GLOBAL WELL-POSEDNESS
OF DISSIPATIVE QUASI-GEOSTROPHIC EQUATIONS
IN CRITICAL SPACES

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Abstract. We prove global well-posedness for the dissipative quasi-geostrophic equation with initial data in critical Besov spaces $B^{1+\frac{2}{p}-2\alpha}_{p,q,0}$, $0 < \alpha \leq 1$, provided that the $B^{1+\frac{2}{p}-2\alpha}_{p,q}$ norm of the initial data is sufficiently small compared with the dissipative coefficient $\kappa$.

1. Introduction

We are concerned with the two dimensional dissipative quasi-geostrophic equation

\[
\begin{align*}
\theta_t + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta &= 0, \\
u &= (-R_2 \theta, R_1 \theta), \\
\theta(x,0) &= \theta_0(x).
\end{align*}
\]

(DQG)

where the scalar $\theta$ represents the potential temperature, $u$ is the fluid velocity, and $R_1, R_2$ are the usual Riesz transform. For the physical background of this equation, one may check [1], [3] and references therein for the details. We solve the open problem given by [1]; namely, with $\theta_0 \in B^{1+\frac{2}{p}-2\alpha}_{p,q}$, for $1 \leq p, q < \infty$, what is the well-posedness of (DQG)? Two crucial estimates were proved in [3], [4], and we use those estimates to get the following result.

Theorem. There exists a constant $\epsilon_0 > 0$ such that for any $\theta_0 \in B^{1+\frac{2}{p}-2\alpha}_{p,q}$ with $\|\theta_0\|_{B^{1+\frac{2}{p}-2\alpha}_{p,q}} < \epsilon \leq \epsilon_0$, (DQG) has a unique global solution $\theta$, which belongs to $C([0,\infty); B^{1+\frac{2}{p}-2\alpha}_{p,q})$.

2. Proof of theorem

Step 1. A priori estimates. Let $\triangle_j$ be the Fourier multiplier given by $\triangle_j f = \Phi_j * f$ ($j = 0, \pm 1, \pm 2, \cdots$) where $\Phi_j(\xi)$ is a smooth function localized around $|\xi| = 2^j$ satisfying $\sum_{k=-\infty}^{\infty} \Phi_k = 1$, except for $\xi = 0$. Applying the operator $\triangle_j$ to the first equation of (DQG), we obtain

\[
\frac{d}{dt}\triangle_j \theta + \triangle_j (u \cdot \nabla \theta) + \kappa (-\Delta)^\alpha \triangle_j \theta = 0.
\]
Multiplying by $\frac{1}{p} \Delta_j \theta \cdot |\Delta_j \theta|^{p-2}$ in the above equation and then integrating with respect to $x$, we have
\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p}^p + \frac{1}{p} \int (-\Delta)^\alpha \cdot \Delta_j \theta \cdot \Delta_j \theta \cdot |\Delta_j \theta|^{p-2} = -\frac{1}{p} \int \Delta_j (u \cdot \nabla \theta) \cdot \Delta_j \theta \cdot |\Delta_j \theta|^{p-2}.
\]
Wu [4] proved the following lower bound estimate:
\[
\int (-\Delta)^\alpha \cdot \Delta_j \theta \cdot \Delta_j \theta \cdot |\Delta_j \theta|^{p-2} \geq C \cdot 2^{2j\alpha} \cdot ||\Delta_j \theta||_{L^p}^p.
\]
So we obtain that
\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p}^p + C \cdot \kappa \cdot 2^{2j\alpha} \cdot ||\Delta_j \theta||_{L^p}^p \leq C \cdot \int \Delta_j (u \cdot \nabla \theta) \cdot \Delta_j \theta \cdot |\Delta_j \theta|^{p-2} \, dx.
\]
We decompose $(u \cdot \nabla \theta)$ as a paraproduct. (We obtain estimates of this product term. See the appendix.) Then,
\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p}^p + C \cdot \kappa \cdot 2^{2j\alpha} \cdot ||\Delta_j \theta||_{L^p}^p \leq C \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \cdot \alpha_j ||\Delta_j \theta||_{L^p}^{p-1} ||\theta||^{2} B_{p,q}^{1+\frac{2}{p}-2\alpha}.
\]
Dividing both sides by $||\Delta_j \theta||_{L^p}^{p-1}$,
\[
\frac{d}{dt} ||\Delta_j \theta||_{L^p} + C \cdot \kappa \cdot 2^{2j\alpha} \cdot ||\Delta_j \theta||_{L^p} \leq C \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \cdot \alpha_j ||\theta||^{2} B_{p,q}^{1+\frac{2}{p}-2\alpha}.
\]
By solving the above differential equation of time, we get
\[
||\Delta_j \theta(t)||_{L^p} \leq e^{-t2^{2j\alpha} \kappa} \cdot ||\Delta_j \theta_0||_{L^p} + C \cdot a_j \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \int_0^t e^{-(t-s)2^{2j\alpha} \kappa} ||\theta||^{2} B_{p,q}^{1+\frac{2}{p}-2\alpha} \, ds.
\]
By Young’s inequality in time,
\[
||\Delta_j \theta(t)||_{L^p} \leq ||\Delta_j \theta_0||_{L^p} + \frac{C}{\kappa} \cdot a_j \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \cdot ||\theta||^{2} L^\infty_{t} B_{p,q}^{1+\frac{2}{p}-2\alpha}.
\]
We note that $||\theta(t)||_{L^p} \leq ||\theta_0||_{L^p}$ was proved in [3]. So,
\[
||\theta||_{L^\infty_{t} B_{p,q}^{1+\frac{2}{p}-2\alpha}} \leq ||\theta_0||_{B_{p,q}^{\frac{2}{p}+1-2\alpha}} + \frac{C}{\kappa} ||\theta||^{2} L^\infty_{t} B_{p,q}^{1+\frac{2}{p}-2\alpha}.
\]
Step 2. Iteration and uniform estimates. Because the bicontinuous constant arising in the above estimate does not depend on time, we will look for a solution \( w(x, t) = \theta(x, t) - S_\alpha(t)\theta_0 \), instead of looking for a solution \( \theta(x, t) \), where \( S_\alpha(t)\theta_0 = e^{-\kappa t(\Delta)^\alpha}\theta_0 \). \( w(x, t) \) satisfies

\[
\begin{align*}
  w_t + u \cdot \nabla (w + S_\alpha(t)\theta_0) + \kappa(\Delta)^\alpha w &= 0, \\
  u &= (-R_2(w + S_\alpha(t)\theta_0), R_1(w + S_\alpha(t)\theta_0)), \\
  w(x, 0) &= 0.
\end{align*}
\]

We define the following sequences:

\[
\begin{align*}
  w^{n+1} + u^n \cdot \nabla (w^{n+1} + S_\alpha(t)\theta_0) + \kappa(\Delta)^\alpha w^{n+1} &= 0, \\
  u^n &= (-R_2(w^n + S_\alpha(t)\theta_0), R_1(w^n + S_\alpha(t)\theta_0)), \\
  w^{n+1}(x, 0) &= 0.
\end{align*}
\]

Similarly to a priori estimates, we have

\[
\begin{align*}
  &||w^{n+1}||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \\
  &\leq \frac{C}{\kappa} ||w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (||w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} + ||S_\alpha(t)\theta_0||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}}).
\end{align*}
\]

Let \( \epsilon_0 \leq \frac{\kappa}{4C} \), and fix \( \eta \) such that \( \eta < \epsilon_0 \). If \( ||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}} \leq \epsilon < \epsilon_0 \), then \( ||w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \) are uniformly bounded by \( ||w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \leq \eta \).

Step 3. Equations of difference, existence, and uniqueness. Let \( \delta w^n = w^n - w^{n-1}, \delta u^n = u^n - u^{n-1} \). Then we have the following system of difference equations:

\[
\begin{align*}
  \delta w^{n+1} + u^n \cdot \nabla \delta w^{n+1} + \kappa(\Delta)^\alpha \delta w^{n+1} + \delta u^n \cdot \nabla (w^n + S_\alpha(t)\theta_0) &= 0, \\
  u^n &= (-R_2(w^n + S_\alpha(t)\theta_0), R_1(w^n + S_\alpha(t)\theta_0)), \\
  \delta w^{n+1}(x, 0) &= 0.
\end{align*}
\]

Then, as before, we get

\[
\begin{align*}
  ||\delta w^{n+1}||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} &\leq \frac{C}{\kappa} ||\delta w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (||w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} + ||\theta_0||_{B_{p,q}^{1+\frac{2}{p}-2\alpha}}) \\
  &< \frac{C}{\kappa} ||\delta w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}} \cdot (\eta + \epsilon) < \frac{1}{2} ||\delta w^n||_{L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha}}.
\end{align*}
\]

So, \( w^n \) converges to \( w \) in \( L_T^\infty B_{p,q}^{1+\frac{2}{p}-2\alpha} \). Furthermore, we can take \( \eta \) as small as we want. Hence \( w^n \) converges to \( w \) in \( C([0, T); B_{p,q}^{1+\frac{2}{p}-2\alpha}) \). Uniqueness can be proved similarly. This completes the proof of theorem.
3. Appendix

We decompose \((u \cdot \nabla \theta)\) as a paraproduct:

\[
\triangle_j(u \cdot \nabla \theta) = \sum_{|j-l| \leq N} \triangle_j(S_{l-1}u \cdot \triangle_l \nabla \theta) + \sum_{|j-l| \leq N} \triangle_j(\triangle_l u \cdot S_{l-1} \nabla \theta) + \sum_{l \geq j-N} \sum_{|l-m| \leq 1} \triangle_j(\triangle_l u \cdot \triangle_m \nabla \theta).
\]

(1)

So we have three terms to the right-hand side of (1). Motivated by [2], we decompose \(I\) defined below as

\[
I = \sum_{|l-j| \leq N} \int \triangle_j(S_{l-1}u \cdot \triangle_l \nabla \theta) \cdot \triangle_j \theta \cdot |\triangle_j \theta|^{p-2} dx
\]

\[
\leq \sum_{|l-j| \leq N} \int [\triangle_j, S_{l-1}u] \nabla \triangle_l \theta \cdot \triangle_j \theta |\triangle_j \theta|^{p-2}
\]

\[
+ \sum_{|l-j| \leq N} \int (S_{l-1}u - S_{j-1}u) \nabla \triangle_j \triangle_l \theta \cdot \triangle_j \theta |\triangle_j \theta|^{p-2}
\]

\[
+ \sum_{|l-j| \leq N} \sum_{|l-m| \leq 1} S_{j-1}u \cdot \nabla \triangle_j \triangle_l \theta \cdot \triangle_j \theta |\triangle_j \theta|^{p-2} dx
\]

\[
= I_1 + I_2 + I_3.
\]

\(I_3\) disappears when integrated, by the divergence free condition of \(u\). (From now on, we repeatedly use Bernstein’s inequalities.) By Hölder inequality,

\[
I_1 = \sum_{|l-j| \leq N} \int [\triangle_j, S_{l-1}u] \nabla \triangle_l \theta \cdot \triangle_j \theta |\triangle_j \theta|^{p-2}
\]

\[
\leq C \cdot ||[\triangle_j, S_{l-1}u] \nabla \triangle_l \theta||_{L^p} \cdot ||\triangle_j \theta||_{L^p}^{p-1}
\]

\[
\leq C \cdot 2^{-j} ||\nabla S_{j-1}u||_{L^\infty} ||\nabla \triangle_j \theta||_{L^p} ||\triangle_j \theta||_{L^p}^{p-1}.
\]

But, by the Calderon-Zygmund theorem, we have

\[
||\nabla S_{j-1}u||_{L^\infty} \leq C \cdot 2^{2j\alpha} ||\theta||_{B^{1+\frac{2}{p}-2\alpha}}
\]

Therefore

\[
I_1 \leq C \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \cdot a_j ||\triangle_j \theta||_{L^p}^{p-1} ||\theta||^2_{B^{1+\frac{2}{p}-2\alpha}}
\]

where \(\{a_j\} \in l^q\) such that \(\sum_{j \geq -1} a_j^q = 1\). Similarly

\[
I_2 \leq C \cdot ||\triangle_j u||_{L^\infty} ||\nabla \triangle_j \theta||_{L^p} ||\triangle_j \theta||_{L^p}^{p-1}
\]

\[
\leq C \cdot 2^{(1+\frac{2}{p})} ||\triangle_j u||_{L^p} ||\triangle_j \theta||_{L^p}^{p}
\]

\[
\leq C \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{2}{p}-2\alpha)} \cdot a_j ||\triangle_j \theta||_{L^p}^{p-1} ||\theta||^2_{B^{1+\frac{2}{p}-2\alpha}}.
\]
In the same way, we get the estimate for the second term of (1). The third term, denoted by $III$, is given by

$$III = \left| \sum_{l \geq j-N} \sum_{l-m \leq 1} \int \nabla \cdot \left( \nabla u \cdot \nabla \nabla \theta \right) \Delta_j \theta \Delta_j \theta \right|^{p-2} \right|$$

$$\leq C \cdot \sum_{l \geq j-N} \left\| \Delta_l \theta \right\|_{L^p} \left\| \Delta_l \theta \right\|_{L^p} \left\| \nabla \left( \nabla \Delta_j \theta \right) \right\|_{L^\infty}$$

$$\leq C \cdot 2^{j(1+\frac{p}{2})} \cdot \left\| \Delta_j \theta \right\|_{L^p} \sum_{l \geq j-N} \left\| \Delta_l \theta \right\|^2_{L^p}$$

$$\leq C \cdot 2^{j(1+\frac{p}{2})} \cdot \left\| \Delta_j \theta \right\|_{L^p} \sum_{l \geq j-N} 2^{-2l(1+\frac{\alpha}{p}+2-2\alpha)} \cdot 2^{l(1+\frac{p}{2}+2-2\alpha)} \left\| \Delta_l \theta \right\|_{L^p}$$

$$\leq C \cdot 2^{2j\alpha} \cdot 2^{-j(1+\frac{p}{2}+2\alpha)} \cdot a_j \left\| \Delta_j \theta \right\|_{L^p} \left\| \nabla \theta \right\|^2_{B^{2-2\alpha}_{p,q}}.$$

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**REFERENCES**


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