) such that \( F_p^\text{red} \subset C \). Let \( n_p \) denote the gcd of the multiplicities of the components in \( F_p \) not contained in \( C \). We define the set \( B = \{ p \in \mathbb{P}^1 \setminus A \mid n_p > 1 \} \subseteq \mathbb{P}^1 \). Let us assume that \( \#A = n \) and \( B := \{ p_1, \ldots, p_r \} \), \( n_i := n_{p_i} \).

**Corollary 4.5.** There is a surjection from \( \pi_1(X_C) \) onto
\[
\mathbb{F}_{n_i; (n_1, \ldots, n_r)} := \langle x_1, \ldots, x_n, y_1, \ldots, y_r : \prod_{j=1}^n x_j \cdot \prod_{i=1}^r y_i = y_1^{n_1} = \cdots = y_r^{n_r} = 1 \rangle.
\]

**Remark 4.6.** If \( n'_i | n_i \), then \( \mathbb{F}_{n_i; (n_1, \ldots, n_r)} \) surjects onto \( \mathbb{F}_{n'_i; (n'_1, \ldots, n'_r)} \). Any \( n'_i \) equal 1 will be dropped. By doing so, we only add some ambiguity about the surjection, but this is not relevant for our purposes.

**Example 4.7.** Let \( C_6 \) be a Zariski sextic, that is, of equation \( D_2^3 + D_3^2 = 0 \), where \( D_i \) is a homogeneous polynomial in \( \mathbb{C}[x, y, z] \) of degree \( i \). The pencil generated by \( D_2^3 \) and \( D_3^2 \) has at least these two as special fibers. According to the notation of Corollary 4.5 we have that \( \pi_1(X_{C_6}) \) surjects onto \( \mathbb{F}_{1; (2, 3, n_3, \ldots, n_r)} \) and therefore (Remark 4.6) onto \( \mathbb{F}_{1; (2, 3)} = \mathbb{Z}_2 * \mathbb{Z}_3 \). Zariski proved in [10] that this is an isomorphism for generic choices.

**Proof of Corollary 4.5.** By Theorem 1 the pencil generated by \( G_1^2 \) and \( G_2^2F_1 \) contains \( F_2 \), therefore, using Corollary 4.5 there exists a surjection from \( \pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2)) \) onto \( \mathbb{F}_{1; (2, 2, n_3, \ldots, n_r)} \), and hence, by Remark 4.6 there exists a surjection onto \( \mathbb{F}_{1; (2, 2)} = \mathbb{Z}_2 * \mathbb{Z}_2 \).

5. Examples

**Example 5.1.** Let us suppose that there exists a pencil with three fibers \( 2A_1 + B_1, 2A_2 + B_2, nA_3 + B_3 \). Then the fundamental group of the complement of \( B_1 \cup B_2 \cup B_3 \) surjects onto \( \mathbb{F}_{2, 2, n} = D_2n \). The simplest example is the tricuspid quartic. Zariski proved in [10] that it lives in a pencil as in Example 4.7 if we add the double of the bitangent line. Then we have a surjection onto \( D_6 \) which is not an isomorphism since it is also proved in [10] that its fundamental group has order 12.
Example 5.2. Let $C$ be a smooth conic and $L_1$, $L_2$, $L_3$ tangent lines at three different points $P_1$, $P_2$ and $P_3$ of $C$. The pencil $\mathcal{P}$ generated by $C$ and $L_1 + L_2$ contains as a special fiber $2L$, where $L$ is the line passing through $P_1$ and $P_2$. Let $f_n$ be the cover map $f_n : \mathbb{P}^2 \to \mathbb{P}^2$, $f_n(L_1, L_2, L_3) := [L_1^n : L_2^n : L_3^n]$. The pull-back $f^*\mathcal{P}$ of the pencil $\mathcal{P}$ is generated by $f_n^*C$ and $f_n^*L_1 + f_n^*L_2 = n(L_1 + L_2)$ and contains the curve $2f_n^*L$. A description of the cover $f_n^*C$ and a presentation of its fundamental group $\pi_1(\mathbb{P}^2 \setminus f_n^*C)$ can be found in [5]. By Corollary 4.5 $\pi_1(\mathbb{P}^2 \setminus f_n^*C)$ has a surjection onto $\mathbb{F}_{1;(2,n)} = \mathbb{Z}_2 \ast \mathbb{Z}_n$.

Example 5.3. In [1] we have studied curves having two irreducible components: a quartic $C_1$ having two singular points of types $\mathbb{A}_3$ and $\mathbb{A}_1$ and a smooth conic $C_2$ such that its intersection with $C_1$ produces a singular point of type $\mathbb{A}_{15}$. Let us drop the $\mathbb{A}_1$ point. Then it is easily seen that the moduli space of such curves has three connected components. Let us describe two of them:

- The tangent line $T$ at $\mathbb{A}_{15}$ passes through $\mathbb{A}_3$. In this case there is a pencil of quartics containing $C_1$ and $4T$ such that another element of the pencil is $C_2 + 2L$, where $L$ is the tangent line at $\mathbb{A}_3$. By Corollary 4.3 $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ has a surjection onto $\mathbb{F}_{1;(2,4)} = \mathbb{Z}_2 \ast \mathbb{Z}_4$.
- There exists a smooth conic $Q$ having four infinitely near points in common with $\mathbb{A}_{15}$ and tangent at $\mathbb{A}_3$. In this case there is again a pencil of quartics containing $C_1$, $2Q$ and $C_2 + 2L$. Therefore, $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ has a surjection onto $\mathbb{F}_{1;(2,2)} = \mathbb{Z}_2 \ast \mathbb{Z}_2$.

Example 5.4. Let us consider the family of curves of type $I$ described in [2]. When $D$ is rational, they satisfy the conditions of Theorem 1 and therefore there is a surjection $\pi_1(\mathbb{P}^2 \setminus (D \cup L_1 \cup L_2)) \to \mathbb{Z}_2 \ast \mathbb{Z}_2$ (Corollary 2). Note that there is yet another pencil that produces a surjection $\pi_1(\mathbb{P}^2 \setminus (D \cup L_1 \cup L_2)) \to \mathbb{Z}_2 \ast \mathbb{Z}_2$.

Consider the most general case, that is:

1. $C$ a rational arrangement of degree $2k + 1$ with an ordinary multiple point $P$ of multiplicity $2k - 1$ and at least $2k - 1$ nodes $Q_1, \ldots, Q_{2k-1}$.
2. $L_i$ a line tangent to $C$ at a point $P_i$, $i = 1, 2$.
3. $D_i$ a curve of degree $k$ with an ordinary multiple point at $P$ of multiplicity $k - 1$, passing through $P_i, Q_1, \ldots, Q_{2k-1}$.

The pencil generated by $L_1 + 2D_2$ and $L_2 + 2D_1$ contains $C$. Using a third line $L_3$ and the cover $f_n : \mathbb{P}^2 \to \mathbb{P}^2$ described above, one obtains curves $f_n^*C$ whose fundamental group $\pi_1(\mathbb{P}^2 \setminus f_n^*C)$ surjects onto $\mathbb{Z}_2 \ast \mathbb{Z}_2$ (for $n$ even) and such that $\pi_1(\mathbb{P}^2 \setminus (f_n^*C \cup f_n^*D_1 \cup f_n^*D_2))$ surjects onto $\mathbb{Z}_n \ast \mathbb{Z}_n$.

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Departamento de Matemáticas, Universidad de Zaragoza, Campus Plaza San Francisco s/n, E-50009 Zaragoza, Spain
E-mail address: artal@unizar.es

Departamento de Matemáticas, Universidad de Zaragoza, Campus Plaza San Francisco s/n, E-50009 Zaragoza, Spain
E-mail address: jicogo@unizar.es

Department of Mathematics, Tokyo Metropolitan University, Minamiohsawa Hachioji, 192-0357 Tokyo, Japan
E-mail address: tokunaga@comp.metro-u.ac.jp