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PSEUDODIFFERENTIAL OPERATORS WITH C*-ALGEBRA-VALUED SYMBOLS: ABSTRACT CHARACTERIZATIONS

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Abstract. Given a separable unital C*-algebra C with norm $||\cdot||$, let E_n denote the Banach-space completion of the C-valued Schwartz space on $\mathbb{R}^n$ with norm $||f||_2 = ||\langle f, f\rangle||^{1/2}$, $\langle f, g\rangle = \int f(x)^* g(x)dx$. The assignment of the pseudodifferential operator $A = a(x, D)$ with C-valued symbol $a(x, \xi)$ to each smooth function with bounded derivatives $a \in \mathcal{B}^C(\mathbb{R}^{2n})$ defines an injective mapping O, from $\mathcal{B}^C(\mathbb{R}^{2n})$ to the set $\mathcal{H}$ of all operators with smooth orbit under the canonical action of the Heisenberg group on the algebra of all adjointable operators on the Hilbert module E_n. In this paper, we construct a left-inverse S for O and prove that S is injective if C is commutative. This generalizes Cordes' description of $\mathcal{H}$ in the scalar case. Combined with previous results of the second author, our main theorem implies that, given a skew-symmetric $n \times n$ matrix J and if C is commutative, then any $A \in \mathcal{H}$ which commutes with every pseudodifferential operator with symbol $F(x+J\xi)$, $F \in \mathcal{B}^C(\mathbb{R}^n)$, is a pseudodifferential operator with symbol $G(x - J\xi)$, for some $G \in \mathcal{B}^C(\mathbb{R}^n)$. That was conjectured by Rieffel.

1. Introduction

Let C be a separable unital C*-algebra with norm $||\cdot||$, and let $\mathcal{S}^C(\mathbb{R}^n)$ denote the set of all C-valued smooth functions on $\mathbb{R}^n$ which, together with all their derivatives, are bounded by arbitrary negative powers of $|x|$, $x \in \mathbb{R}^n$. We equip it with the C-valued inner product

$$\langle f, g\rangle = \int f(x)^* g(x)dx\,,$$

which induces the norm $||f||_2 = ||\langle f, f\rangle||^{1/2}$, and we denote by E_n its Banach space completion with this norm. The inner product $\langle\cdot,\cdot\rangle$ turns E_n into a Hilbert module [5]. The set of all (bounded) adjointable operators on E_n is denoted $\mathcal{B}^*(E_n)$.

Let $\mathcal{B}^C(\mathbb{R}^{2n})$ denote the set of all smooth bounded functions from $\mathbb{R}^{2n}$ to C whose derivatives of arbitrary order are also bounded. For each a in $\mathcal{B}^C(\mathbb{R}^{2n})$, a

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linear mapping from $\mathcal{S}^C(\mathbb{R}^n)$ to itself is defined by the formula

$$(Au)(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi, \tag{1}$$

where $\hat{u}$ denotes the Fourier transform,

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-iy\cdot\xi} u(y)dy\,.$$

As usual, we denote $A = a(x,D)$. This operator extends to an element of $\mathcal{B}^*(E_n)$ whose norm satisfies the following estimate. There exists a constant $k > 0$ depending only on n such that

$$||A|| \leq k \sup\{\, ||\partial_x^\alpha \partial_\xi^\beta a(x,\xi)||; (x,\xi) \in \mathbb{R}^{2n} \text{ and } \alpha, \beta \leq (1,\cdots,1)\,\}. \tag{2}$$

This generalization of the Calderón-Vaillancourt Theorem [1] was proven by Merklen [7, 8], following ideas of Hwang [4] and Seiler [11]. The case of $a(x,\xi) = F(x+J\xi)$, where $F \in \mathcal{B}^C(\mathbb{R}^n)$ and J is an $n\times n$ skew-symmetric matrix, had been proven earlier by Rieffel [10, Corollary 4.7].

The estimate (2) implies that the mapping

$$\mathbb{R}^{2n} \ni (z,\zeta) \longmapsto A_{z,\zeta} = T_{-z}M_{-\zeta}AM_\zeta T_z \in \mathcal{B}^*(E_n) \tag{3}$$

is smooth (i.e., C^∞ with respect to the norm topology), where T_z and M_ζ are defined by $T_z u(x) = u(x-z)$ and $M_\zeta u(x) = e^{i\zeta\cdot x}u(x)$, $u \in \mathcal{S}^C(\mathbb{R}^n)$. That follows just as in the scalar case [3, Chapter 8].

Definition 1. We call *Heisenberg smooth* an operator $A \in \mathcal{B}^*(E_n)$ for which the mapping (3) is smooth, and we denote by $\mathcal{H}$ the set of all such operators.

The elements of $\mathcal{H}$ are the smooth vectors for the action of the Heisenberg group on $\mathcal{B}^*(E_n)$ given by the same formula as the standard one in the scalar case (i.e., when C is the algebra $\mathbb{C}$ of complex numbers and then $E_n = L^2(\mathbb{R}^n)$, and we denote $\mathcal{S}^C(\mathbb{R}^n)$ and $\mathcal{B}^C(\mathbb{R}^{2n})$ by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^{2n})$, respectively).

We therefore have a mapping

$$\begin{array}{rcl} O : \mathcal{B}^C(\mathbb{R}^{2n}) & \longrightarrow & \mathcal{H} \\ a & \longmapsto & O(a) = a(x,D). \end{array} \tag{4}$$

In the scalar case, it is well known (this can be proven by a Schwartz-kernel argument) that if a pseudodifferential operator as in (1) vanishes on $\mathcal{S}(\mathbb{R}^n)$, then a must be zero. Let us show that this implies that O is injective for arbitrary C.

Given any complex-valued function u defined on $\mathbb{R}^n$, we denote by $\tilde{u} : \mathbb{R}^n \to C$ the function defined by

$$\tilde{u}(x) = u(x)\mathbf{1}_C, \tag{5}$$

where $\mathbf{1}_C$ denotes the identity of C. If $O(a) = 0$, the fact that $O(a)\tilde{u} = 0$ for every $u \in \mathcal{S}(\mathbb{R}^n)$ and the injectivity of O in the scalar case imply that $(x,\xi) \mapsto \rho(a(x,\xi))$ vanishes identically, for every $\rho \in C^*$, the (Banach space) dual of C. We then get $a \equiv 0$, as we wanted.

Our results in this paper can now be summarized in the following theorem, proven in Sections 2 and 3.

Theorem 1. *Let C be a unital separable C^*-algebra. There exists a linear mapping $S : \mathcal{H} \to \mathcal{B}^C(\mathbb{R}^{2n})$ such that $S \circ O$ is the identity operator. If C is commutative, then S is injective.*

Since an injective left-inverse is an inverse, we get

Corollary 1. *If C is commutative and an operator $A \in \mathcal{B}^*(E_n)$ is given, then the mapping defined in* (3) *is smooth if and only if $A = a(x, D)$ for some $a \in \mathcal{B}^C(\mathbb{R}^{2n})$.*

Theorem 1 and Corollary 1 were proven by Cordes [2] in the scalar case. His construction [3, Chapter 8] of the left-inverse S works also in the general case, if only one is careful enough to avoid mentioning trace-class or Hilbert-Schmidt operators. That is what we show in Section 2. His proof that S is injective, however, strongly depends on the fact that, when $C = \mathbb{C}$, $E_n = L^2(\mathbb{R}^n)$ is a Hilbert space. In the general commutative case, the lack of an orthonormal basis in E_n can be bypassed by still reducing the problem to copies of $L^2(\mathbb{R}^n)$, as shown at the beginning of Section 3. After this reduction, we are then able to follow the steps of Cordes' proof. Crucial for this strategy, our Lemma 5 is essentially [9, Lemma 2.4] specialized to commutative C*-algebras. In Section 4, we explain how Theorem 1 implies, in the commutative case, an abstract characterization, conjectured by Rieffel [10], of a certain class of C*-algebra-valued-symbol pseudodifferential operators.

The assumption of separability of C is needed to justify several results about vector-valued integration (see [8, Apêndice], for example), which are used without further comments throughout the text.

2. Left inverse for O

Given f and g functions from $\mathbb{R}^n$ to X (X will be either C or $\mathbb{C}$), let $f \otimes g : \mathbb{R}^{2n} \to X$ be defined by

$$f \otimes g(x, y) = f(x)g(y). \tag{6}$$

Given a vector space V, we denote by $V \overset{\mathtt{alg}}{\otimes} V$ the algebraic tensor product of V by itself. In case the elements of V are functions from $\mathbb{R}^n$ to X, $V \overset{\mathtt{alg}}{\otimes} V$ is isomorphic to the linear span of all function as in (6) with f and g in V.

Lemma 1. *Given $A \in B^*(E_n)$ mapping $\mathcal{S}^C(\mathbb{R}^n)$ to itself, there exists a unique operator $A \otimes I \in B^*(E_{2n})$ such that, for all f and g in $\mathcal{S}^C(\mathbb{R}^n)$,*

$$(A \otimes I)(f \otimes g) \;=\; Af \otimes g. \tag{7}$$

Proof. Let $L^2(\mathbb{R}^n; C)$ denote the set of equivalence classes (for the equality-almost-everywhere equivalence) of Borel measurable functions $f : \mathbb{R}^n \to C$ such that

$$\int ||f(x)||^2 dx \;<\; \infty,$$

and let $||f||_{L^2}$ denote the square root of the integral above. This is a Banach space containing $\mathcal{S}^C(\mathbb{R}^n)$ as a dense subspace. It follows from the inequality

$$||f||_2 \;\le\; ||f||_{L^2}, \text{ for all } f \in \mathcal{S}^C(\mathbb{R}^n),$$

that $L^2(\mathbb{R}^n; C)$ embeds in E_n as a $|| \cdot ||_2$-dense subspace.

Let $\mathtt{S}_n$ denote the set of all simple measurable functions from $\mathbb{R}^n$ to C. It takes an elementary but messy argument to show that $\mathtt{S}_n \overset{\mathtt{alg}}{\otimes} \mathtt{S}_n$ is $|| \cdot ||_{L^2}$-dense in $\mathtt{S}_{2n}$, which is dense in $L^2(\mathbb{R}^{2n}; C)$. Since $\mathtt{S}_n$ is dense in $L^2(\mathbb{R}^n; C)$, it follows that $L^2(\mathbb{R}^n; C) \overset{\mathtt{alg}}{\otimes} L^2(\mathbb{R}^n; C)$ is dense in $L^2(\mathbb{R}^{2n}; C)$. Since $\mathcal{S}^C(\mathbb{R}^n)$ is dense in

$L^2(\mathbb{R}^n;C)$, it follows that $\mathcal{S}^C(\mathbb{R}^n) \overset{\texttt{alg}}{\otimes} \mathcal{S}^C(\mathbb{R}^n)$ is $||\cdot||_{L^2}$-dense in $L^2(\mathbb{R}^{2n};C)$. Hence it is also $||\cdot||_2$-dense in E_{2n}.

Let $\phi : C \to \mathcal{B}^*(E_n)$ be given by left multiplication on $\mathcal{S}^C(\mathbb{R}^n)$, and denote by $E_n \otimes_\phi E_n$ the interior tensor product (given by ϕ) as defined in [5, page 41]. The fact that $\mathcal{S}^C(\mathbb{R}^n) \overset{\texttt{alg}}{\otimes} \mathcal{S}^C(\mathbb{R}^n)$ is dense in E_{2n} allows us to identify $E_n \otimes_\phi E_n$ with E_{2n} (notice that the space N in [5, Proposition 4.5] consists only of 0 in this case).

Given $A \in \mathcal{B}^*(E_n)$, it now follows from the more general result around [5, (4.6)] that there exists a unique $A \otimes I \in \mathcal{B}^*(E_{2n})$ such that $A \otimes I(f \otimes g) = Af \otimes g$ for all $f \otimes g \in E_n \overset{\texttt{alg}}{\otimes} E_n$. In particular, we get (7) for all f and g in $\mathcal{S}^C(\mathbb{R}^n)$. That (7) uniquely determines $A \otimes I$ also follows from the fact that $\mathcal{S}^C(\mathbb{R}^n) \overset{\texttt{alg}}{\otimes} \mathcal{S}^C(\mathbb{R}^n)$ is dense in E_{2n}. $\square$

Let us denote by $\gamma_1(t)$ and $\gamma_2(t)$, respectively, the fundamental solutions of $(\partial_t + 1)$ and $(\partial_t + 1)^2$ given by

$$\gamma_1(t) = \begin{cases} e^{-t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \qquad \text{and} \qquad \gamma_2(t) = \begin{cases} te^{-t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

We then define u and v in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by

$$v(\xi, \eta) = \gamma_1(\xi - \eta)/(1 + i\xi)^2 \tag{8}$$

and

$$u(x, \eta) = (1 + \partial_\eta)[(1 - i\eta)^2 \gamma_2(-x)\gamma_2(-\eta)e^{ix\eta}]. \tag{9}$$

The following lemma can be proven exactly as in the scalar case [3, Section 8.3].

Lemma 2. *If a and b in $\mathcal{B}^C(\mathbb{R}^2)$ are such that $(1 + \partial_z)^2(1 + \partial_\zeta)^2 a(z, \zeta) = b(z, \zeta)$, then we have, for all $(z, \zeta) \in \mathbb{R}^2$,*

$$a(z, \zeta) = \int_{\mathbb{R}^3} \overline{u(x, \eta)} e^{ix\xi} b(x + z, \xi + \zeta) v(\xi, \eta) d\xi\, dx\, d\eta\,. \tag{10}$$

We also omit the proof of the following lemma.

Lemma 3. *There exists a sequence v_l in $\mathcal{S}(\mathbb{R}) \overset{\texttt{alg}}{\otimes} \mathcal{S}(\mathbb{R})$ such that $v_l \to v$ in $L^2(\mathbb{R}^2)$ and*

$$\lim_{l \to \infty} \int_{\mathbb{R}^3} |u(x, \eta)| \cdot |v(\xi, \eta) - v_l(\xi, \eta)|\, d\xi\, dx\, d\eta \;=\; 0.$$

We are ready to define S when $n = 1$. Given $A \in \mathcal{H}$, let $B = f(0, 0)$, where $f : \mathbb{R}^2 \to \mathcal{B}^*(E_1)$ denotes the smooth function

$$f(z, \zeta) \;=\; (1 + \partial_z)^2(1 + \partial_\zeta)^2 A_{z,\zeta}\,, \tag{11}$$

with $A_{z,\zeta}$ as in (3). The group property allows one to show that $f(z, \zeta) = B_{z,\zeta}$ for all $(z, \zeta) \in \mathbb{R}^2$. We then define

$$(SA)(z, \zeta) \;=\; \sqrt{2\pi}\, \langle\, \tilde{u}, (B_{z,\zeta} F^* \otimes I)\tilde{v}\,\rangle, \tag{12}$$

where $F \in \mathcal{B}^*(E_1)$, $F^* = F^{-1}$, denotes the Fourier transform and $\langle \cdot, \cdot \rangle$ denotes the inner product of E_2. The meaning of $\tilde{}$ was defined in (5) and we are regarding, as explained in the proof of Lemma 1, $L^2(\mathbb{R}^2; C)$ as a subspace of E_2.

It is not hard to see that S maps $\mathcal{H}$ to $\mathcal{B}^C(\mathbb{R}^2)$ (this uses the inequality $||A\otimes I|| \leq ||A||$, which follows from [5, (4.6)]). Given $a \in \mathcal{B}^C(\mathbb{R}^2)$, let $c = SOa$. To prove that $S \circ O$ is the identity on $\mathcal{B}^C(\mathbb{R}^2)$, it is enough to show that

$$\int_{\mathbb{R}^2} [a(z,\zeta) - c(z,\zeta)] f(z,\zeta)\, dz\, d\zeta \; = 0, \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^2).$$

Indeed, if this is the case, then $(z,\zeta) \mapsto \rho(a(z,\zeta) - c(z,\zeta))$ vanishes identically for all $\rho \in C^*$, and the equality $a = c$ will therefore hold.

For each $l \in \mathbb{N}$, define $c_l(z,\zeta) = \sqrt{2\pi}\langle\, \tilde{u}, (B_{z,\zeta}F^* \otimes I)\tilde{v}_l\,\rangle$, where v_l is the sequence given by Lemma 3 and $B_{z,\zeta}$ is what one gets in (11) making $A = O(a)$. Since, for every $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned} &|| \int [c_l(z,\zeta) - c(z,\zeta)] f(z,\zeta) dz\, d\zeta\, || \\ &\qquad \leq ||u||_{L^2} \cdot ||B|| \cdot ||v - v_l||_{L^2} \cdot \int |f(z,\zeta)|\, dz\, d\zeta \;\longrightarrow\; 0, \end{aligned}$$

as $l \to \infty$, it is enough to show that

$$\lim_{l\to\infty} \int [c_l(z,\zeta) - a(z,\zeta)] f(z,\zeta) dz\, d\zeta \; = 0.$$

It follows from (2) that $B = O(b)$, for $b(x,\xi) = (1+\partial_x)^2(1+\partial_\xi)^2 a(x,\xi)$. We then get $B_{z,\zeta} = O(b_{z,\zeta})$, for $b_{z,\zeta}(x,\xi) = b(x+z,\xi+\zeta)$. Hence, if φ and ψ belong to $\mathcal{S}(\mathbb{R})$, then

$$[(B_{z,\zeta}F^* \otimes I)(\varphi \otimes \psi)](x,\eta) \;=\; \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} b(x+z,\xi+\zeta)\varphi(\xi)\psi(\eta) d\xi\,.$$

Using that $v_l \in \mathcal{S}(\mathbb{R}^n) \overset{\texttt{alg}}{\otimes} \mathcal{S}(\mathbb{R}^n)$, we then get

$$c_l(z,\zeta) = \int_{\mathbb{R}^3} \overline{u(x,\xi)} e^{ix\xi} b(x+z,\xi+\zeta) v_l(\xi,\eta) d\xi\, dx\, d\eta\,.$$

By Lemma 2, we then have

$$\begin{aligned} &\int [c_l(z,\zeta) - a(z,\zeta)] f(z,\zeta) dz\, d\zeta \\ &\quad = \int_{\mathbb{R}^2} [\int_{\mathbb{R}^3} \overline{u(x,\eta)} e^{ix\xi} b(x+z,\xi+\zeta)(v(\xi,\eta) - v_l(\xi,\eta)) d\xi\, dx\, d\eta\,] f(z,\zeta) dz\, d\zeta\,. \end{aligned}$$

Since $(x,\xi,\eta) \mapsto \overline{u(x,\eta)}(v(\xi,\eta) - v_l(\xi,\eta))$ belongs to $L^1(\mathbb{R}^3)$, we may interchange the order of integration and obtain that the above expression is bounded by

$$\sup_{x,\xi} ||b(x,\xi)|| \cdot ||f||_{L^1} \cdot \int_{\mathbb{R}^3} |u(x,\eta)| \cdot |v(\xi,\eta) - v_l(\xi,\eta)|\, d\xi\, dx\, d\eta\,,$$

which tends to zero, by Lemma 3, as we wanted.

This proves that S is a left-inverse for O when $n = 1$. We now comment on some of the changes needed to extend these ideas for arbitrary n. We have to replace u and v, respectively, by $u_n(x,\eta) = u(x_1,\eta_1)\cdots u(x_n,\eta_n)$ and $v_n(\xi,\eta) = v(\xi_1,\eta_1)\cdots v(\xi_n,\eta_n)$. In the definitions of S and c_l, we replace $\sqrt{2\pi}$ by $(2\pi)^{n/2}$,

$\langle\cdot,\cdot\rangle$ denotes the inner product of E_{2n} and $F \in \mathcal{B}^*(E_n)$. The new $B_{z,\zeta}$ is defined by

$$B_{z,\zeta} = [\prod_{j=1}^{n}(1+\partial_{z_j})^2(1+\partial_{\zeta_j})^2]A_{z,\zeta}. \tag{13}$$

The integral in Lemma 2 is now an integral over $\mathbb{R}^{3n}$ and the equality in (10) holds for all $(z,\zeta) \in \mathbb{R}^{2n}$. The integral in Lemma 3 is also over $\mathbb{R}^{3n}$, and v_l belongs to $\mathcal{S}(\mathbb{R}^n) \overset{\texttt{alg}}{\otimes} \mathcal{S}(\mathbb{R}^n)$.

3. Commutative case

In this section, we assume that C is equal to $C(\Omega)$, the algebra of continuous functions on a Hausdorff compact topological space Ω. For each $\lambda \in \Omega$ and each $f \in \mathcal{S}^C(\mathbb{R}^n)$, we define $V_\lambda f \in \mathcal{S}(\mathbb{R}^n)$ by

$$(V_\lambda f)(x) \;=\; [f(x)](\lambda), \quad x \in \mathbb{R}^n.$$

It is clear that V_λ extends to a continuous linear mapping $V_\lambda : E_n \longrightarrow L^2(\mathbb{R}^n)$, with $||V_\lambda|| \le 1$.

Lemma 4. *Let there be given $T \in \mathcal{B}^*(E_n)$, $f \in E_n$ and $\lambda \in \Omega$. If $V_\lambda f = 0$, then $V_\lambda T f = 0$.*

Proof. The equality $\langle V_\lambda g, V_\lambda g\rangle_{L^2(\mathbb{R}^n)} = \langle g,g\rangle(\lambda)$ holds for all $g \in \mathcal{S}^C(\mathbb{R}^n)$, hence also for all $g \in E_n$. We then have

$$\begin{aligned}\langle V_\lambda Tf, V_\lambda Tf\rangle &= \langle Tf,Tf\rangle(\lambda) = \langle f, T^*Tf\rangle(\lambda) = |\langle f, T^*Tf\rangle(\lambda)| \\ &\le \sqrt{\langle f,f\rangle(\lambda)}\sqrt{\langle T^*Tf, T^*Tf\rangle(\lambda)} \\ &= \sqrt{\langle V_\lambda f, V_\lambda f\rangle_{L^2(\mathbb{R}^n)}}\sqrt{\langle V_\lambda T^*Tf, V_\lambda T^*Tf\rangle_{L^2(\mathbb{R}^n)}}.\end{aligned}$$

This implies our claim. □

Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\tilde\varphi \in \mathcal{S}^C(\mathbb{R}^n)$ be defined by $[\tilde\varphi(x)](\lambda) = \varphi(x)$, for all $\lambda \in \Omega$ and all $x \in \mathbb{R}^n$. It is obvious that $V_\lambda\tilde\varphi = \varphi$. Given $T \in \mathcal{B}^*(E_n)$ and $\lambda \in \Omega$, let T_λ denote the unique linear mapping defined by the requirement that the diagram

$$\begin{array}{ccc} \mathcal{S}^C(\mathbb{R}^n) & \xrightarrow{T} & E_n \\ \Big\downarrow V_\lambda & & \Big\downarrow V_\lambda \\ \mathcal{S}(\mathbb{R}^n) & \xrightarrow{T_\lambda} & L^2(\mathbb{R}^n) \end{array}$$

commutes. This is well defined by Lemma 4 and because the left vertical arrow in the above diagram is surjective.

Lemma 5. *For each $T \in \mathcal{B}^*(E_n)$ and each $\lambda \in \Omega$, T_λ extends to a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, we have*

$$||T|| \;=\; \sup\{||T_\lambda||;\, \lambda \in \Omega\}. \tag{14}$$

Proof. Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\tilde{\varphi}$ denote the element of E_n defined after Lemma 4. We have

$$||T_\lambda \varphi||_{L^2(\mathbb{R}^n)} = ||V_\lambda T\tilde{\varphi}||_{L^2(\mathbb{R}^n)} \leq ||T\tilde{\varphi}||_2 \leq ||T|| \cdot ||\tilde{\varphi}||_2 = ||T|| \cdot ||\varphi||_{L^2(\mathbb{R}^n)}.$$

This implies that T_λ extends to a bounded operator on $L^2(\mathbb{R}^n)$ with norm bounded by $||T||$.

Let M denote the right-hand side of (14). For each $\lambda \in \Omega$ and each $f \in \mathcal{S}^C(\mathbb{R}^n)$, using Lemma 4 and the first statement in its proof, we get

$$\begin{aligned} |\langle Tf, Tf\rangle(\lambda)| &= |\langle V_\lambda Tf, V_\lambda Tf\rangle_{L^2(\mathbb{R}^n)}| \\ &= |\langle T_\lambda V_\lambda f, T_\lambda V_\lambda f\rangle_{L^2(\mathbb{R}^n)}| \leq ||T_\lambda|| \cdot ||V_\lambda f||_{L^2(\mathbb{R}^n)} \leq M||f||_2. \end{aligned}$$

Taking the supremum in λ on the left, we get $||Tf||_2 \leq M||f||_2$. □

Our goal in this section is to prove that the mapping S defined in the previous section is injective for $C = C(\Omega)$. This will finish the proof of Theorem 1.

Given $A \in \mathcal{H}$ such that $SA = 0$, we want to show that $A = 0$. In view of the following lemma, it suffices to show that $B = 0$, where $B = B_{0,0}$ ($B_{z,\zeta}$ as defined on (13)). Lemma 6 is [3, Proposition 3.1] when $C = \mathbb{C}$. The same proof works for any C*-algebra C.

Lemma 6. *If* $Y \in \mathcal{H}$, $Y_{z,\zeta} = T_{-z}M_{-\zeta}YM_\zeta T_z$ $(z, \zeta \in \mathbb{R}^n)$, *and either* $(1+\partial_{z_j})Y_{z,\zeta} \equiv 0$ *or* $(1+\partial_{\zeta_j})Y_{z,\zeta} \equiv 0$ *for some* j, *then* $Y = 0$.

By Lemma 5, in order to prove that $B = 0$, it suffices to show that $B_\lambda = 0$ for each $\lambda \in \Omega$. For z and ζ in $\mathbb{R}^n$, define $E_{z,\zeta} = M_\zeta T_z$. We then have $B_{z,\zeta} = E^*_{z,\zeta} B E_{z,\zeta}$. Using that $E_{z,\zeta}F^* = e^{iz\cdot\zeta}F^* E_{\zeta,-z}$, we may rewrite equation $SA = 0$ as

$$e^{iz\cdot\zeta}\langle (E_{z,\zeta} \otimes I)\tilde{u}_n, (BF^* E_{\zeta,-z} \otimes I)\tilde{v}_n\rangle = 0, \quad \text{for all} \quad (z, \zeta).$$

Evaluating this equation at λ gives

$$e^{iz\zeta}\langle (E_{z,\zeta} \otimes I)u_n, (B_\lambda F^* E_{\zeta,-z} \otimes I)v_n\rangle_{L^2(\mathbb{R}^n)} = 0, \quad \text{for all} \quad (z, \zeta). \tag{15}$$

For a fixed $\varphi \in C_c^\infty(\mathbb{R}^{2n})$ to be chosen soon and for each bounded operator D on $L^2(\mathbb{R}^n)$, define

$$\Xi(D) = \int \varphi(z, \zeta)e^{iz\zeta}\langle (E_{z,\zeta} \otimes I)u_n, (DF^* E_{\zeta,-z} \otimes I)v_n\rangle_{L^2(\mathbb{R}^n)} dz\, d\zeta\,. \tag{16}$$

In case D is finite-rank, and hence we may take $b^1, \cdots, b^k, c^1, \cdots, c^k$ in $L^2(\mathbb{R}^n)$ such that, for all $f \in L^2(\mathbb{R}^n)$,

$$DF^* f = \sum_{j=1}^k b^j \langle c^j, f\rangle_{L^2(\mathbb{R}^n)},$$

we have

$$\Xi(D) =$$

$$\sum_{j=1}^k \iint b^j(x)\bar{c}^j(\xi) \iiint e^{iz\cdot\zeta}\varphi(z,\zeta)e^{-ix\cdot\zeta}\bar{u}_n(x - z, \eta)e^{-iz\cdot\xi}v_n(\xi - \zeta, \eta) dz\, d\zeta\, d\eta\, d\xi\, dx\,.$$

Making the change of variables $x - z = z'$, $\xi - \zeta = \zeta'$ on the inner triple integral above, we get

(17)

$$\Xi(D)=\sum_{j=1}^{k}\iint b^j(x)\bar{c}^j(\xi)e^{-ix\cdot\xi}\iiint e^{iz\cdot\zeta}\varphi(x-z,\xi-\zeta)\bar{u}_n(z,\eta)v_n(\zeta,\eta)dz\,d\zeta\,d\eta\,d\xi\,dx\,.$$

For arbitrary χ and ψ in $C_c^\infty(\mathbb{R}^n)$, let φ be defined by

$$(1+\partial_x)^2(1+\partial_\xi)^2[e^{ix\xi}\bar{\chi}(-x)\psi(-\xi)] = \varphi^\sharp(x,\xi), \quad \varphi(x,\xi) = \varphi^\sharp(-x,-\xi).$$

Using the higher dimensional version of Lemma 2 mentioned at the end of Section 2, the right side of (17) becomes

$$\sum_{j=1}^{k}\iint b^j(x)\bar{c}^j(\xi)\bar{\chi}(x)\psi(\xi)dx\,d\xi \;=\langle\chi, DF^*\psi\rangle_{L^2(\mathbb{R}^n)}.$$

This shows that, for this choice of φ,

$$\Xi(D) = \langle\chi, DF^*\psi\rangle_{L^2(\mathbb{R}^n)}, \tag{18}$$

whenever D has finite rank.

Let $\{\phi_1,\phi_2,\cdots\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. For each positive integer j, let P_j denote the orthogonal projection onto the spam of $\{\phi_1,\cdots,\phi_j\}$. Cordes proved ([3, Chapter 8], between equations (3.27) and (3.29)) that, for any bounded operator T on $L^2(\mathbb{R}^n)$, one has $\lim_{j\to\infty}\Xi(P_jTP_j) = \Xi(T)$. Applying this to $T = B_\lambda$ and using (18), we get

$$\Xi(B_\lambda) = \lim_{j\to\infty}\Xi(P_jB_\lambda P_j) = \lim_{j\to\infty}\langle\chi, P_jB_\lambda P_jF^*\psi\rangle_{L^2(\mathbb{R}^n)} \;=\; \langle\chi, B_\lambda F^*\psi\rangle_{L^2(\mathbb{R}^n)}.$$

By (15), the left-hand side of this equality vanishes. Since χ and ψ are arbitrary test functions, this shows that $B_\lambda = 0$. This finishes the proof of Theorem 1 (recall our remarks before and after the statement of Lemma 6).

4. Rieffel's conjecture

Given a skew-symmetric $n\times n$ matrix J and $F\in\mathcal{B}^C(\mathbb{R}^n)$ (i.e., $F:\mathbb{R}^n\to C$ is smooth and, together with all its derivatives, is bounded), let us denote by L_F the pseudodifferential operator $a(x,D)\in\mathcal{B}^*(E_n)$ with symbol $a(x,\xi) = F(x+J\xi)$. At the end of Chapter 4 in [10], Rieffel made a conjecture that, in the commutative case, may be rephrased as follows: any operator $A\in\mathcal{B}^*(E_n)$ that is Heisenberg-smooth and commutes with every operator of the form $R_G = b(x,D)$, where $b(x,\xi) = G(x-J\xi)$ with $G\in\mathcal{B}^C(\mathbb{R}^n)$, is of the form $A = L_F$ for some $F\in\mathcal{B}^C(\mathbb{R}^n)$.

Using Cordes' characterization of the Heisenberg-smooth operators in the scalar case, we have shown [6] that Rieffel's conjecture is true when $C = \mathbb{C}$. It has been further proven by the second author [7] that Rieffel's conjecture is true for any separable C^*-algebra C for which the operator O defined in (4) is a bijection. Under this assumption, a result actually stronger than what was conjectured by Rieffel was proven in [7, Theorem 3.5]: To get $A = L_F$ for some $F\in\mathcal{B}^C(\mathbb{R}^n)$, one only needs to require that a given $A\in\mathcal{B}^*(E_n)$ is "translation-smooth" (i.e., the mapping $\mathbb{R}^n\ni z\mapsto T_{-z}AT_z\in\mathcal{B}^*(E_n)$ is smooth) and commutes with every R_G with $G\in\mathcal{S}^C(\mathbb{R}^n)$. Combining this result with our Theorem 1, we then get

Theorem 2. *Let C be a unital commutative separable C^*-algebra. If a given $A \in \mathcal{B}^*(E_n)$ is translation-smooth and commutes with every R_G, $G \in \mathcal{S}^C(\mathbb{R}^n)$, then $A = L_F$ for some $F \in \mathcal{B}^C(\mathbb{R}^n)$.*

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