UNIFORM PERIODIC POINT GROWTH IN ENTROPY RANK ONE

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Abstract. We show that algebraic dynamical systems with entropy rank one have uniformly exponentially many periodic points in all directions.

1. Introduction

Let $\alpha$ be an action of $\mathbb{Z}^d$ by continuous automorphisms of a compact metrizable abelian group $X$ (such a system is called an algebraic $\mathbb{Z}^d$-action). For a continuous map $\beta : X \to X$ write $h(\beta)$ for the topological entropy and

$$F(\beta) = \{ x \in X \mid \beta x = x \}$$

for the set of fixed points. The action $\alpha$ is said to have entropy rank one if, for each $n \in \mathbb{Z}^d$, $h(\alpha^n) < \infty$.

If $\alpha$ is a mixing entropy rank one action and the topological dimension $\dim(X)$ is finite, $F(\alpha^n)$ is finite for all $n \in \mathbb{Z}^d$. Our purpose here is to show that under natural conditions $|F(\alpha^n)|$ exhibits uniform exponential growth. Write $n_j \to \infty$ for a sequence in $\mathbb{Z}^d$ if for any finite set $F \subset \mathbb{Z}^d$ there is some $J$ for which $j > J$ implies that $n_j \notin F$. Equivalently, this means $\|n_j\| \to \infty$ as $j \to \infty$ where $\| \cdot \|$ is the Euclidean norm on $\mathbb{Z}^d$. The Noetherian condition mentioned in Theorem 1.1 is explained in Section 2.

Theorem 1.1. Let $\alpha$ be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group $X$. Then there exist constants $C_1, C_2 \geq 0$ such that

$$\limsup_{n \to \infty} \frac{1}{\|n\|} \log |F(\alpha^n)| = C_1 < \infty$$

and

$$\liminf_{n \to \infty} \frac{1}{\|n\|} \log |F(\alpha^n)| = C_2.$$ 

If $\dim(X) > 0$ and the action is Noetherian, then $C_2 > 0$.

An affirmative answer to Lehmer’s problem (see Remark 2.3) would render the finite-dimension assumption in Theorem 1.1 redundant. The assumption is also not required for a mixing Noetherian entropy rank one algebraic action; in that setting...

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it is a consequence of the Noetherian condition. Essentially, the only non-trivial conclusion in the theorem is that $C_2 > 0$.

The action $\alpha$ is **expansive** if there exists a neighbourhood $U$ of $0_X$ such that
\[
\bigcap_{n \in \mathbb{Z}^d} \alpha^n(U) = \{0_X\}.
\]

For $d = 1$ and $\alpha$ expansive, the growth rate of periodic points exists, so $C_1 = C_2$ in Theorem 1.1. The constant coincides with the entropy and is non-zero provided $X$ is infinite.

The $\mathbb{Z}^2$-action on a zero-dimensional group in Example 1.3 is expansive yet has $C_2 = 0$. Expansive actions on connected groups are more indicative of the import of a positive value for $C_2$. In terms of expansive subdynamics (see [3] and [6]), there are sequences $n \to \infty$ converging to non-expansive lines; along such sequences $|F(\alpha^n)|$ is much smaller than the same expression with $n$ of similar Euclidean size and far from non-expansive directions. Nonetheless, there is a uniform exponential growth in all directions.

**Example 1.2.** Consider the $\mathbb{Z}^2$-action $\alpha$ dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto 2r$ and $r \mapsto 3r$ on $\mathbb{Z}[\mathbb{Z}_2^2]/\langle 2, 1 + u_1 + u_2 \rangle$ (this is an example of the type introduced by Ledrappier [12]; $X$ is a zero-dimensional group). As shown in [4], $|F(\alpha(n,0))| = 2^{n - 2^{\text{ord}_2(n)}}$. In particular,
\[
\lim_{n \to \infty} \frac{1}{\| (2^n, 0) \|} \log |F(\alpha(2^n,0))| = 0,
\]
showing that some assumption on the topological dimension of $X$ is needed to have $C_2 > 0$ in Theorem 1.1. If the map dual to $r \mapsto u_2 r$ is ignored, then this

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**Table 1.** Periodic point counts for $\times 2, \times 3$. 

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example is a non-expansive (and non-Noetherian) $\mathbb{Z}$-action on a zero-dimensional group with $C_2 = 0$.

**Example 1.4.** The Noetherian condition is needed to have $C_2 > 0$ even for $d = 1$ on connected groups. For example, the automorphism $\alpha$ dual to the map $r \mapsto 2r$ on $\mathbb{Q}$ has $|F(\alpha^n)| = 1$ for all $n$. For rings between $\mathbb{Z}$ and $\mathbb{Q}$ a variety of exotic periodic point behavior is possible (see [20] or [21] for the details).

The full statement of Theorem 1.1 does however apply to Noetherian non-expansive systems. Thus, for example, the genuinely partially hyperbolic systems such as that described by Damjanović and Katok [5, Ex. 7.3] satisfy the hypotheses.

2. **Proof of Theorem 1.1**

The case $d = 1$ is covered by [4] and [16], so we may assume $d \geq 2$. Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [11].

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \ldots, u_d$ with integer coefficients. If $\alpha$ is an algebraic $\mathbb{Z}^d$-action of a compact abelian group $X$, the character group $\hat{X}$ has the structure of a discrete countable $\mathbb{R}^d$-module, obtained by first identifying the dual automorphism $\hat{\alpha}^n$ with multiplication by the monomial $u^n = u_1^{n_1} \cdots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, any countable $\mathbb{R}^d$-module $M$ has an associated algebraic $\mathbb{Z}^d$-action obtained by dualizing the action induced by multiplying by monomials on $M$. The action $\alpha = \alpha_M$ is described as *Noetherian* if the module $M$ is Noetherian. A full account of this correspondence and the resulting theory is given in Schmidt’s monograph [19].

Entropy rank one actions are described in [7] and developments concerning their periodic points may be found in the papers [16] and [18]. By a slight abuse of notation, write $h(\cdot)$ for the topological entropy of maps and for the extension of the entropy function to all of $\mathbb{R}^d$ in the sense explained below.

**Proposition 2.1.** Let $\alpha_M$ be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group. Then

1. The set of associated primes of the associated $R_d$-module, $\text{Asc}(M)$, is finite. For each $p \in \text{Asc}(M)$, the domain $R_d/p$ has Krull dimension 1 and its field of fractions $\mathbb{K}(p)$ is a global field.
2. There exist positive integers $m(p)$, $p \in \text{Asc}(M)$, such that for every non-zero $n \in \mathbb{Z}^d$,
   
   $h(\alpha^n_M) = \sum_{p \in \text{Asc}(M)} m(p)h(\alpha^n_{R_d/p})$

   and

   $|F(\alpha^n_M)| \leq \prod_{p \in \text{Asc}(M)} |F(\alpha^n_{R_d/p})|^{m(p)}$,

   with equality if $M$ is Noetherian.
3. For each $p \in \text{Asc}(M)$, there is a finite set of places $S(p)$ of $\mathbb{K}(p)$ such that

   $h(\alpha^n_{R_d/p}) = \sum_{v \in S(p)} \max\{1_v \cdot n, 0\} > 0$
and

\[ |F(\alpha_{R_d/p}^n)| = \prod_{v \in S(p)} |\xi_p^v - 1|, \]

where \( \xi_p = (\xi_1, \ldots, \xi_d) \) denotes the image of \( u \) in \((R_d/p)^d\) and \( L_v = (\log |\xi_1|, \ldots, \log |\xi_d|) \).

**Proof.** Let \( n \in \mathbb{Z}^d \) be non-zero. Since \( \alpha_M \) is mixing, [19 Prop. 6.6] shows that for each \( p \in \text{Asc}(M) \), \( \alpha_{R_d/p}^n \) is ergodic, so \( h(\alpha_{R_d/p}^n) > 0 \). It follows from [7] that for each \( p \in \text{Asc}(M) \), \( R_d/p \) has Krull dimension 1 and \( \mathbb{K}(p) \) is a global field. If \( \text{char}(R_d/p) > 0 \), then \( h(\alpha_{R_d/p}^n) \geq \log 2 \). Via Yuzvinskiı’s formula (see [19 Th. 14.1] or [23]), this implies there are only finitely many such \( p \in \text{Asc}(M) \), as \( h(\alpha_{R_d/p}^n) < \infty \).

Also, there can be only finitely many \( p \in \text{Asc}(M) \) with \( \text{char}(R_d/p) = 0 \), as

\[ \dim(X) \geq |\{p \in \text{Asc}(M) : \text{char}(R_d/p) = 0\}|. \]

This establishes (1).

The method of [7] Lem. 8.2] shows that in any prime filtration of a Noetherian submodule of \( M \), each prime \( p \in \text{Asc}(M) \) appears with a maximum multiplicity

\[ m(p) = \dim_{\mathbb{K}(p)}(M \otimes_{R_d} \mathbb{K}(p)), \]

which is finite by similar reasoning to the proof of (1). By adopting an algorithm for obtaining a prime filtration that selects the associated primes of \( M \) first, one obtains a Noetherian submodule \( L \subset M \) such that each prime \( p \in \text{Asc}(M) \) appears with multiplicity \( m(p) \) in a filtration of \( L \). Furthermore, if \( L \neq M \), then each \( q \in \text{Asc}(M/L) \) is maximal and \( R_d/q \) is a finite field. Hence, Yuzvinskiı’s formula shows that \( h(\alpha_{M/L}^n) = 0 \) and \( h(\alpha_{R_d}^n) = h(\alpha_{L}^n) \); the formula (2.1) then follows from [15 Lem. 4.3].

If \( M \) is Noetherian, equality in (2.2) is given by [16 Th. 3.2]. If \( M \) is not Noetherian, (2.2) follows from [16 Th. 3.2] applied to \( L \), together with the inequality

\[ |F(\alpha_M^n)| \leq |F(\alpha_L^n)|, \]

which is established using a similar method to the proof of [17 Lem. 2.6].

Finally, the entropy formula (2.3) follows from [7 Prop. 8.5] and the periodic point counting formula (2.4) is [16 Lem 3.1]. \( \square \)

**Proof of Theorem 1.1.** Let \( M = \hat{X} \) denote the dual \( R_d \)-module and let \( p \in \text{Asc}(M) \). For a fixed \( p \in \text{Asc}(M) \) and any non-zero \( n \in \mathbb{Z}^d \), set

\[ f(n) = \frac{1}{\|n\|} \log |F(\alpha_{R_d/p}^n)|. \]

Let \( h : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) denote the directional entropy function for the action \( \alpha_{R_d/p} \). This is the function obtained by extending the entropy formula (2.3) to values of \( \mathbb{R}^d \) (see [7 Sec. 8] for further details).

For any vector \( v \in \mathbb{R}^d \setminus \{0\} \), write \( \tilde{v} \in S_{d-1} \) for the unique vector of unit length with the property that \( v = \lambda \tilde{v} \) for some scalar \( \lambda > 0 \). From (2.3) and (2.4) it follows that

\[ f(n) = g(n) + h(\tilde{n}) \]
where
\[ g(n) = \frac{1}{|n|} \sum_{v \in \mathcal{S}(p)} \log |1 - \phi_v(n)|_v \]
and
\[ \phi_v(n) = \begin{cases} \xi_p^n & \text{if } |\xi_p^n|_v > 1; \\ \xi_p^n & \text{if } |\xi_p^n|_v \leq 1. \end{cases} \]

Notice that \( \phi_v(n) \neq 1 \) if \( n \neq 0 \) by the assumption that the action is mixing.

To establish the lower bound \( C_2 > 0 \) in Theorem 1.1, first note that equality in (2.2) gives the expression
\[ \frac{1}{|n|} \log |F(\alpha^{n}_{R_d})| \]
as a finite sum of terms of the form (2.5) and, crucially, the assumption \( \dim(X) > 0 \) means at least one of these terms arises from a prime \( p \) such that \( \mathbb{K}(p) \) is an algebraic number field (rather than a function field of positive characteristic). For the lower bound, it is therefore enough by (2.2) to consider only the asymptotic behavior of \( f \) with the assumption that it arises from such a prime.

We need to know that \( g \) does not make an asymptotic contribution. It is clear that \( g \) cannot be too large, since
\[ \prod_{v \in \mathcal{S}(p)} |1 - \phi_v(n)|_v \leq 2^{|\mathcal{S}(p)|}. \]

On the other hand, it cannot be large and negative for the following reason. If \( v \) is an infinite (Archimedean) place, then Baker’s theorem [1] can be used to find constants \( A, B \geq 0 \) such that
\[ |1 - \phi_v(n)|_v \geq \frac{A}{\max\{n_i\}^B} \]
(see [4] or [9] for similar arguments; roughly speaking, the issue is to bound the proximity of \( \xi_p^n \) and 1 in terms of \( n \)). If \( v \) is a finite place, then Yu’s \( p \)-adic bounds for linear forms in logarithms [22] give a similar lower bound (note that \( d \geq 2 \) by assumption). Since \( \mathcal{S}(p) \) is finite, this shows that
\[ g(n) \to 0 \text{ as } n \to \infty. \]

Assume, for a contradiction, that \( C_2 = 0 \). Then there is a sequence \( n_j \to \infty \) as \( j \to \infty \) with the property that
\[ \lim_{j \to \infty} f(n_j) = 0. \]

It follows from (2.6) and (2.7) that
\[ h(n_j) \to 0 \text{ as } j \to \infty. \]

Now the entropy function \( h \) restricted to the unit sphere \( S_{d-1} \) is a continuous function on a compact set, so (2.9) implies that there is some \( v \in S_{d-1} \) for which
\[ h(v) = 0. \]

If the ray through \( v \) happens to meet a point \( m \in \mathbb{Z}^d \), then we have an immediate contradiction: the automorphism \( \alpha_m \mathbb{R}_d/p \) would have zero entropy, contradicting (2.3). In order to show that any point \( v \) with (2.10) gives a contradiction we need to make a quantitative version of that argument.
The explicit formula (2.3) for $h$ means there is a list of vectors 
\[ a_1, \ldots, a_r \in \mathbb{R}^d \]
with the property that for any $u \in \mathbb{R}^d$,
\[ h(u) = u \cdot a_k, \]
for some $k$, $1 \leq k \leq r$. It follows that for any $n \in \mathbb{Z}^d$ and $u \in \mathbb{R}^d$,
\[ |h(n) - h(u)| \leq \max_{1 \leq k \leq r} \max_{1 \leq i \leq d} |a_{k,i}| \|n - u\|, \]
where $a_k = (a_{k,1}, \ldots, a_{k,d})$. In particular, since $h(\lambda v) = 0$ for all $\lambda > 0$ we may find a sequence $(m_j)$ of vectors in $\mathbb{Z}^d$ with $\|m_j - \lambda_j v\| \to 0$ as $j \to \infty$ (for some sequence $(\lambda_j)$ of scalars) and hence have
\begin{equation}
(2.11) \quad h(m_j) \to 0 \text{ as } j \to \infty.
\end{equation}
On the other hand, the dual group of $R_d/p$ is connected and finite-dimensional. Hence, by Yuzvinskii’s formula, for any $n \neq 0$,
\[ h(n) \geq m(P) > 0 \]
where $m(P) = \int_0^1 \log |P(e^{2\pi iv})| \, dv$ denotes the logarithmic Mahler measure of some polynomial $P$ of degree no greater than $\dim_{\mathbb{Q}}(\mathbb{K}(P))$. It follows that there is a constant $C > 0$ (depending only on $p$) for which
\[ h(n) > C > 0 \text{ for any } n \neq 0 \]
(the existence of a lower bound for the non-zero logarithmic Mahler measure of polynomials of bounded degree is well known; see [2] or [8] for the background). This lower bound contradicts (2.11), so (2.8) is impossible. Thus $C_2 > 0$.

The upper bound is clear: the inequality (2.2) together with the explicit formula (2.4) gives a uniform constant $C_1$ with the property that
\[ |F(a^n)| \leq C_1^{\max_{1 \leq i \leq d} \{n_i\}}. \]
\[ \square \]

**Remark 2.2.** Baker’s theorem provides the key estimates in many dynamical problems; see [1] and [9] for examples. As pointed out by Lind [14], in order to establish the logarithmic growth rate of periodic points for a quasihyperbolic toral automorphism (a typical application), sometimes all that is needed is a weaker and earlier result due to Gel’fond [10]. Here we need something closer to the full weight of the theorems of Baker and Yu, because we are in a higher rank.

**Remark 2.3.** Lehner’s problem [13] asks if there is a uniform lower bound for all positive Mahler measures. As shown by Lind (see [15]) this is equivalent to a uniform lower bound for the topological entropy of any mixing compact group automorphism. If ‘Lehner’s conjecture’ (that there is such a bound and that it is attained by the expected polynomial) holds, then the topological entropy of any compact group automorphism with positive entropy is at least
\[ 0.162 \cdots = m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1). \]
This does not imply a uniform bound for $C_2$ in Theorem 1.1 because $C_2$ is influenced by the geometry of the acting group as well as the collection of maps in its image. For example, the $\mathbb{Z}^2$-action $\alpha$ corresponding to the module $R_2/\langle u_1^2 - 2, u_1^7u_2^9 \rangle$ has the property that $h(\alpha^{(k,1)}) = \log 3$, so the corresponding constant $C_2$ cannot exceed $\log 3/\sqrt{1 + k^2}$.
References


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